Transformations of trigonometric functions

In this presentation we discuss transformations of trigonometric functions. In particular we will look at the most general forms of four trigonometric functions:

$$f(x) = a\sin(b(x-c)) + d$$

$$f(x) = a\cos(b(x-c)) + d$$

$$f(x) = a\tan(b(x-c)) + d$$

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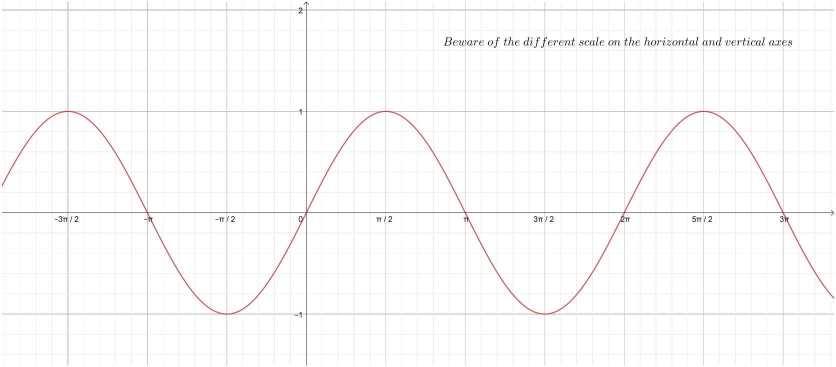
Before you start you need to be familiar with graphs of $\sin x$, $\cos x$, $\tan x$, $\cot x$ and transformations of function - in particular: translations, dilations and reflections.

We will start with a brief review of the graphs of the four trig functions.

Sine function

Graph:



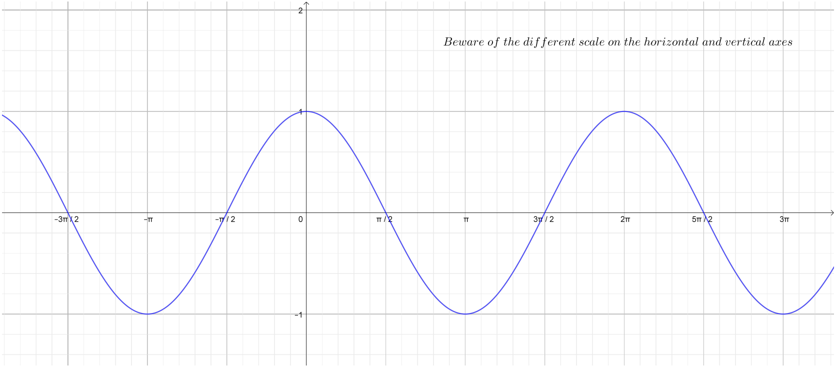
Properties:

Domain: $x \in \mathbb{R}$. Range $y \in [-1, 1]$. Period: 2π.

Zeroes: $x = k\pi$, where $k \in \mathbb{Z}$.

Cosine function

Graph:



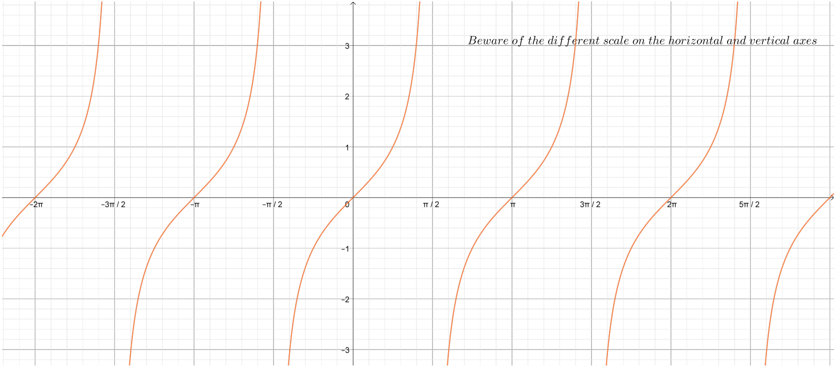
Properties:

Domain: $x \in \mathbb{R}$. Range $y \in [-1, 1]$. Period: 2π.

Zeroes: $x = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$.

Tangent function

Graph:



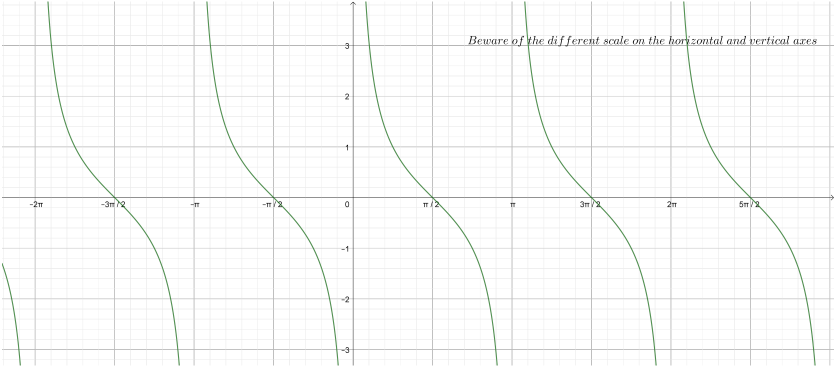
Properties:

Domain: $x \in \mathbb{R} - \{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\}$. Range $y \in \mathbb{R}$. Period: π.

Zeroes: $x = k\pi$, where $k \in \mathbb{Z}$. Asymptotes: $x = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$.

