7.1 EXPONENTS

In §5.3.3 and §5.3.4 we looked at the exponential function, $f(x) = a^x, a > 0$ and the logarithmic function, $f(x) = \log_a x, a > 0$ and considered their general behaviour. In this chapter we will look in more detail at how to solve exponential and logarithmic equations as well as applications of both the exponential and logarithmic functions.

7.1.1 BASIC RULES OF INDICES

We start by looking at the notation involved when dealing with indices (or exponents).
The expression

$$\underbrace{a \times a \times a \times \ldots \times a}_{n \text{ times}}$$

can be written in index form, a^n, where **n is the index** (or power or **exponent**) and a is the **base**. This expression is read as "*a to the power of n*." or more briefly as "*a to the n*".

For example, we have that $3^5 = 3 \times 3 \times 3 \times 3 \times 3$ so that 3 is the base and 5 the exponent (or index).

The laws for positive integral indices are summarised below.

If a and b are real numbers and m and n are positive integers, we have that

Law	Rule	Example
1. Multiplication [same base]	: $a^m \times a^n = a^{m+n}$	$3^4 \times 3^6 = 3^{4+6} = 3^{10}$
2. Division [same base]	: $a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}$	$7^9 \div 7^5 = 7^{9-5} = 7^4$
3. Power of a power [same base]	: $(a^m)^n = a^{m \times n}$	$(2^3)^5 = 2^{3 \times 5} = 2^{15}$
4. Power of a power [same power]	: $a^m \times b^m = (ab)^m$	$3^4 \times 7^4 = (3 \times 7)^4 = 21^4$
5. Division [same power]	: $a^m \div b^m = \frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$	$5^3 \div 7^3 = \left(\frac{5}{7}\right)^3$
6. Negative one to a power	: $(-1)^n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$	$(-1)^3 = (-1)^5 = \ldots = -1$ $(-1)^2 = (-1)^4 = \ldots = 1$

There are more laws of indices that are based on rational indices, negative indices and the zero index. A summary of these laws is provided next.

Law	Rule	Example
1. Fractional index Type 1 [nth root]	: $a^{1/n} = \sqrt[n]{a}, n \in \mathbb{N}$. Note: if n is even, then $a \geq 0$. if n is odd, then $a \in \mathbb{R}$.	$8^{1/3} = \sqrt[3]{8} = 2$ $(-27)^{1/3} = \sqrt[3]{-27} = -3$
2. Fractional index Type 2	: $a^{m/n} = \sqrt[n]{a^m}$ Note: if n is even, then $a^m \geq 0$ if n is odd, then $a \in \mathbb{R}$.	$16^{3/4} = \sqrt[4]{16^3} = 8$
3. Negative index	: $a^{-1} = \dfrac{1}{a}, a \neq 0$ $a^{-n} = \dfrac{1}{a^n}, a \neq 0, n \in \mathbb{N}$	$2^{-1} = \dfrac{1}{2} = 0.5$ $3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$
4. Zero index	: $a^0 = 1, a \neq 0$ Note: $0^n = 0, n \neq 0$	$12^0 = 1$

We make the following note about fractional indices:

As $\frac{m}{n} = m \times \frac{1}{n} = \frac{1}{n} \times m$, we have that for $b \geq 0$ i. $b^{\frac{m}{n}} = b^{m \times \frac{1}{n}} = (b^m)^{\frac{1}{n}} = \sqrt[n]{b^m}$

ii. $b^{\frac{m}{n}} = b^{\frac{1}{n} \times m} = \left(b^{\frac{1}{n}}\right)^m = (\sqrt[n]{b})^m$

Then, If $b \geq 0$, then $b^{\frac{m}{n}} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m, m \in \mathbb{Z}, n \in \mathbb{N}$

If $b < 0$, then $b^{\frac{m}{n}} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m, m \in \mathbb{Z}, n \in \{1, 3, 5, \ldots\}$

EXAMPLE 7.1

Simplify the following

(a) $\left(\frac{4x^2}{5y^4}\right)^2 \times (2x^3y)^3$ (b) $\frac{3^{n+1} + 3^2}{3}$

Solution

(a) $$\left(\frac{4x^2}{5y^4}\right)^2 \times (2x^3y)^3 = \frac{4^2x^{2\times 2}}{5^2y^{4\times 2}} \times 2^3x^{3\times 3}y^{1\times 3}$$
$$= \frac{16x^4}{25y^8} \times 8x^9y^3$$
$$= \frac{128}{25}x^{4+9}y^{3-8}$$
$$= \frac{128}{25}x^{13}y^{-5}$$
$$= \frac{128x^{13}}{25y^5}$$

(b) $$\frac{3^{n+1} + 3^2}{3} = \frac{3(3^n + 3)}{3}$$
$$= 3^n + 3$$

EXAMPLE 7.2

Simplify the following

(a) $\dfrac{4x^2(-y^{-1})^{-2}}{(-2x^2)^3(y^{-2})^2}$ (b) $\dfrac{x^{-1}+y^{-1}}{x^{-1}y^{-1}}$

Solution

(a)
$$\frac{4x^2(-y^{-1})^{-2}}{(-2x^2)^3(y^{-2})^2} = \frac{4x^2 \times y^{-1\times -2}}{-8x^{2\times 3}\times y^{-2\times 2}} = -\frac{x^2y^2}{2x^6y^{-4}}$$
$$= -\frac{y^{2-(-4)}}{2x^{6-(2)}}$$
$$= -\frac{y^6}{2x^4}$$

(b)
$$\frac{x^{-1}+y^{-1}}{x^{-1}y^{-1}} = \frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{xy}} = \left(\frac{1}{x}+\frac{1}{y}\right)\times\frac{xy}{1} = \frac{xy}{x}+\frac{xy}{y}$$
$$= y + x$$

EXAMPLE 7.3

Simplify the following

(a) $\dfrac{2^{n-3}\times 8^{n+1}}{2^{2n-1}\times 4^{2-n}}$ (b) $\dfrac{(a^{1/3}\times b^{1/2})^{-6}}{\sqrt[4]{a^8b^9}}$

Solution

(a)
$$\frac{2^{n-3}\times 8^{n+1}}{2^{2n-1}\times 4^{2-n}} = \frac{2^{n-3}\times (2^3)^{n+1}}{2^{2n-1}\times (2^2)^{2-n}} = \frac{2^{n-3}\times 2^{3n+3}}{2^{2n-1}\times 2^{4-2n}}$$
$$= \frac{2^{n-3+(3n+3)}}{2^{2n-1+(4-2n)}}$$
$$= \frac{2^{4n}}{2^3}$$
$$= 2^{4n-3}$$

(b)
$$\frac{(a^{1/3}\times b^{1/2})^{-6}}{\sqrt[4]{a^8b^9}} = \frac{a^{\frac{1}{3}\times -6}\times b^{\frac{1}{2}\times -6}}{(a^8b^9)^{\frac{1}{4}}} = \frac{a^{-2}\times b^{-3}}{a^2b^{\frac{9}{4}}}$$
$$= a^{-2-2}\times b^{-3-\frac{9}{4}}$$
$$= a^{-4}b^{-\frac{21}{4}}$$
$$= \frac{1}{a^4b^{\frac{21}{4}}}$$

EXERCISES 7.1.1

1. Simplify the following

(a) $\left(\frac{3y^2}{4x^3}\right)^3 \times (2x^2y^3)^3$ (b) $\left(\frac{2}{3a^2}\right)^3 + \frac{1}{8a^6}$ (c) $\frac{2^{n+1}+2^2}{2}$

