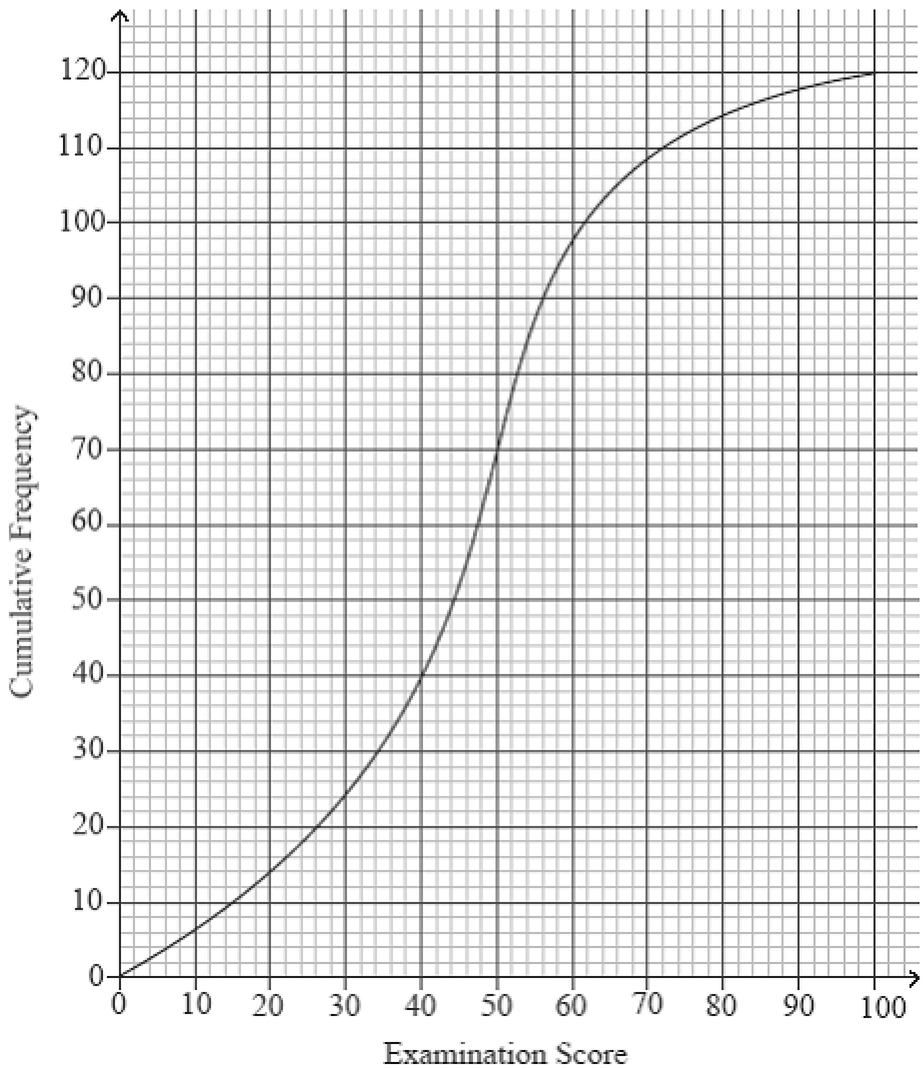
Name:

1. (6 points) 120 Mathematics students in a school sat an examination. Their scores (given as a percentage) were summarized on a cumulative frequency diagram. This diagram is given below.



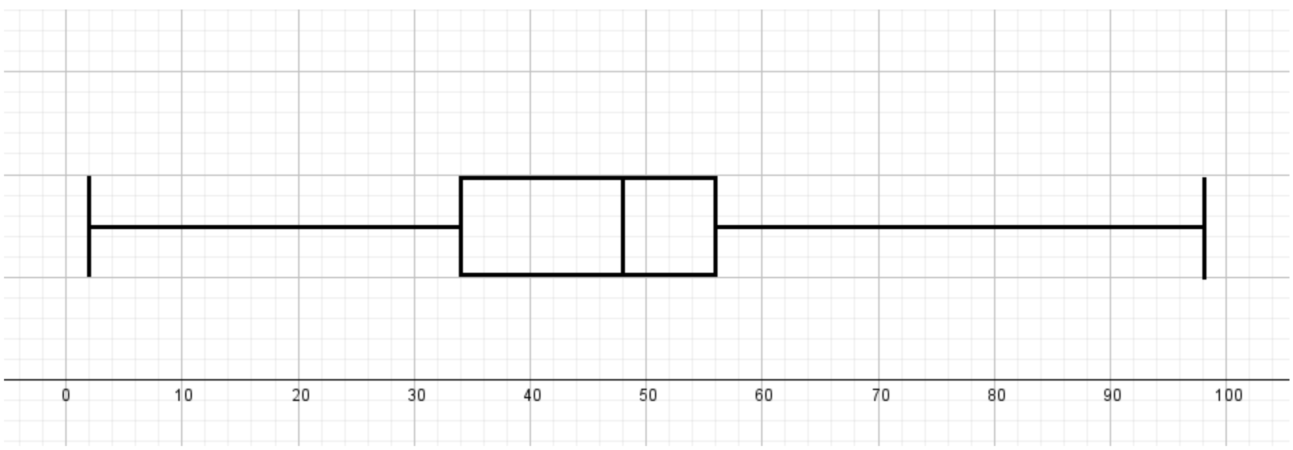
(a) (1 point) A score of at least 30% is required to pass the examination. Estimate the number of students who passed the exam.