Cotangent function

Graph:

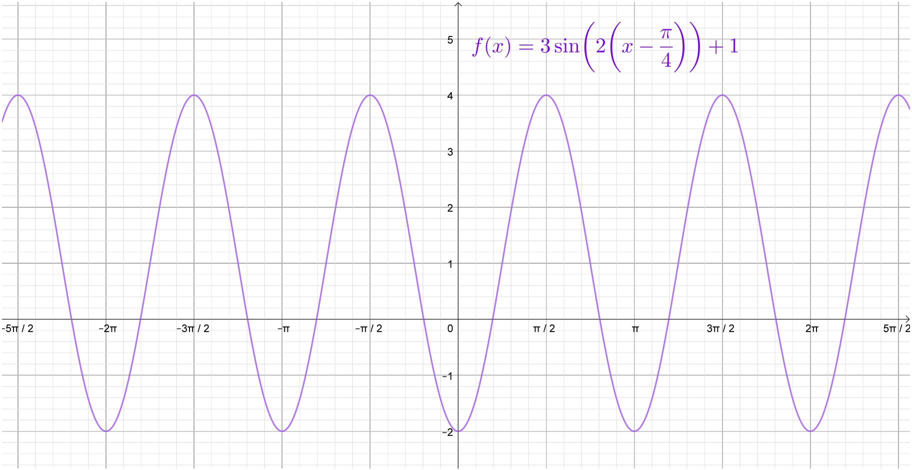


Properties:

Domain: $x \in \mathbb{R} - \{k\pi : k \in \mathbb{Z}\}$. Range $y \in \mathbb{R}$. Period: π.

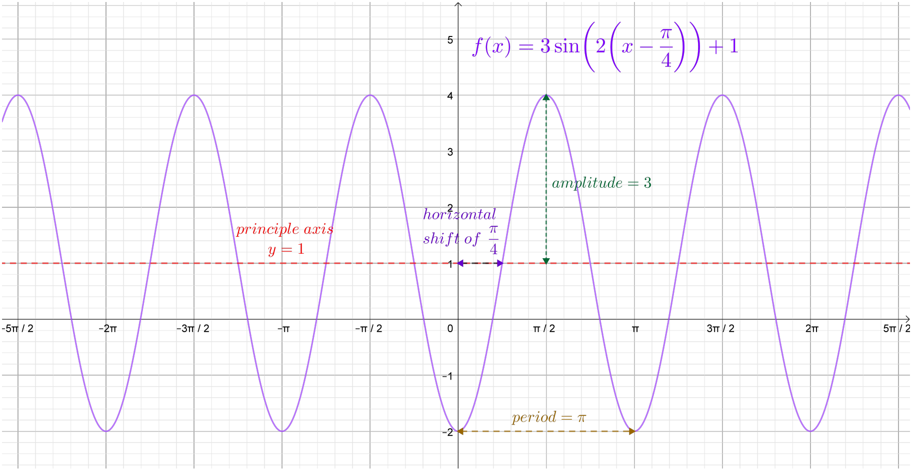
Zeroes: $x = \frac{\pi}{2} + k\pi$, where $k \in \mathbb{Z}$. Asymptotes: $x = k\pi$, where $k \in \mathbb{Z}$.

Consider the following function and its graph:



Intro to transformations of sine and cosine

Now I will add some useful features of the graph on the diagram:



Intro to transformations of sine and cosine

The above feature help us deduce the equation from the graph.
If we have a function of the form $f(x) = a\sin(b(x-c)) + d$

- a corresponds to the amplitude, we need to be careful however. We have $|a| = amplitude$ i.e. we need to check if the graph has been reflected.

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- b corresponds to the period. Note that if $b = 2$, then the graph has been stretched (squeezed) horizontally by a factor of $\frac{1}{2}$. It is useful to use the following formula $\frac{old\ period}{b} = new\ period$, where the old period is the period of the original function, so in case of sine and cosine we have $\frac{2\pi}{b} = new\ period$.

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- d corresponds to the principle axis (or "middle line").

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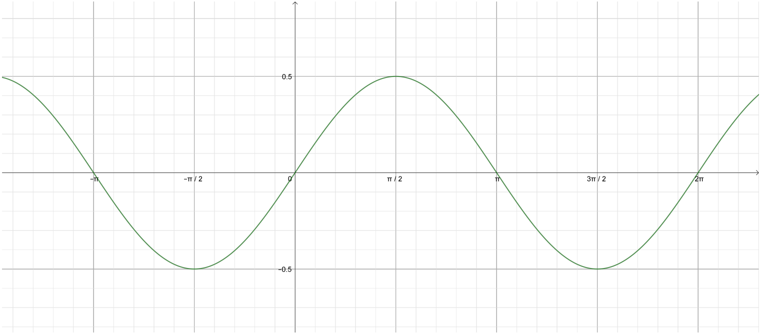
- c corresponds to the horizontal shift.
- d corresponds to the principle axis (or "middle line"). An easy way to find the middle line is to find the average of *max* and *min*.

We will now practice deducing the equation from the graph.

Example 1
The graph of the function $f(x) = a \sin x$ is shown below. Find the value of a.