(d) $\left(\frac{2x^3}{3y^2}\right)^3 \times (xy^2)^2$ (e) $\left(\frac{2x^3}{4y^2}\right)^2 \times \frac{12y^6}{8x^4}$ (f) $\frac{3^{n+2}+9}{3}$

(g) $\frac{4^{n+2}-16}{4}$ (h) $\frac{4^{n+2}-16}{2}$ (i) $\left(\frac{1}{2b}\right)^4 - \frac{b^2}{16}$

2. Simplify the following

(a) $\frac{20^6}{10^6}$ (b) $\frac{12^{2x}}{(6^3)^x}$ (c) $\frac{16^{2y+1}}{8^{2y+1}}$

(d) $\frac{(ab)^{2x}}{a^{2x}b^{4x}}$ (e) $\frac{(xy)^6}{64x^6}$ (f) $\frac{27^{n+2}}{6^{n+2}}$

3. Simplify the following

(a) $\left(\frac{x}{y}\right)^3 \times \left(\frac{y}{z}\right)^2 \times \left(\frac{z}{x}\right)^4$ (b) $3^{2n} \times 27 \times 243^{n-1}$ (c) $\frac{25^{2n} \times 5^{1-n}}{(5^2)^n}$

(d) $\frac{9^n \times 3^{n+2}}{27^n}$ (e) $\frac{2^n \times 4^{2n+1}}{2^{1-n}}$ (f) $\frac{2^{2n+1} \times 4^{-n}}{(2^n)^3}$

(g) $\frac{x^{4n+1}}{(x^{n+1})^{(n-1)}}$ (h) $\frac{x^{4n^2+n}}{(x^{n+1})^{(n-1)}}$ (i) $\frac{(3^x)(3^{x+1})(3^2)}{(3^x)^2}$

4. Simplify $\frac{(x^m)^n(y^2)^m}{(x^m)^{(n+1)}y^2}$.

5. Simplify the following, leaving your answer in positive power form

(a) $\frac{(-3^4) \times 3^{-2}}{(-3)^{-2}}$ (b) $\frac{9y^2(-x^{-1})^{-2}}{(-2y^2)^3(x^{-2})^3}$ (c) $\frac{x^{-1}-y^{-1}}{x^{-1}y^{-1}}$

(d) $\frac{x^{-2}+2x^{-1}}{x^{-1}+x^{-2}}$ (e) $\frac{(-2)^3 \times 2^{-3}}{(x^{-1})^2 \times x^2}$ (f) $\frac{(-a)^3 \times a^{-3}}{(b^{-1})^{-2}b^{-3}}$

6. Simplify the following

(a) $\frac{(x^{-1})^2+(y^2)^{-1}}{x^2+y^2}$ (b) $\frac{(x^2)^{-2}+2y}{1+2yx^4}$ (c) $\frac{(x+h)^{-1}-x^{-1}}{h}$

(d) $(x^2-1)^{-1} \times (x+1)$ (e) $\frac{(x-1)^{-3}}{(x+1)^{-1}(x^2-1)^2}$ (f) $\frac{y(x^{-1})^2+x^{-1}}{x+y}$

7. Simplify the following

(a) $5^{n+1} - 5^{n-1} - 2 \times 5^{n-2}$ (b) $a^{x-y} \times a^{y-z} \times a^{z-x}$ (c) $\left(\frac{a^{-\frac{1}{2}}b^3}{ab^{-1}}\right)^2 \times \frac{1}{ab}$

(d) $\left(\frac{a^{m+n}}{a^n}\right)^m \times \left(\frac{a^{n-m}}{a^n}\right)^{m-n}$ (e) $\frac{p^{-2}-q^{-2}}{p^{-1}-q^{-1}}$ (f) $\frac{1}{1+a^{\frac{1}{2}}} - \frac{1}{1-a^{\frac{1}{2}}}$

(g) $\frac{2^{n+4}-2(2^n)}{2(2^{n+3})}$ (h) $\sqrt{a\sqrt{a\sqrt{a}}}$

8. Simplify the following

(a) $\frac{\sqrt{x}\times\sqrt[3]{x^2}}{\sqrt[4]{x}}$ (b) $\frac{b^{n+1}\times 8a^{2n-1}}{(2b)^2(ab)^{-n+1}}$ (c) $\frac{2^n-6^n}{1-3^n}$

(d) $\frac{7^{m+1}-7^m}{7^n-7^{n+2}}$ (e) $\frac{5^{2n+1}+25^n}{5^{2n}+5^{1+n}}$ (f) $\left(x-2x^{\frac{1}{2}}+1\right)^{\frac{1}{2}} \times \frac{x+1}{\sqrt{x}-1}$

7.1.2 INDICIAL EQUATIONS

Solving equations of the form $x^{\frac{1}{2}} = 3$, where the **variable is the base**, requires that we square both sides of the equation so that $\left(x^{\frac{1}{2}}\right)^2 = 3^2 \Rightarrow x = 9$. However, when the **variable is the power** and not the base we need to take a different approach.

> Indicial (exponential) equations take on the general form $b^x = a$, where the unknown (variable), x, is the power

Consider the case where we wish to solve for x given that $2^x = 8$.

In this case we need to think of a value of x so that when 2 is raised to the power of x the answer is 8. Using trial and error, it is not too difficult to arrive at $x = 3$ $(2^3 = 2\times2\times2 = 8)$.

Next consider the equation $3^{x+1} = 27$.
Again, we need to find a number such that when 3 is raised to that number, the answer is 27. Here we have that $27 = 3^3$. Therefore we can rewrite the equation as $3^{x+1} = 3^3$.
As the base on both sides of the equality is the same we can then equate the powers, that is,

$$\begin{aligned} 3^{x+1} = 27 &\Leftrightarrow 3^{x+1} = 3^3 \\ &\Leftrightarrow x+1 = 3 \\ &\Leftrightarrow x = 2 \end{aligned}$$

Such an approach can be used for a variety of equations. We summarise this process for simple exponential equations:

	Solve for x : $b^x = N$	Example: Solve $5^x = 625$
Step 1:	Express the number N in the form b^{number}	$625 = 5^4$
Step 2:	Write the equation $b^x = b^{\text{number}}$	$\therefore 5^x = 5^4$
Step 3:	Equate exponents, x = number	$\Leftrightarrow x = 4$

EXAMPLE 7.4

Solve the following

(a) $3^x = 81$ (b) $2 \times 5^u = 250$ (c) $2^x = \frac{1}{32}$

Solution

(a) $3^x = 81 \Leftrightarrow 3^x = 3^4$

$\Leftrightarrow x = 4$

(b) $2 \times 5^u = 250 \Leftrightarrow 5^u = 125$

$\Leftrightarrow 5^u = 5^3$

$\Leftrightarrow u = 3$

(c) $2^x = \frac{1}{32} \Leftrightarrow 2^x = \frac{1}{2^5}$

$\Leftrightarrow 2^x = 2^{-5}$

$\Leftrightarrow x = -5$

EXAMPLE 7.5

Find

(a) $\left\{x \mid \left(\frac{1}{2}\right)^x = 16\right\}$ (b) $\{x \mid 3^{x+1} = 3\sqrt{3}\}$ (c) $\{x \mid 4^{x-1} = 64\}$

Solution

(a) $\left(\frac{1}{2}\right)^x = 16 \Leftrightarrow (2^{-1})^x = 16$

$\Leftrightarrow 2^{-x} = 2^4$

$\Leftrightarrow -x = 4$

$\Leftrightarrow x = -4$

i.e., solution set is $\{-4\}$.

