

# Sum and difference of angles

You should know the identities for the sine and cosine of a sum or a difference of two angles.

# Identities

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \sin \beta \cos \alpha$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

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Easy consequences of the above identities:

$$\sin(2\alpha) = 2 \sin \alpha \cos \alpha$$

$$\cos(2\alpha) = \cos^2 \alpha - \sin^2 \alpha = 2 \cos^2 \alpha - 1 = 1 - 2 \sin^2 \alpha$$

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If you don't think that the above are easy consequences, let me know and I will explain in class.

# Identities

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$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

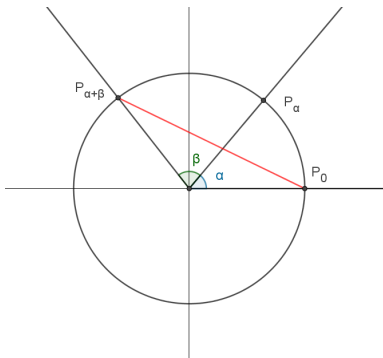
# Proof

Let's put  $\alpha + \beta$  on the unit circle.



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We then have from the definition of cosine that  $\cos(\alpha + \beta)$  is the x-coordinate of point  $P_{\alpha+\beta}$ . We will calculate the length of  $P_0P_{\alpha+\beta}$ .

# Proof

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$$|P_0 P_{\alpha+\beta}|^2 = (1 - \cos(\alpha + \beta))^2 + \sin^2(\alpha + \beta)$$

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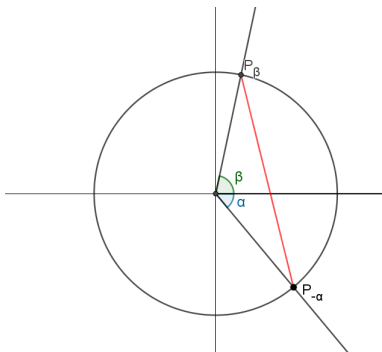
$$|P_0 P_{\alpha+\beta}|^2 = (1 - \cos(\alpha + \beta))^2 + \sin^2(\alpha + \beta)$$

We get:

$$|P_0 P_{\alpha+\beta}|^2 = 1 - 2 \cos(\alpha + \beta) + \cos^2(\alpha + \beta) + \sin^2(\alpha + \beta) = 2 - 2 \cos(\alpha + \beta)$$

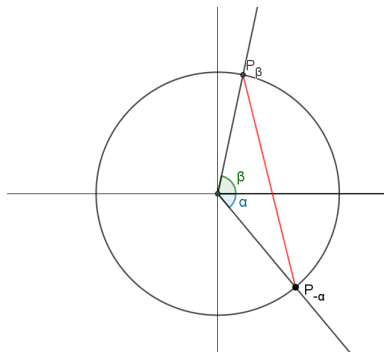
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Now if we rotate our triangle we get:



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Of course the triangle did not change, so the length of the red segment is the same:

$$|P_0 P_{\alpha+\beta}| = |P_{-\alpha} P_{\beta}|$$

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$$\begin{aligned} |P_{-\alpha}P_{\beta}|^2 &= \cos^2 \beta - 2 \cos \beta \cos(-\alpha) + \cos^2(-\alpha) + \\ &\quad + \sin^2(-\alpha) - 2 \sin(-\alpha) \sin \beta + \sin^2 \beta = \\ &= 2 - 2 \cos \beta \cos \alpha + 2 \sin \alpha \sin \alpha \end{aligned}$$

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So finally we got:

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So:

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

# Summary

Please try and understand this proof. If you think some details are missing, let me know.

# Applications

Let's calculate  $\sin 105^\circ$ .

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$$\begin{aligned}\sin(105^\circ) &= \sin(45^\circ + 60^\circ) = \\ &= \sin 45^\circ \cos 60^\circ + \sin 60^\circ \cos 45^\circ = \\ &= \frac{\sqrt{2}}{2} \times \frac{1}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \\ &= \frac{\sqrt{2} + \sqrt{6}}{4}\end{aligned}$$

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We could've predicted the result, since:

$$\sin 105^\circ = \sin \frac{7\pi}{12} = \sin \left( \frac{\pi}{2} + \frac{\pi}{12} \right) = \cos \frac{\pi}{12}$$

In case of any questions, you can email me at [T.J.Lechowski@gmail.com](mailto:T.J.Lechowski@gmail.com).