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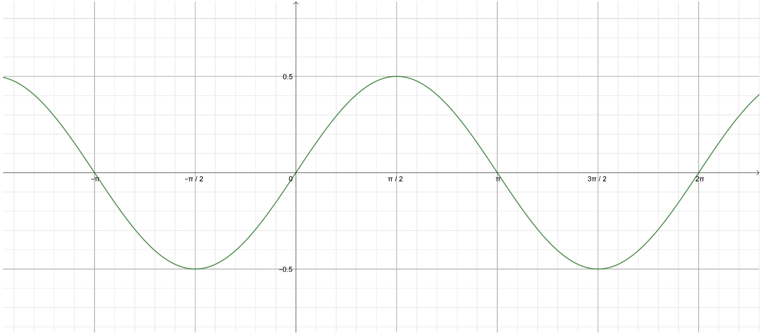
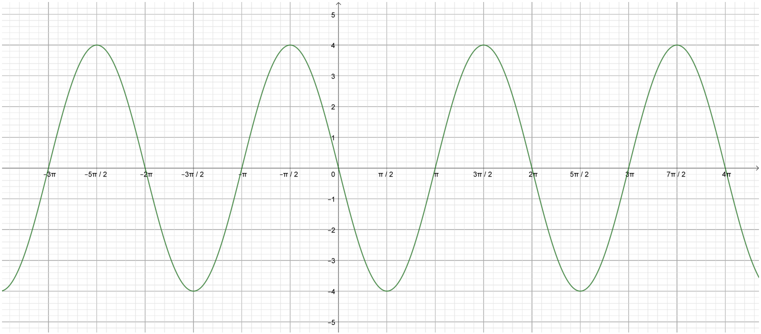


The amplitude of the graph is $\frac{1}{2}$, the graph has not been reflected in the x-axis, so $a = \frac{1}{2}$.

Example 2

The graph of the function $f(x) = a\sin x$ is shown below. Find the value of a.



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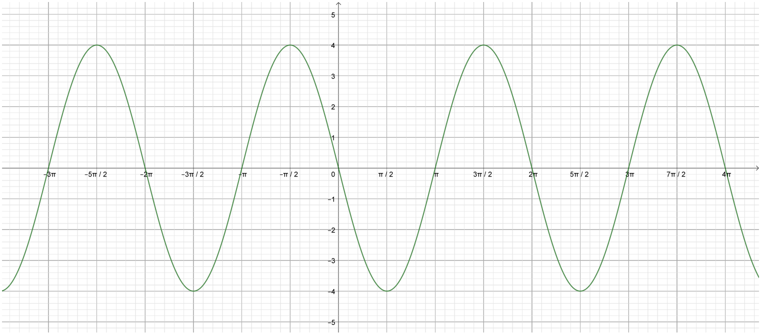
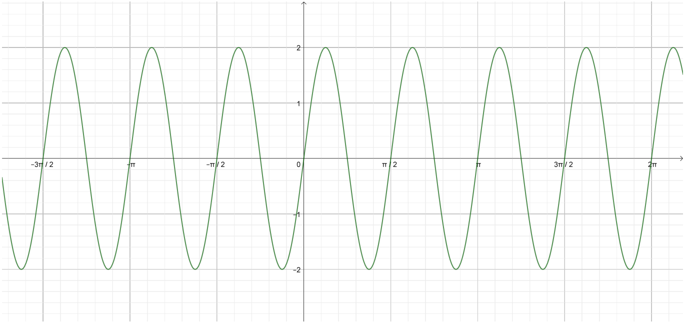


The amplitude of the graph is 4, this time the graph has been reflected in the x-axis, so $a = -4$. (To see this note that if we start at the origin and move right, the graph of sine goes up, here the graph goes down.)

Example 3

The graph of the function $f(x) = a\sin bx$ is shown below. Find the values of a and b.



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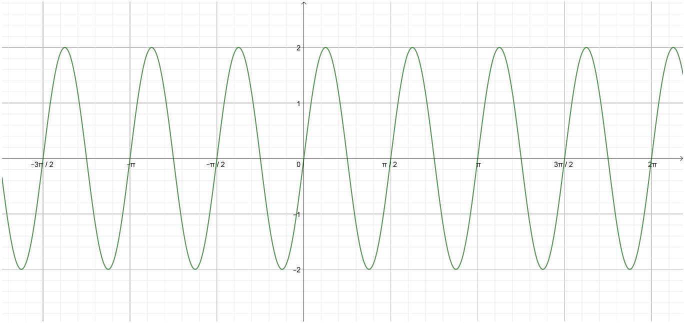

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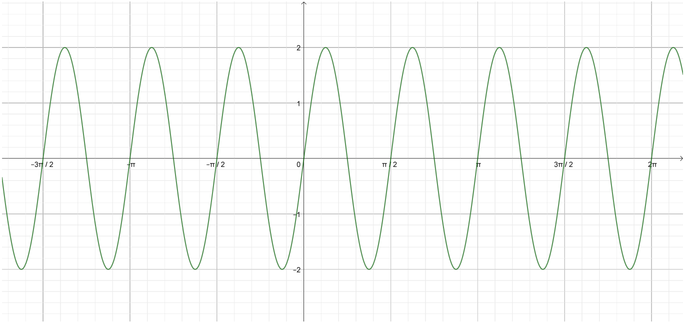

The amplitude of the graph is 2, the graph has not been reflected in the x-axis, so $a = 2$. The period of the new function is $\frac{\pi}{2}$. You can see this by looking at the zeroes of the function. This means that the graph of sine has been stretched by a factor of $\frac{1}{4}$, so $b = 4$.