(b) $3^{x+1} = 3\sqrt{3} \Leftrightarrow 3^{x+1} = 3 \times 3^{1/2}$

$\Leftrightarrow 3^{x+1} = 3^{3/2}$

$\Leftrightarrow x + 1 = \frac{3}{2}$

$\Leftrightarrow x = \frac{1}{2}$

i.e., solution set is $\{0.5\}$.

(c) $4^{x-1} = 64 \Leftrightarrow (2^2)^{x-1} = 2^6$

$\Leftrightarrow 2^{2x-2} = 2^6$

$\Leftrightarrow 2x - 2 = 6$

$\Leftrightarrow 2x = 8$

$\Leftrightarrow x = 4$

i.e., solution set is $\{4\}$.

EXERCISES 7.1.2

1. Solve the following equations.

(a) $\{x|\ 4^x = 16\}$ (b) $\left\{x|\ 7^x = \frac{1}{49}\right\}$ (c) $\{x|\ 8^x = 4\}$

(d) $\{x|\ 3^x = 243\}$ (e) $\{x|\ 3^{x-2} = 81\}$ (f) $\left\{x|\ 4^x = \frac{1}{32}\right\}$

(g) $\{x|\ 3^{2x-4} = 1\}$ (h) $\{x|\ 4^{2x+1} = 128\}$ (i) $\{x|\ 27^x = 3\}$

2. Solve the following equations.

(a) $\{x|\ 7^{x+6} = 1\}$ (b) $\left\{x|\ 8^x = \frac{1}{4}\right\}$ (c) $\{x|\ 10^x = 0.001\}$

(d) $\{x|\ 9^x = 27\}$ (e) $\{x|\ 2^{4x-1} = 1\}$ (f) $\{x|\ 25^x = \sqrt{5}\}$

(g) $\left\{x|\ 16^x = \frac{1}{\sqrt{2}}\right\}$ (h) $\{x|\ 4^{-x} = 32\sqrt{2}\}$ (i) $\{x|\ 9^{-2x} = 243\}$

7.1.3 EQUATIONS OF THE FORM $b^{f(x)} = b^{g(x)}$.

This is an extension of the previous section, in that now we will consider exponential equations of the form $b^{f(x)} = N$ where N can be expressed as a number having base b so that $N = b^{g(x)}$.

Consider the equation $2^{x^2-1} = 8$. Our first step is to express 8 as 2^3 so that we can then write

$$2^{x^2-1} = 8 \Leftrightarrow 2^{x^2-1} = 2^3$$

Then, equating powers we have: $\Leftrightarrow x^2 - 1 = 3$

So that, $\Leftrightarrow x^2 - 4 = 0$

$$\Leftrightarrow (x-2)(x+2) = 0$$

$$\therefore x = 2 \text{ or } x = -2$$

Checking these values by substituting back into the origial equation shows them to be correct.

i.e., when $x = 2$, L.H.S $= 2^{2^2-1} = 2^{4-1} = 2^3 = 8 =$ R.H.S

when $x = -2$, L.H.S $= 2^{(-2)^2-1} = 2^{4-1} = 2^3 = 8 =$ R.H.S

However, had the equation been, $2^{x^2-1} = 2^{5-x}$, then the solution would have been

$$2^{x^2-1} = 2^{5-x} \Leftrightarrow x^2 - 1 = 5 - x \text{ [equating powers]}$$

$$\Leftrightarrow x^2 + x - 6 = 0$$

$$\Leftrightarrow (x-2)(x+3) = 0$$

$$\therefore x = 2 \text{ or } x = -3$$

Again, we can check that these solutions satisfy the original equation.

The thing to note here is that the solution process has not altered. Rather than having one of the powers represented by a constant, we now have both powers containing the variable.

EXAMPLE 7.6

Find $\{x|\ 3^{x^2-5x+2} = 9^{x+1}\}$.

Solution

We need to first express the equation in the form $b^{f(x)} = b^{g(x)}$ where, in this case, $b = 3$:

$$\begin{aligned} 3^{x^2-5x+2} = 9^{x+1} \Leftrightarrow 3^{x^2-5x+2} &= (3^2)^{x+1} \\ \Leftrightarrow 3^{x^2-5x+2} &= 3^{2x+2} \\ \Leftrightarrow x^2-5x+2 &= 2x+2 \\ \Leftrightarrow x^2-7x &= 0 \\ \Leftrightarrow x(x-7) &= 0 \\ \Leftrightarrow x &= 0 \text{ or } x = 7 \end{aligned}$$

Again, checking our solutions we have, $x = 0$: L.H.S $= 3^{0-0+2} = 9 = 9^{0+1}$ = R.H.S

$x = 7$: L.H.S $= 3^{7^2-5\times7+2} = 3^{16} = 9^{7+1}$ = R.H.S

Therefore, the solution set is $\{0, 7\}$

We now have a more general statement for solving exponential equations:

$b^{f(x)} = b^{g(x)} \Leftrightarrow f(x) = g(x)$, where $b > 0$ and $b \neq 1$.

It is important to realise that this will only be true if **the base is the same on both sides** of the equality sign.

EXERCISES 7.1.3

1. Solve the following for the unknown

(a) $9^{2x-1} = 3^{2x+5}$ (b) $4^{x+1} = 8^{2x-4}$ (c) $25^{2x+3} = 125^{x+1}$

(d) $2^{4x+1} = 4^{x+2}$ (e) $16^{2x-1} = 8^{2x+1}$ (f) $\sqrt{3} \times 27^{x+1} = 9^{2x+1}$

(g) $(\sqrt{3})^{x-1} = 9^{-x+2}$ (h) $8^x = \dfrac{1}{16^{x+1}}$ (i) $4^{x+2} \times 8^{x-1} = 2$

2. Solve for the unknown

(a) $8^{x+1} = \dfrac{1}{2^x}$ (b) $8^{x+1} = 2^{x^2-1}$ (c) $3^{x-1} = 3^{x^2-1}$

(d) $4^{x^2-7x+12} = 1$ (e) $6^{\sqrt{n^2-3n}} = 36$ (f) $(5^x)^2 = 5^{x^2}$

3. Solve the following

(a) $(x^2-x-1)^{x^2} = x^2-x-1$

(b) $(x-2)^{x^2-x-12} = 1$

(c) $(3x-4)^{2x^2} = (3x-4)^{5x-2}$

(d) $|x|^{x^2-2x} = 1$

(e) $(x^2+x-57)^{3x^2+3} = (x^2+x-57)^{10x}$

7.1.4 WHAT IF THE BASE IS NOT THE SAME?

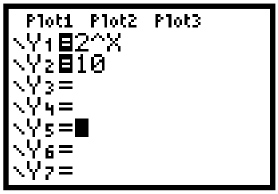
Consider the equation $2^x = 10$. It is not possible (at this stage) to express the number 10, in exponent form with a base of 2. This means that our previous methods will not work.

However, we could try a numerical or even graphical approach to this problem. Clearly the value of x must be somewhere in the range [3,4] as $2^3 = 8$ and $2^4 = 16$.We explore this problem using the graphics calculator.