$$N \approx 120 - 24 = 96$$

(b) (1 point) The highest grade is awarded to the top 10% of the students. Write down the score required to get the highest grade.

$$Grade_{min} = 70\%$$

(c) (4 points) Given that the minimal score was 2 and maximal score was 98, draw a box & whisker diagram to represent the exam scores of the students:



2. (7 points) Consider the polynomial

$$P(x) = 2x^3 + Ax^2 + Bx - 10$$

where $A, B \in \mathbb{R}$. One of the roots of this polynomial is $1 + 3i$.

(a) (2 points) Find the other two roots.

By conjugate root theorem we have $1 - 3i$ as another root

Using product of roots formula we have:

$$(1 + 3i)(1 - 3i)\alpha = (-1)^3 \frac{-10}{2}$$

which gives:

$$10\alpha = 5$$

so the third root is $\frac{1}{2}$.

(b) (2 points) Show that $A = -5$ and $B = 22$.

We can multiply:

$$2\left(x - (1 + 3i)\right)\left(x - (1 - 3i)\right)\left(x - \frac{1}{2}\right)$$

but it's probably quicker to apply the formulae:

$$\frac{-A}{2} = \alpha + \beta + \gamma = \frac{5}{2}$$
$$\frac{B}{2} = \alpha\beta + \alpha\gamma + \beta\gamma = 11$$

both methods give the required answer.

(c) (2 points) Show that

$$2(x+1)^3 - 5(x+1)^2 + 22(x+1) - 10 \equiv 2x^3 + x^2 + 18x + 9$$

$$\begin{aligned} LHS &= 2(x+1)^3 - 5(x+1)^2 + 22(x+1) - 10 = \\ &= 2(x^3 + 3x^2 + 3x + 1) - 5(x^2 + 2x + 1) + 22x + 12 = \\ &= 2x^3 + 6x^2 + 6x + 2 - 5x^2 - 10x - 5 + 22x + 12 = \\ &= 2x^3 + x^2 + 18x + 9 = RHS \end{aligned}$$

□

(d) (1 point) Write down the solutions to the equation:

$$2x^3 + x^2 + 18x + 9 = 0$$

The solutions to

$$2x^3 + x^2 + 18x + 9 = 0$$

are (using part (c)) the same as the solutions to:

$$2(x+1)^3 - 5(x+1)^2 + 22(x+1) - 10 = 0$$

which are one smaller than the roots of $P(x)$.

So the solutions are: $\frac{1}{2}$, $3i$ and $-3i$

3. (7 points)

   (a) (3 points) A polynomial $P(x) = x^3 + px^2 + qx + 3$ is divisible by $(x+1)$ and leaves a remainder of -3 when divided by $(x-2)$. Calculate the values of p and q.

   We have:

$$\begin{cases} P(-1) = 0 \\ P(2) = -3 \end{cases}$$

$$\begin{cases} -1 + p - q + 3 = 0 \\ 8 + 4p + 2q + 3 = -3 \end{cases}$$

$$\begin{cases} p - q = -2 \\ 2p + q = -7 \end{cases}$$

   Solving this gives $p = -3$ and $q = -1$

(b) (4 points) Another polynomial $Q(x)$ is also divisible by $(x+1)$ and leaves a remainder of -3 when divided by $(x-2)$. Find the remainder when $Q(x)$ is divided by $x^2 - x - 2$.

Recall that when dividing by a quadratic the remainder is of the form $mx + c$.

We have $Q(-1) = 0$ and $Q(2) = -3$ as before.

Note that $x^2 - x - 2 = (x+1)(x-2)$

Now we write:

$$Q(x) = S(x)(x^2 - x - 2) + mx + c$$

Plugging $x = -1$ and $x = 2$ into the above equation gives two equations:

$$\begin{cases} 0 = -m + c \\ -3 = 2m + c \end{cases}$$

Solving gives $m = -1 = c$.

So the remainder is $-x - 1$.