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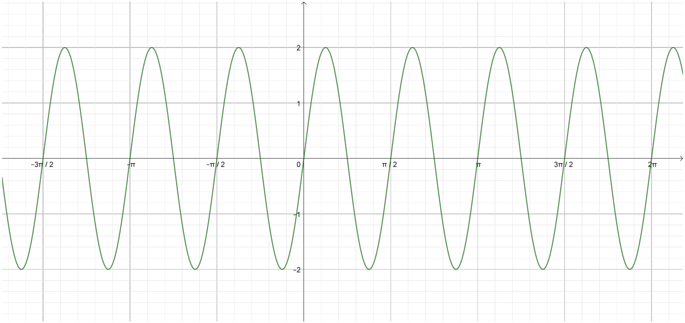

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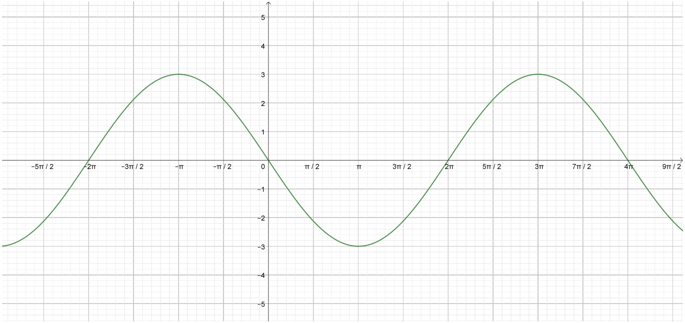

Example 4

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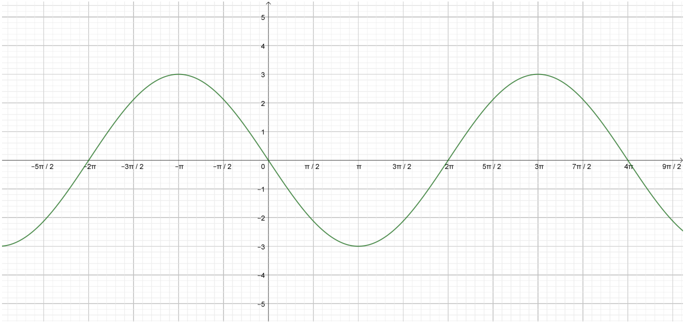

The amplitude of the graph is 3, the graph is reflected in the x-axis, so $a = -3$.

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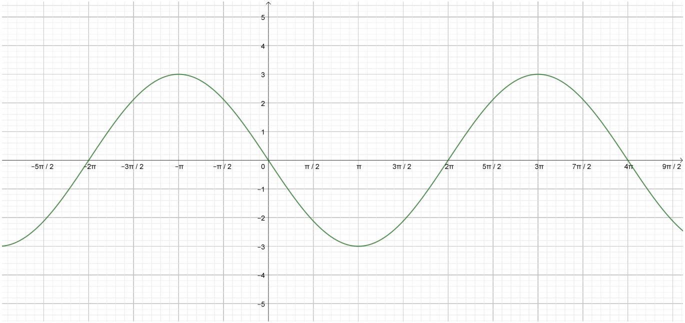

The amplitude of the graph is 3, the graph is reflected in the x-axis, so $a = -3$. The period of the new function is 4π. You can see this by looking at the zeroes or the peaks of the function. This means that the graph of sine has been stretched by a factor of 2, so $b = \frac{1}{2}$.

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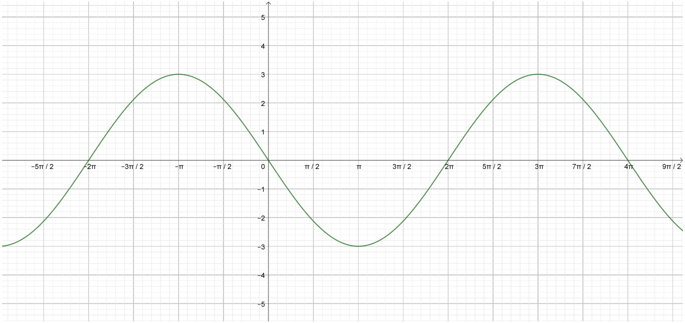

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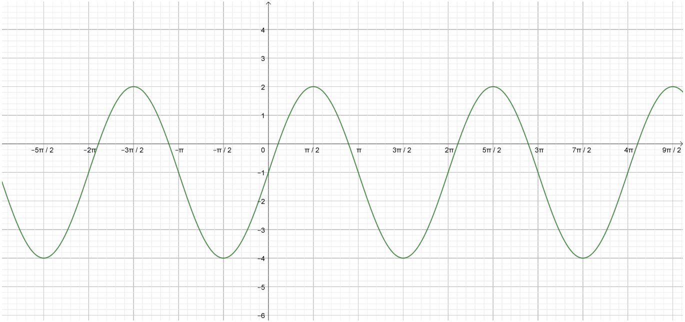
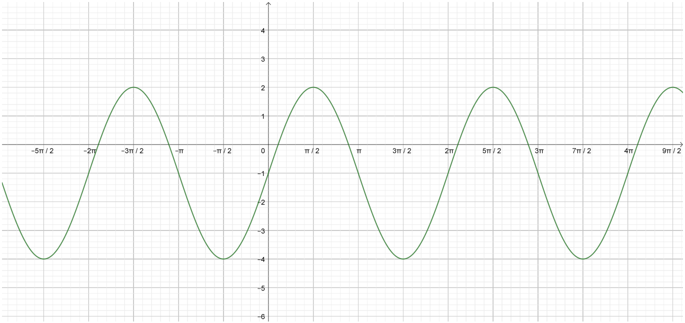
Example 5

The graph of the function $f(x) = a\sin x + d$ is shown below. Find the values of a and d.



Example 5

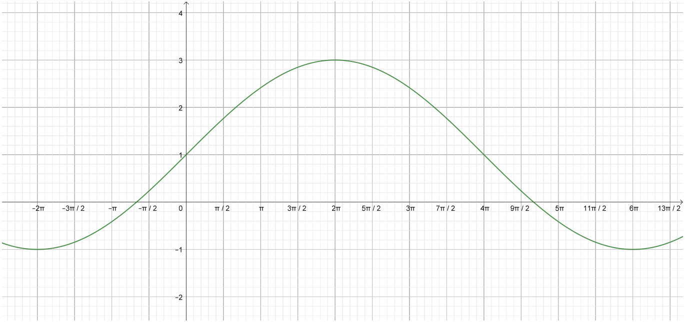
The graph of the function $f(x) = a \sin x + d$ is shown below. Find the values of a and d.