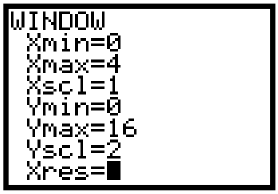
We begin by defining the two relevant equations,

$$y = 2^x \text{ and } y = 10 .$$

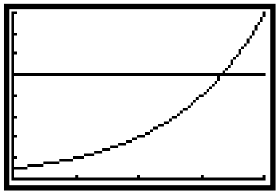
Then we enter these functions using the equation editor screen:



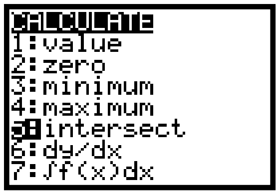
Next, we set our domian and range.
As we have already decided that $x \in [3, 4]$, we can set the domain to be $0 \le x \le 4$.



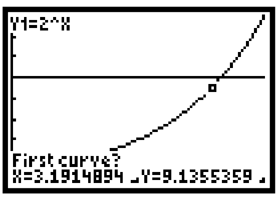
We can now obtain a graphical display of the equation $2^x = 10$.

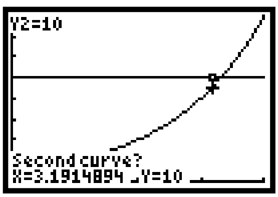


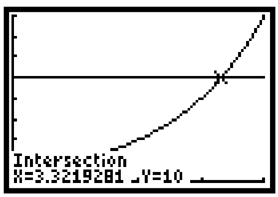
We can now find the point of intersection.
To do this we use the **CALC** menu and choose option **5: int** (this will determine the intersection of the two curves).



When asked for **First curve?** press **ENTER**. Similarly, for **Second curve?** When asked to **Guess?** move the cursor as close as possible to the point where both graphs meet and press **ENTER**.







We have a solution (to four decimal places) for x, i.e., $x = 3.3219$. We could also have used the **ZOOM** facility to obtain the same result. We do this in the next example.

At this stage, the key to being able to solve equations of the form $a^x = b$ (where b cannot be easily expressed as a number having base 'a'), is to accurately sketch the graphs of $y = a^x$ and $y = b$, and then to determine where the two graphs meet.

EXAMPLE 7.7

Solve for x, $2^x = 12$

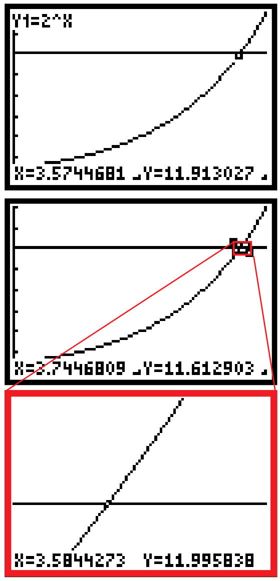
Solution

Let $y_1 = 2^x$ and $y_2 = 12$.

We enter the functions: **Y_1 = 2^X**

Y_2 = 12

Using the TI–83 to sketch the given graphs, we have:

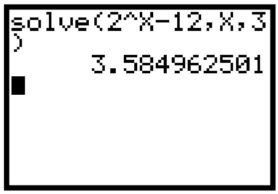


Using the **TRACE** key, we can move the cursor along the graph so that the square lies at the point where the two graphs meet. At this stage, our 'best solution' is $x = 3.57$

We can obtain a more accurate answer by using the **ZOOM** facility and then selecting **1:ZBOX** option (to 'close–in' on the point of intersection). Repeated use of the **Zoom** facility will continue to provide a more accurate solution.

After using the zoom facility once we have $x = 3.5844$
(The actual answer is $x = 3.5849...$).

Note: We can use the **0:solve** function on the TI–83,
i.e., **solve** (2^x–12,x,3) = 3.5849. . . .



However, it is the graphical approach that we wish to emphaise at this stage, so that the relationship between the solution, the roots and the graphical representation can be clearly observed.

EXERCISES 7.1.4

1. Use a graphical approach to solve the following (give your answer correct to 2 d.p.).

(a)	i.	$2^x = 40$	ii.	$2^x = 1020$	iii.	$2^x = 6$
(b)	i.	$3^x = 12$	ii.	$3^x = 80$	iii.	$3^x = 500$
(c)	i.	$\left(\frac{1}{2}\right)^x = \frac{1}{10}$	ii.	$\left(\frac{1}{2}\right)^x = 20$	iii.	$\left(\frac{1}{2}\right)^x = 80$
(d)	i.	$5^x = 0.1$	ii.	$5^x = 15$	iii.	$5^x = 0.01$

2. Use a graphical approach to solve the following (give your answer correct to 2 d.p.).

(a) $\{x \mid 2^x = 1 - x\}$ (b) $\{x \mid 2^x = -x + 2\}$

(c) $\{x \mid 2^{-x} = 4x + 2\}$ (d) $\{x \mid 3^x = 1 - x^2\}$

(e) $\{x \mid 3^{-x} = x^2 - 1\}$ (f) $\{x \mid 5^x = 2 - (x-1)^2\}$

7.1.5 A SPECIAL BASE (e)

Of all the expressions a^x, that for which $a = e$ is known as **the exponential function**. The exponential function is also known as the natural exponential function, in recognition of the important role that the value 'e' has. The importance of 'e' is that it occurs in many applications that arise as a result of natural phenomena. The question then remains; **What is 'e'**?

We consider how an investment can earn continuously compounded interest:

If a principal amount $\$P$ is invested at an annual percentage rate r, compounded once a year, the amount in the balance, $\$A$, after one year is given by $A = P + P \times r = P(1 + r)$. We can then have more frequent (quarterly, monthly, daily) compounding interest.

For example, if we have quarterly compounding interest then each quarter will have an effective rate of $\frac{r}{4}$, which will be compounded 4 times. This means that by the end of the year, the balance will be given by $A = \left(1 + \frac{r}{4}\right)^4$.

If we next consider the situation where there are n compoundings per year, so that the rate per compounding becomes $\frac{r}{n}$, we then have that the amount in the balance after a year (i.e., after n compoundings) is given by $A = \left(1 + \frac{r}{n}\right)^n$.

If we allow the number of compoundings n, to increase without bound, we obtain what is known as continuous compounding. We can set up a table of values for the case when $r = 1$.

n	$\left(1 + \frac{1}{n}\right)^n$
1	$\left(1 + \frac{1}{1}\right)^1 = 2$
10	$\left(1 + \frac{1}{10}\right)^{10} = 2.593742\ldots$
100	$\left(1 + \frac{1}{100}\right)^{100} = 2.704813\ldots$
1000	$\left(1 + \frac{1}{1000}\right)^{1000} = 2.716923\ldots$
10 000	$\left(1 + \frac{1}{10000}\right)^{10000} = 2.718145\ldots$

From the table of values, we have that as the value of n increases, the value of $\left(1 + \frac{1}{n}\right)^n$ approaches a fixed number. This number is given by 2.718145. . ., which happens to be an approximate value for the number 'e'.

That is, $$\lim_{n \to \infty}\left(1+\frac{1}{n}\right)^n = e = 2.71828\ldots$$

This limiting expression is also known as Euler's number.

This means that the natural base, e, is an irrational number just as the number π is. Notice then, the number e can be used in the same way that π is used in calculations.

EXAMPLE 7.8

Evaluate the following to 4 decimal places

(a) e^2 (b) $\sqrt{e}$ (c) $e^3 - 2$

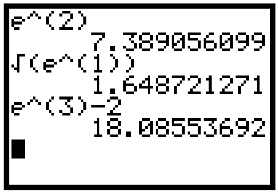
Solution

Remember, treat 'e' as you would any other number.