The principle axis is $y = -1$, so $d = -1$. The amplitude is 3 and the graph has not been reflected so $a = 3$.

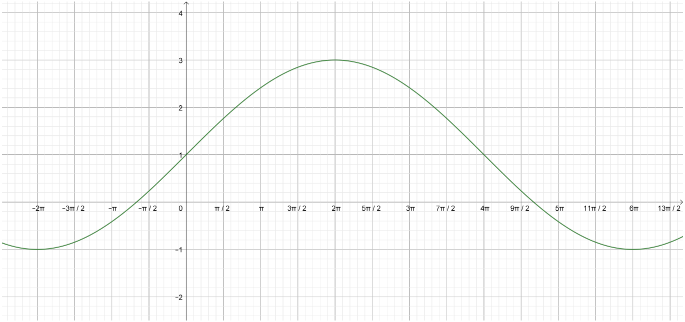
Example 6

The graph of the function $f(x) = a\sin bx + d$ is shown below. Find the values of a, b and d.



Example 6

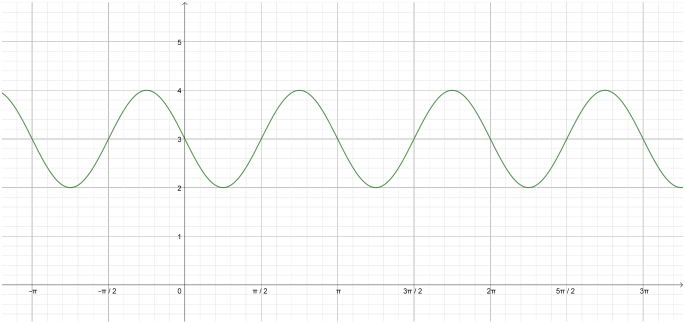
The graph of the function $f(x) = a \sin bx + d$ is shown below. Find the values of a, b and d.



The principle axis is $y = 1$, so $d = 1$. The amplitude is 2 and the graph has not been reflected so $a = 2$. The period is 8π (we can see clearly that half of the period is 4π). This means that the graph has been stretched by a factor of 4, so $b = \frac{1}{4}$.

Example 7

The graph of the function $f(x) = a \sin bx + d$ is shown below. Find the values of a, b and d.



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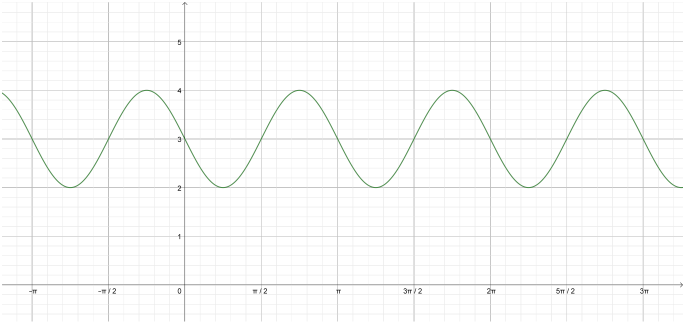
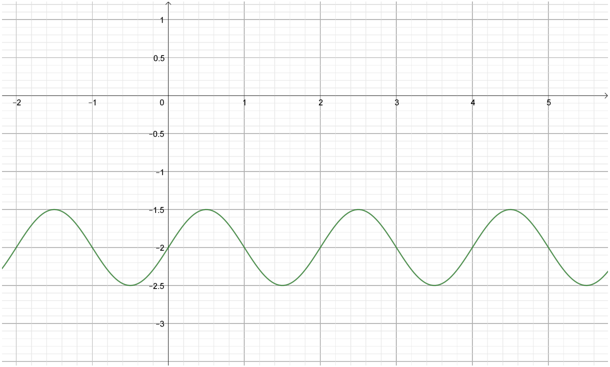


The principle axis is $y = 3$, so $d = 3$. The amplitude is 1 but the graph has been reflected so $a = -1$. The period is π so the graph has been stretched by a factor of $\frac{1}{2}$, so $b = 2$.

Example 8

The graph of the function $f(x) = a \sin bx + d$ is shown below. Find the values of a, b and d.



Example 8

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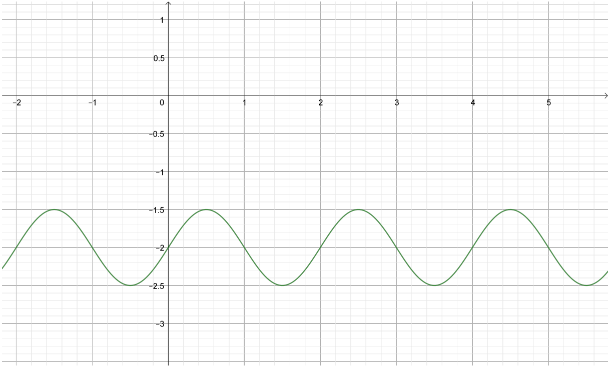
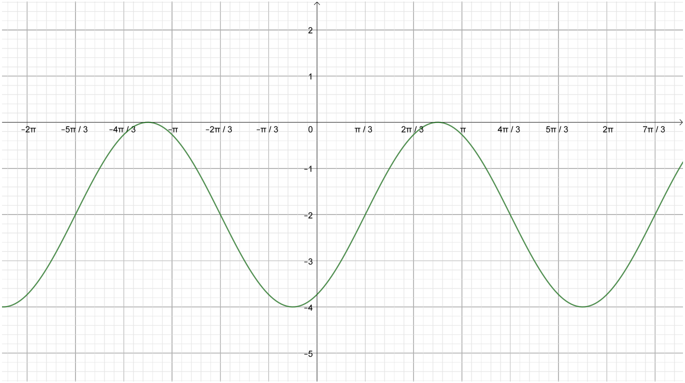


The principle axis is $y = -2$, so $d = -2$. The amplitude is $\frac{1}{2}$, the graph is not reflected so $a = \frac{1}{2}$. The period is 2, it maybe less obvious how the graph was stretched, so let's go straight to the formula $\frac{2\pi}{b} = 2$ and we get that $b = \pi$.

Example 9

The graph of the function $f(x) = a\sin(x - c) + d$ is shown below. Find the values of a, c and d.



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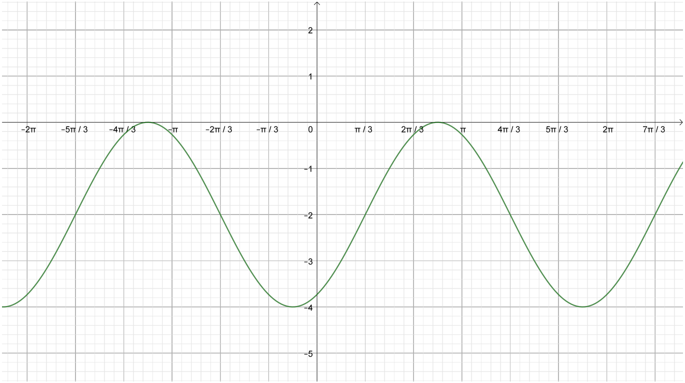
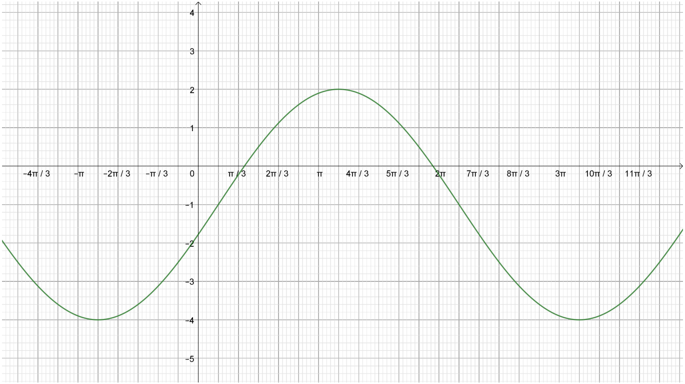


The principle axis is $y = -2$, so $d = -2$. The amplitude is 2, the graph is not reflected so $a = 2$. The graph has been shifted to the right by $\frac{\pi}{3}$, so $c = \frac{\pi}{3}$ (note the negative sign in front of c in the equation).

Example 10

The graph of the function $f(x) = a\sin(b(x-c)) + d$ is shown below. Find the values of a, b, c and d.



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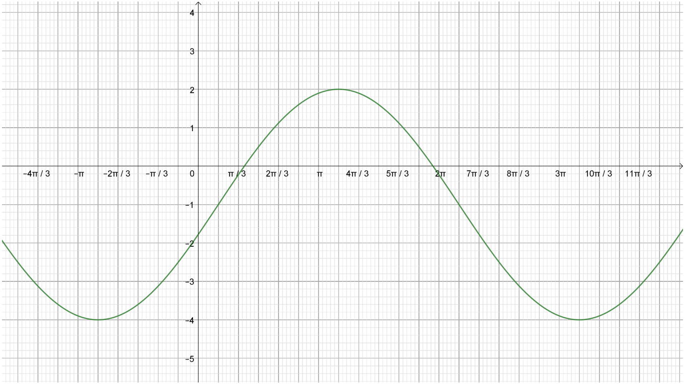

The principle axis $y = -1$, so $d = -1$. Amplitude is 3, not reflected so $a = 3$. Period is 4π so $b = \frac{1}{2}$. Horizontal shift is $\frac{\pi}{6}$ units to the right so $c = \frac{\pi}{6}$ (again because we're subtracting c from x).

You may have noticed that the previous example had more possible answers.

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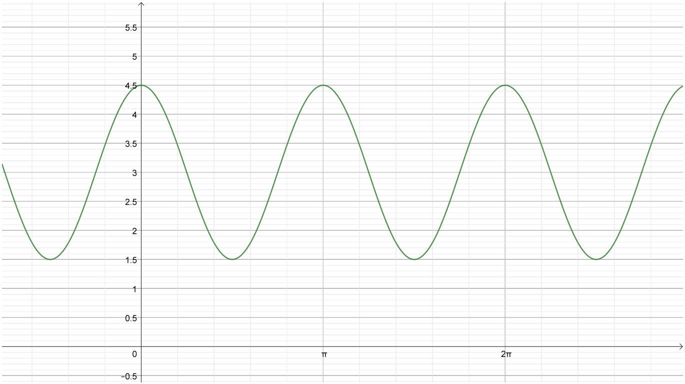
Now we move on to a few examples on cosine functions, but these are very similar in nature.

Example 11

The graph of the function $f(x) = a\cos bx + d$ is shown below. Find the values of a, b and d.