(a) $e^2 = 7.3891$

(b) $\sqrt{e} = 1.6487$

(c) $e^3 - 2 = 18.0855$



EXAMPLE 7.9

Solve the following

(a) $e^{2x} = e$ (b) $e^{2x-3} = \frac{1}{e}$ (c) $e^{x^2-1} = e^{3x-3}$

Solution

(a) $e^{2x} = e \Leftrightarrow e^{2x} = e^1 \Leftrightarrow 2x = 1$

$\Leftrightarrow x = \frac{1}{2}$

(b) $e^{2x-3} = \frac{1}{e} \Leftrightarrow e^{2x-3} = e^{-1} \Leftrightarrow 2x - 3 = -1$

$\Leftrightarrow 2x = 2$

$\Leftrightarrow x = 1$

(c) $e^{x^2-1} = e^{3x-3} \Leftrightarrow x^2 - 1 = 3x - 3$

$\Leftrightarrow x^2 - 3x + 2 = 0$

$\Leftrightarrow (x-1)(x-2) = 0$

$\Leftrightarrow x = 1$ or $x = 2$

EXERCISES 7.1.5

1. Solve for x

(a) $e^x = e^2$ (b) $e^x = \frac{1}{e}$ (c) $e^x = \sqrt{e}$ (d) $e^{-2x} = \frac{1}{e}$

2. Solve for x

(a) $e^{2x} = e^{3-x}$ (b) $e^{2+x} = e^{5-4x}$ (c) $\frac{1}{e^{3x+1}} = e^{2x-1}$

3. Solve for x

(a) $(e^x)^2 = e^x$ (b) $(e^x)^2 = \sqrt{e^{x+2}}$

4. Solve for x

(a) $e^{x^2-x} = e^2$ (b) $e^{x^2-x} = e^6$ (c) $e^{x^2+3x-1} = e^{x-2}$

(d) $e^{x^2+4x} = e^{6-x}$ (e) $\left(\frac{1}{e}\right)^x = e^{-x^2}$ (f) $e^{-x+2} = e^{\frac{1}{x}}$

5. Solve for x, giving your answer correct to 4 decimal places

(a) $e^x = 4$ (b) $e^x = 9$ (c) $e^x = 25$ (d) $e^x = -4$

6. Solve for x, giving your answer correct to 4 decimal places

(a) $3e^{2x} = 7$ (b) $4e^{2x} = 9$ (c) $2e^{3x} = 5$ (d) $7e^{3x} = 2$

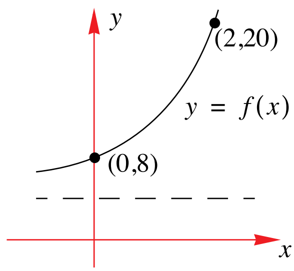
7. Solve for x, giving your answer correct to 4 decimal places

(a) $e^x = -x+1$ (b) $2e^x = -3x-1$ (c) $e^x = x+1$

8. (a) Show that if $e^{2x}+6 = 5e^x$ then $e^x = 3$ or $e^x = 2$.

(b) Show that if $e^{2x}-5e^x = 6$ then $e^x = 6$.

9. The graph of the function $f(x) = a \times 2^x + b$ is shown alongside.
Find the value of $f(3)$.



10. Find the values of a and k if the graph with equation $f(x) = ae^{-kx}$ passes through the points $(1, e)$ and $(-1, 2e)$.

7.2 EXPONENTIAL MODELLING

There are many situations and examples where an exponentional function is an appropriate function to model a particular **growth** or **decay** process. For example:

1. When looking at the bacteria count of an experiment, the growth in the number of bacteria present in the colony is accurately represented by an exponential growth model.

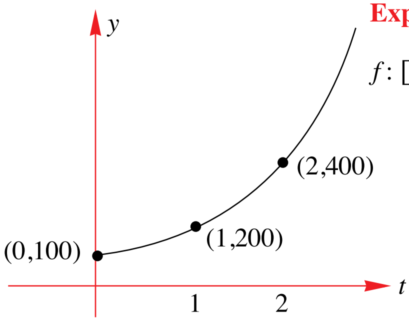
If there are initially 100 bacteria in a colony and the size doubles every day, we model this situation by making use of the exponential function,

$$f: [0, a[\mapsto \mathbb{R}\text{, where } f(t) = 100 \times 2^t, a \in \mathbb{R}.$$

The graph of such a model is given below.

Exponential growth

$f: [0, a[\mapsto \mathbb{R}$, where $f(t) = 100 \times 2^t, a \in \mathbb{R}$



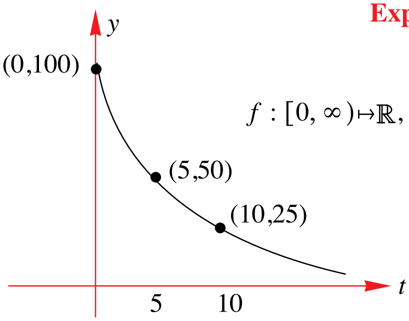
2. Certain physical quantities decrease exponentially, for example, the decay of a radioactive substance, or isotope. Associated with this is the half–life, that is, the time that it takes for the substance to decay to one half of its original amount.

A radioactive bismuth isotope has a half–life of 5 days. If there is 100 milligrams initially, then we can model this situation by making use of the exponential function,

$$f : [0, \infty) \mapsto \mathbb{R}, \text{ where } f(t) = 100 \times \left(\frac{1}{2}\right)^{t/5}$$

Exponential decay

$f : [0, \infty) \mapsto \mathbb{R}$, where $f(t) = 100 \times \left(\frac{1}{2}\right)^{t/5}$



Other areas where the use of exponential modelling appears include, medicine (drug dosage), economics (compound interest), oceanography (light penetration in an ocean), environment (endangered species) and many more. We shall look at a few examples of exponential modelling in detail.

Notice that whenever making use of an exponential function to model a real life situation, the domain of consideration is always restricted to $[0, \infty)$. Corresponding to time, $t = 0$ (or $x = 0$), there exists an initial amount. This initial amount is usually denoted by a capital letter with a subscript of '0'. For example, if we are referring to the population size of bacteria, N or the number of radioactive particles P, then their initial amounts would be represented by N_0 and P_0 respectively, so that when $t = 0$, $N = N_0$ and $P = P_0$

Such equations would then be given by

1.	$N = N_0 \times a^t, t \geq 0, a > 1$ [growth]
2.	$P = P_0 \times a^{-t}, t \geq 0, a > 1$ [decay]

EXAMPLE 7.10

During the chemical processing of a particular type of mineral, the amount M kg of the mineral present at time t hours since the process started, is given by

$$M(t) = M_0(2)^{kt}, t \ge 0, k < 0$$

where M_0 is the original amount of mineral present. If 128 kilograms of the mineral are reduced to 32 kilograms in the first six hours of the process find,

(a) i. the value of k.
ii. the quantity of the mineral that remains after 10 hours of processing.

(b) Sketch a graph of the amount of mineral present at time t hours after the process started.