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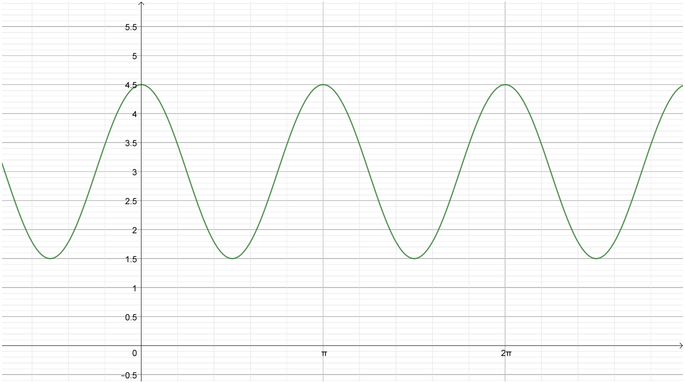
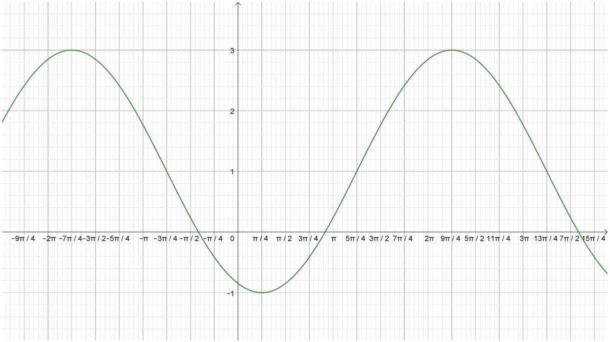


The principle axis is $y = 3$, so $d = 3$. The amplitude is $\frac{3}{2}$, the graph has not been reflected (if we go right from the y-axis, the cosine function starts at 1 and goes down, our function also goes down) so $a = \frac{3}{2}$. The period is π, so $b = 2$.

Example 12

The graph of the function $f(x) = a\cos(b(x - c)) + d$ is shown below. Find the values of a, b, c and d.



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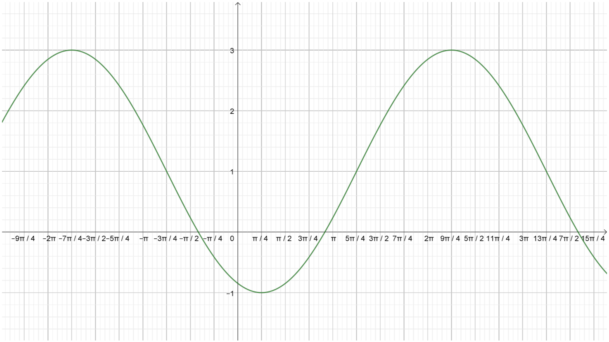

The principle axis is $y = 1$, so $d = 1$. The amplitude is 2, but the graph has been reflected so $a = -2$. The period is 4π (we can see that half the period is 2π), so $b = \frac{1}{2}$. The graph has been shifted $\frac{\pi}{4}$ units to the right, so $c = \frac{\pi}{4}$. You can look at the bottom peak to see this.

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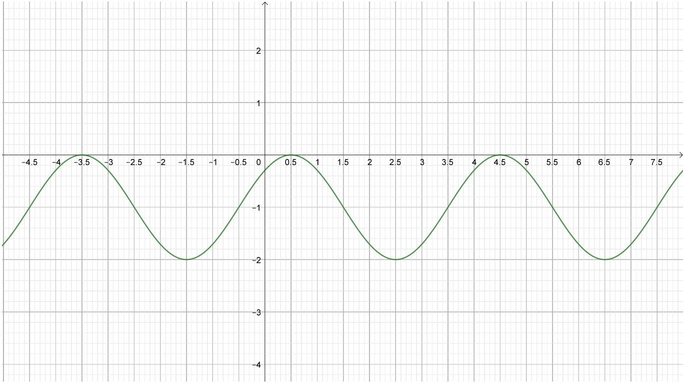
The graph of the function $f(x) = a\cos(b(x-c)) + d$ is shown below. Find the values of a, b, c and d.



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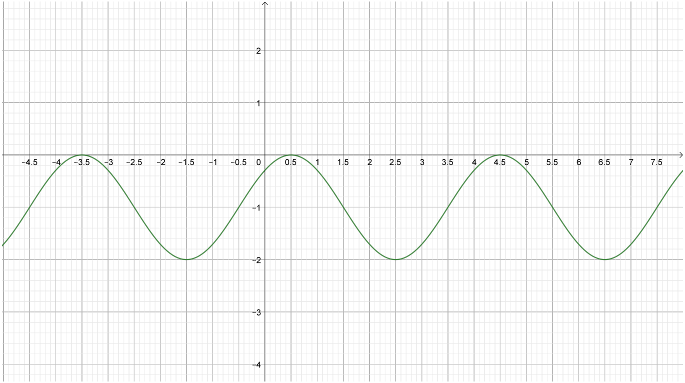

The principle axis is $y = -1$, so $d = -1$. The amplitude is 1, the graph has not been reflected so $a = 1$. The period is 4, we solve $\frac{2\pi}{b} = 4$ to get $b = \frac{\pi}{2}$. Finally the graph has been shifted 0.5 units to the right, so $c = \frac{1}{2}$.

Again notice that the last two examples had multiple solutions. It's a good practice to come up with other solutions, but make sure that you check your answers (by drawing appropriate graph on for instance desmos.com)

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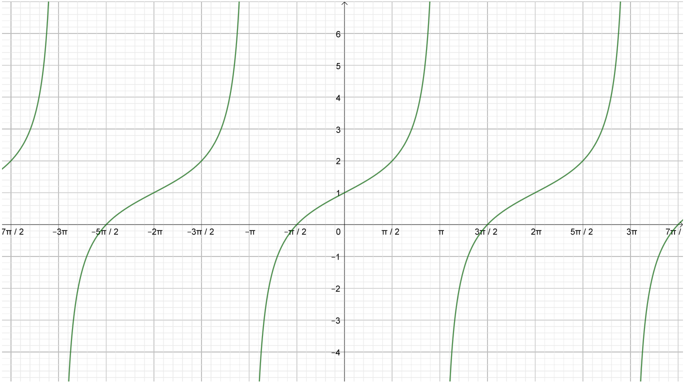
We now turn to tangent and cotangent functions.