Solution

(a) i. We have that when $t = 0, M = 128 \Rightarrow M_0 = 128$ (the initial amount of mineral). The equation then becomes

$$M(t) = 128 \times (2)^{kt}, t \ge 0, k < 0 .$$

Next, when $t = 6, M = 32$, so that when we substitute this information into the equation, we have, $32 = 128 \times (2)^{6k} \Leftrightarrow 2^{6k} = \frac{1}{4}$

$$\Leftrightarrow 2^{6k} = 2^{-2}$$
$$\Leftrightarrow 6k = -2$$
$$\Leftrightarrow k = -\frac{1}{3}$$

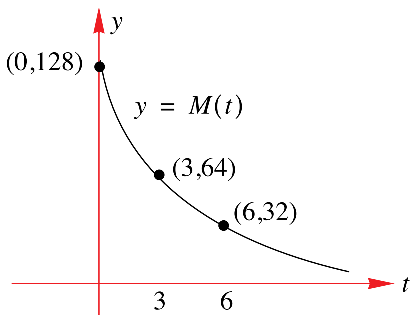
Therefore, the equation is given by, $M(t) = 128 \times (2)^{-\frac{1}{3}t}, t \ge 0$

ii. After 10 hours, we have, $M(10) = 128 \times (2)^{-\frac{1}{3} \times 10}$
$= 12.699$

That is, there is approximately 12.70 kg of mineral left after 10 hours of processing.

(b) We notice that the equation is of the form $f : [0, \infty) \mapsto \mathbb{R}$,where $f(t) = a^{-x}, a > 1$ i.e., an exponential decay.

Hence, we have a decreasing function:



EXAMPLE 7.11

The scrap value , \$V, of some machinery after t years is given by

$$V(t) = 50000(0.58)^t, t \ge 0 .$$

(a) What was the initial cost of the machine?
(b) What is the scrap value of the machine after 4 years?
(c) How long would it be before the scrap value reaches \$20000?
(d) The machine needs to be sold at some time when the scrap value of the machine lies somewhere between 10000 and 15000. What time–frame does the owner have?

Solution

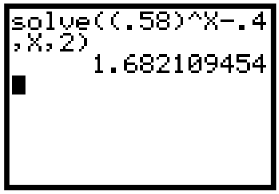
(a) When $t = 0$, we have $V(0) = 50000(0.58)^0 = 50000$.
That is, the machine initially cost \$50000.

(b) After 4 years, we have $V(4) = 50000(0.58)^4 = 5658.25$
That is, after 4 years, the scrap value of the machine would be \$5658.25.

(c) We need to determine the value of t when $V = 20000$:

$$20000 = 50000(0.58)^t \Leftrightarrow 0.4 = 0.58^t$$

Then, using the TI–83 we have (using the solve facility):
That is, $t = 1.68$ (to 2 d.p)



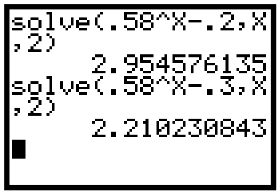
(d) This time we want to solve for t where $10000 \le V(t) \le 15000$.

Now, $10000 \le V(t) \le 15000 \Leftrightarrow 10000 \le 50000(0.58)^t \le 15000$

$$\Leftrightarrow 0.2 \le (0.58)^t \le 0.3$$

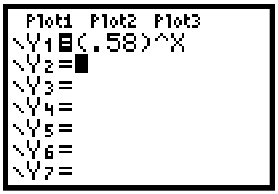
Solving the corresponding equalities, we have:

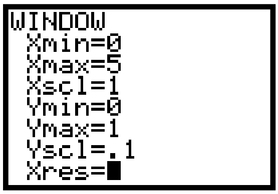
Giving $\quad 2.21 \le t \le 2.95$.

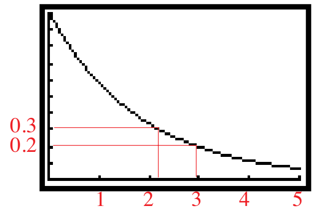


Notice that the graph helped in guessing the values of t that were used in determining the solutions.

Using the TI–83, we can easily sketch the graph of $y = (0.58)^t, t \ge 0$:







EXERCISES 7.2

1. The number of bacteria in a culture, N, is modelled by the exponential function

$$N = 1000 \times 2^{0.2t}, t \ge 0$$

where t is measured in days.

(a) Find the initial number of bacteria in this culture.

(b) Find the number of bacteria after i 3 days.
ii. 5 days.

(c) How long does it takes for the number of bacteria to grow to 4000?

2. The 'growth' of crystals, measured in kilograms, in a chemical solution, has been approximately modelled by the exponential function $W = 2 \times 10^{kt}, t \ge 0$, where W is measured in kilograms and t in years. After 1 year in a chemical solution, the amount of crystal in the chemical increased by 6 grams.

(a) Find the value of k.

(b) Find the amount of crystal in the chemical solution after 10 years.

(c) How long does it takes for this crystal to double in 'size'?

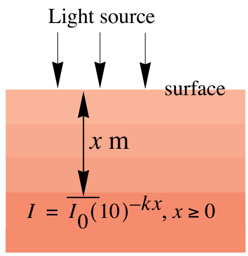
(d) Sketch the graph showing the amount of crystal in the chemical solution at time t.

3. It is found that the intensity of light decreases as it passes through water. The intensity I units at a depth x metres from the surface is given by

$$I = I_0(10)^{-kx}, x \geq 0$$

where I_0 units is the intensity at the surface.

Based on recordings taken by a diving team, it was found that $I = 0.2I_0$ at a depth of 50 metres.



(a) Find the value of k (to 5 d.p.).
(b) Find the percentage of light remaining at a depth of 20 metres.
(c) How much further would the divers need to descend, to reach a level at which the intensity of light would be given by $I = 0.1I_0$?
(d) Find the depth at which the intensity would be a half of that at the surface.
(e) Sketch the graph representing the intensity of light at a depth of x metres.

4. An endangered species of animal is placed into a game reserve. 150 such animals have been introduced into this reserve. The number of animals, $N(t)$, alive t years after being placed in this reserve is predicted by the exponential growth model $N(t) = 150 \times 1.05^t$.
(a) Find the number of animals that are alive after
i. 1 year ii. 2 years iii. 5 years
(b) How long will it take for the population to double?
(c) How long is it before there are 400 of this species in the reserve?
(d) Sketch a graph depicting the population size of the herd over time.
Is this a realistic model?

5. The processing of a type of mineral in a chemical solution has been found to reduce the amount of that mineral left in the solution. Using this chemical process, the amount W kg of the mineral left in the solution at time t hours is modelled by the exponential decay function $W = W_0 \times 10^{-kt}$, where W_0 kg is the original amount of mineral.
It is found that 50 kilograms of mineral are reduced to 30 kilograms in 10 hours.
(a) Write down the value of W_0 .
(b) Find the value of k (to 4 decimal places).
(c) How much of the mineral will be in the solution after 20 hours?
(d) Sketch the graph representing the amount of mineral *left* in the solution.
(e) Sketch the graph representing the amount by which the mineral is *reduced*.

6. The temperatures of distant dying stars have been modelled by exponential decay functions. A distant star known to have an initial surface temperature of 15000˚C, is losing heat according to the function $T = T_0 \times 10^{-0.1t}$, where T_0 ˚C is its present temperature, and T ˚C the temperature at time t (in millions of years).
(a) Determine the value of T_0 .
(b) Find the temperature of this star in
i. one million years,
ii. 10 million years.
(c) How long will it be before the star reaches a temperature that is half its original surface temperature?
(d) Sketch a graph representing this situation.

7. The amount of radioactive material, Q grams, decays according to the model given by the equation $Q = 200 \times 10^{-kt}, t \geq 0$, where t is measured in years. It is known that after 40 years, the amount of radioactive material present is 50 grams.
(a) Find the value of k (to 4 d.p.).
(b) Find the amount of radioactive material present after 80 years.
(c) What is the half life for this radioactive substance ? *The half–life is the time taken for the radioactive material to reach half its original amount.*
(d) Sketch the graph representing the amount of radioactive material present as a function of time, t years.

8. The resale value, V dollars, of a structure, decreases according to the function

$$V = 2000000(10)^{-0.01t}, t \geq 0$$

where t is the number of years since the structure was built.
(a) How much would the structure have sold for upon completion?
(b) How much would the structure have sold for 10 years after completion?
(c) How long will it take for the structure to lose half its value? (Answer to 1 d.p)
(d) Sketch the graph of the structure's value since completion.

9. The population number N in a small town in northern India is approximately modelled by the equation $N = N_0 \times 10^{kt}, t \geq 0$, where N_0 is the initial popluation and t is the time in years since 1980.
The population was found to increase from 100,000 in 1980 to 150,000 in 1990.
(a) Show that $N_0 = 100000$ and that $1.5 = 10^{10k}$.
(b) Hence find the value of k (to 5 d.p.).
(c) Find the population in this town in 1997.
(d) How long (since 1980) will it be before the population reaches 250,000?

10. The healing process of certain types of wounds is measured by the decrease in the surface area that the wound occupies on the skin. A certain skin wound has its surface area modelled by the equation $S = 20 \times 2^{-0.01t}, t \geq 0$ where S square centimetres is the unhealed area t days after the skin received the wound.
(a) What area did the wound originally cover?
(b) What area will the wound occupy after 2 days?
(c) How long will it be before the wound area is reduced by 50%?
(d) How long will it be before the wound area is reduced by 90%?

11. In a certain city the number of inhabitants, N, at time t years since the 1st of January 1970, is modelled by the equation $N = 120000(1.04)^{kt}, t \geq 0, k > 0$.
On the 1st of January 1980, the inhabitants numbered 177629.
(a) Determine the value of k.
(b) How many people will be living in this city by
i. 1st January 2007? ii. 1st April 2007?
(c) How long will it take for the population to reach 1000000?

12. Suppose you deposited \$700 into an account that pays 5.80% interest per annum.
(a) How much money will you have in the account at the end of 5 years if
i. the interest is compounded quarterly?
ii. the interest is compounded continuously?
(b) With continuous compounding, how long will it take to double your money?
(c) Sketch the graph showing the amount of money in the account for (b).

13. On the 1st of January 1988, a number of antelopes were introduced into a wildlife reservation, free of predators. Over the years, the number of antelopes in the reservation was recorded:

Date (day/month/year)	1/1/88	1/1/90	1/6/94	1/1/98	1/6/02	1/6/04
Number of antelopes	–	120	190	260	400	485

Although the exact number of antelopes that were placed in the reserve was not available, it is thought that an exponential function would provide a good model for the number of antelopes present in the reserve.

Assume an exponential growth model of the form $N = N_0 \times 2^{kt}, t \geq 0, k > 0$, where N represents the number of antelopes present at time t years since 1/1/80, an N_0 is the initial population size of the herd, and k is a positive real constant.

(a) Determine the number of antelopes introduced into the reserve.

(b) Determine the equation that best models this situation.

(c) Based on this model, predict the number of antelopes that will be present in the reserve by the year 2008.

14. Betty, the mathematician, has a young baby who was recently ill with fever. Betty noticed that the baby's temperature, T, was increasing linearly, until an hour after being given a dose of penicillin. It peaked, then decreased very quickly, possibly exponentially.

Betty approximated the baby's temperature, above 37˚C by the function

$$T(t) = t \times 0.82^t, t \geq 0$$

where t refers to the time in hours after 7.00pm.

(a) Sketch the graph of $T(t)$.

(b) Determine the maximum temperature and the time when this occured (giving your answer correct to to 2 d.p)

15. An equation of the form $N(t) = \dfrac{a}{1 + be^{-ct}}, t \geq 0$, where a, b and c are positive constants represents a logistic curve. Logistic curves have been found useful when describing a population N that initially grows rapidly, but whose growth rate decreases after t reaches a certain value.

A study of the growth of protozoa was found to display these characteristics. It was found that the population was well described if $c = 1.12$, $a = 100$, and t measured time in days.

(a) If the initial population was 5 protozoa, find the value of b.

(b) It was found that the growth rate was a maximum when the population size reached 50. How long did it take for this to occur ?

(c) Determine the optimum population size for the protozoa.

16. The height of some particular types of trees can be approximately modelled by the logistic function $h = \dfrac{36}{1 + 200e^{-0.2t}}, t \geq 0$ where h is the height of the tree measured in metres and t the age of the tree (in years) since it was planted.

(a) Determine the height of the tree when planted.

(b) By how much will the tree have grown in the first year ?

(c) How tall will the tree be after 10 years ?

(d) How tall will it be after 100 years ?

(e) How long will it take for the tree to grow to a height of
   i. 10 metres ?
   ii. 20 metres ?
   iii. 30 metres ?
(f) What is the maximum height that a tree, whose height is modelled by this equation, will reach? Explain your answer.
(g) Sketch a graph representing the height of trees against time for trees whose height can be modelled by the above function.

17. Certain prescription drugs, e.g. tablets that are taken orally, which enter the bloodstream at a rate R, are approximately modelled by the equation $R = a \times b^t, t \geq 0$ where t is measured in minutes and a and b are appropriate constants.

When an adult is administered a 100-milligram tablet, the rate is modelled by the function $R = 5 \times 0.95^t, t \geq 0$ mg/min.
The amount A mg of the drug in the bloodstream at time t minutes can then be approximated by a second function, $A = 98(1 - 0.95^t)$ mg.
(a) What is the initial rate at which the drug enters the bloodstream?
(b) How long will it take before the rate at which the drug enters the bloodstream is halved?
(c) How long does it takes for
   i. 10 milligrams of the drug to enter the bloodstream.
   ii. 50 milligrams of the drug to enter the bloodstream.
   iii. 95 milligrams of the drug to enter the bloodstream.
(d) How much of the drug is in the bloodstream when the drug is entering at a rate of 4 mg/min.
(e) Sketch the graph of R and A, on the same set of axes.
(f) Will the patient ever feel the full effects of the 100-milligram drug?

18. As consumers, we know from experience that the demand for a product tends to decrease as the price increases. This type of information can be represented by a demand function. The demand function for a particular product is given by $p = 500 - 0.6 \times e^{0.0004x}$, where p is the price per unit and x is the total demand in number of units.

(a) Find the price p to the nearest dollar for a demand of
   i. 1000 units.
   ii. 5000 units.
   iii. 10 000 units.
(b) Sketch the graph of this demand function.
(c) What level of demand will produce a price per unit of $200?

The total revenue, R, obtained by selling x units of this product is given by $R = xp$.

(d) Find the revenue by selling
   i. 1000 units
   ii. 5000 units
   iii. 10 000 units
(e) Sketch the graph of the revenue equation.
(f) Find the number of units that must be sold in order to maximize the total revenue.
(g) Determine the maximum revenue. Giving your answer to 2 d.p.

7.3 LOGARITHMS

7.3.1 WHAT ARE LOGARITHMS?

Consider the following sequence of numbers:

Sequence N	1	2	3	4	5	6	7	8	9	10	. . .
Sequence y	2	4	8	16	32	64	128	256	512	1024	

The relationship between the values of N and y is given by $y = 2^N$.

Using the above table, evaluate the product 16×64. Using the above table? What for? Clearly there is no use for such a table. Surely this can be done using mental arithmetic (or even using a calculator!). There really is no need to use the above table. However, let's explore this question further, in the hope that we might find something more than the answer.