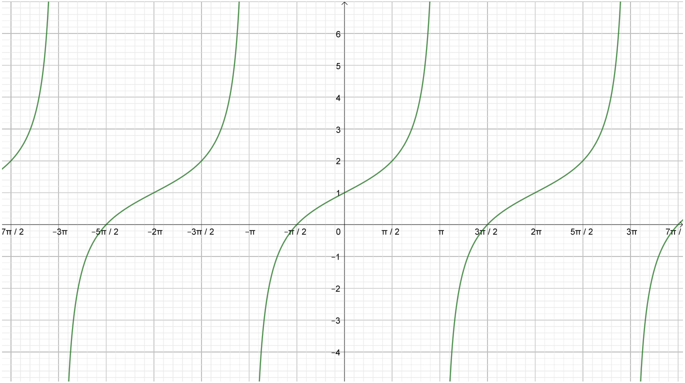
Example 14

The graph of the function $f(x) = \tan(bx) + d$ is shown below. Find the values of b and d.



Example 14

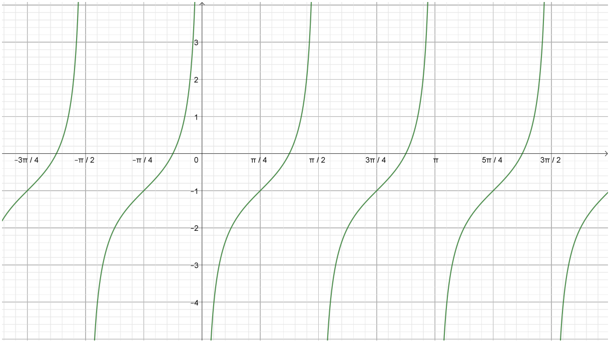
The graph of the function $f(x) = \tan(bx) + d$ is shown below. Find the values of b and d.



This may be less obvious, but still we can identify the vertical translation one unit upwards, so $d = 1$. The period is 2π, the period of $\tan x$ is π, so the it has been stretched by a factor of 2, which means that $b = \frac{1}{2}$.

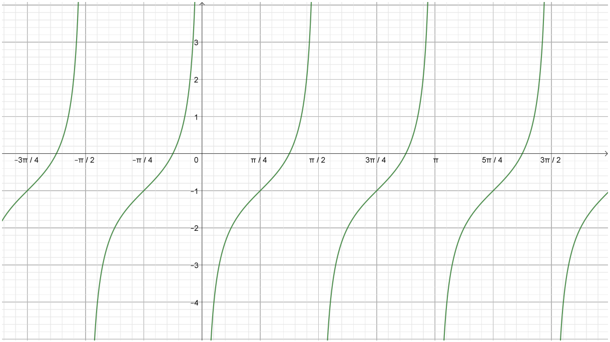
Example 15

The graph of the function $f(x) = \tan(b(x - c)) + d$ is shown below. Find the values of b, c and d.



Example 15

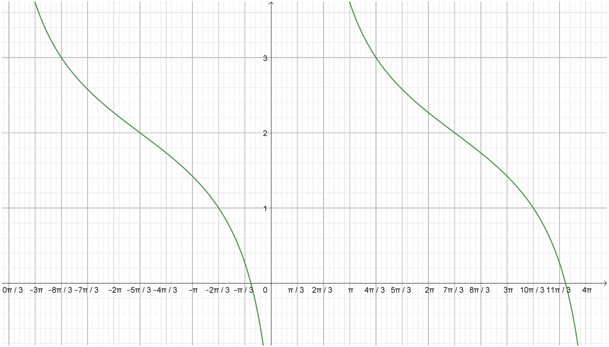
The graph of the function $f(x) = \tan(b(x-c)) + d$ is shown below. Find the values of b, c and d.



The period is $\frac{\pi}{2}$, so the graph of $\tan x$ has been stretched by a factor of $\frac{1}{2}$, which means that $b = 2$. Now we can see the translation by a vector $\begin{pmatrix} \frac{\pi}{4} \\ -1 \end{pmatrix}$, so $c = \frac{\pi}{4}$ and $d = -1$.

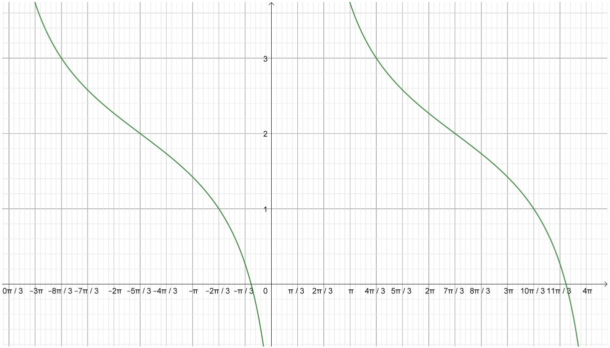
Example 16

The graph of the function $f(x) = \cot(b(x - c)) + d$ is shown below. Find the values of b, c and d.



Example 16

The graph of the function $f(x) = \cot(b(x-c)) + d$ is shown below. Find the values of b, c and d.



The period is 4π, so the graph of $\cot x$ has been stretched by a factor of 4, which means that $b = \frac{1}{4}$. By looking at the image of the point $(\frac{\pi}{2}, 0) \to (\frac{7\pi}{3}, 2)$, we can identify the translation by a vector $\begin{pmatrix} \frac{\pi}{3} \\ 2 \end{pmatrix}$, so $c = \frac{\pi}{3}$ and $d = 2$.

In case of any questions you can email me at T.J.Lechowski@gmail.com.