We start by setting up a table of values that correspond to the numbers in question:

N	4	6	Sum(4 + 6) = 10
y	16	64	Product = 1024

From the first table of sequences, we notice that the sum of the 'N–sequence' (i.e., 10), corresponds to the value of the 'y–sequence' (i.e., 1024).

We next consider the product 8×32, again. Setting up a table of values for the numbers in the sequences that are under investigation we have:

N	3	5	Sum(3 + 5) = 8
y	8	32	Product = 256

What about 4×64 ? As before, we set up the required table of values:

N	2	6	Sum(2 + 6) = 8
y	4	64	Product = 256

In each case the **product** of two terms of the sequence y corresponds to the **sum** obtained by adding corresponding terms of the sequence N.

Notice then that **dividing** two numbers from the sequence y corresponds to the result when **subtracting** the two corresponding numbers from the sequence N,

e.g., for the sequence y: $512 \div 32 = 16$.

for the sequence N: $9 - 5 = 4$

This remarkable property was observed as early as 1594 by **John Napier**. John Napier was born in 1550 (when his father was all of sixteen years of age!) He lived most of his life at the family estate of Merchiston Castle, near Edinburgh, Scotland. Although his life was not without controversy, in matters both religious and political, Napier (when relaxing from his political and religious polemics) would endulge in the study of mathematics and science. His amusement with the study of mathematics led him to the invention of logarithms. In 1614 Napier published his discussion of logarithms in a brochure entitled Mirifici logarithmorum canonis descriptio (A description of the Wonderful Law of Logarithms). Napier died in1617.

It is only fair to mention that the Swiss instrument maker **Jobst Bürgi** (1552–1632) conceived and constructed a table of logarithms independently of Napier, publishing his results in 1620, six years after Napier had announced his discovery.

One of the anomalies in the history of mathematics is the fact that logarithms were discovered before exponents were in use.

Although in this day and age of technology, the use of electronic calculators and computers, render the evaluation of products and quotients to a task that involves the simple push of a few buttons, logarithms are an efficient means of converting a product to a sum and a quotient to a difference. So, what are logarithms?

Nowadays, a logarithm is universally regarded as an exponent.

Thus, if $y = b^N$ we say that N is the logarithm of y to the base b.

From the sequence table, we have that $2^7 = 128$, so that 7 is the **logarithm** of 128 to the **base** 2. Similarly, $3^4 = 81$, and so 4 is the **logarithm** of 81 to the **base** 3.

We use the following notation when using logarithms: $y = b^N \Leftrightarrow N = \log_N y$

That is, ***N* is the logarithm of *y* to the base *b***, which corresponds to the power that the base b must be raised so that the result is y.

EXAMPLE 7.12

Find the following logarithms

(a) $\log_2 32$ (b) $\log_{10} 1000$ (c) $\log_3 729$ (d) $\log_2 \frac{1}{16}$

Solution

(a) To determine the number $\log_2 32$, we ask ourselves the following question:
"To what power must we raise the number 2, so that our result is 32?"
Letting $x = \log_2 32$, we must find the number x such that $2^x = 32$.
Clearly then, $x = 5$, and so we have that $\log_2 32 = 5$.
One convention in setting out such questions is: $\log_2 32 = x \Leftrightarrow 2^x = 32$
$$\Leftrightarrow x = 5$$

(b) As in part (a), we ask the question
"To what power must we raise the number 10, so that our result is 1000 ?"
That is, $x = \log_{10} 1000 \Leftrightarrow 10^x = 1000$
$$\Leftrightarrow x = 3$$
So that $\log_{10} 1000 = 3$.

(c) Now, $x = \log_3 729 \Leftrightarrow 3^x = 729$

$\Leftrightarrow x = 6$ [This was obtained by trial and error.]

Therefore, $\log_3 729 = 6$.

(d) Although we have a fraction, this does not alter the process:

$$x = \log_2 \frac{1}{16} \Leftrightarrow 2^x = \frac{1}{16}$$

$$\Leftrightarrow 2^x = \frac{1}{2^4}(= 2^{-4})$$

$$\Leftrightarrow x = -4$$

7.3.2 CAN WE FIND THE LOGARITHM OF A NEGATIVE NUMBER?

To evaluate $\log_a(-4)$ for some base $a > 0$, we need to solve the equivalent statement:

$$x = \log_a(-4) \Leftrightarrow a^x = -4 .$$

However, the value of a^x where $a > 0$, will always be positive, therefore there is no value of x for which $a^x = -4$. This means that

we can not evaluate the logarithm of a negative number.

We can now make our definition a little stronger: $N = \log_b y \Leftrightarrow y = b^N, y > 0$

Note: we also require that $b \neq 1$, otherwise we will have that $y = 1^N = 1$ for any value of N.

As we saw earlier, there exists a natural exponential whose base is 'e'. In the same way we also have the natural logarithm, whose base is also 'e'.

In this instance, we refer to this logarithm base 'e' as the **natural logarithm**.

We denote the natural logarithm by $\log_e x$ or $\ln x$ (read as '*el n*'). i.e., $y = e^N \Leftrightarrow N = \log_e y$.

For example, $\log_e e^2 = 2$. That is, $N = \log_e e^2 \Leftrightarrow e^N = e^2$

$$\Leftrightarrow N = 2$$

EXAMPLE 7.13

Find the value of x given that

(a) $\log_e x = 3$ (b) $\log_e(x-2) = 0.5$ (c) $\log_e 5 = x$

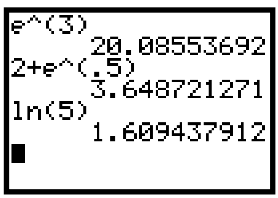
Solution

(a) $\log_e x = 3 \Leftrightarrow x = e^3 \approx 20.09$

(b) $\log_e(x-2) = 0.5 \Leftrightarrow x - 2 = e^{0.5}$

$$\Leftrightarrow x = 2 + \sqrt{e}$$

$$\therefore x \approx 3.65$$



(c) $x = \log_e 5 \approx 1.61$

Notice that your calculator has two logarithmic functions:
The **log** button stands for $\log_{10} x$.
The **ln** button stands for $\log_e x$.

EXERCISES 7.3

1. Use the definition of a logarithm to determine

(a) $\log_6 36$ (b) $\log_7 49$ (c) $\log_3 243$ (d) $\log_4 64$

(e) $\log_2\left(\frac{1}{8}\right)$ (f) $\log_3\left(\frac{1}{9}\right)$ (g) $\log_4 1$ (h) $\log_{10} 1$

(i) $\log_{\frac{1}{2}} 2$ (j) $\log_{\frac{1}{3}} 9$ (k) $\log_3 \sqrt{3}$ (l) $\log_{10} 0.01$

2. Change the following exponential expressions into their equivalent logarithmic form.

(a) $10^4 = 10000$ (b) $10^{-3} = 0.001$ (c) $10^y = x + 1$
(d) $10^7 = p$ (e) $2^y = x - 1$ (f) $2^{4x} = y - 2$

3. Change the following logarithmic expressions into their equivalent exponential form.

(a) $\log_2 x = 9$ (b) $\log_b y = x$ (c) $\log_b t = ax$
(d) $\log_{10} z = x^2$ (e) $\log_{10} y = 1 - x$ (f) $\log_2(ax - b) = y$

4. Solve for x in each of the following.

(a) $\log_2 x = 4$ (b) $\log_3 9 = x$ (c) $\log_4 x = \frac{1}{2}$

(d) $\log_x 3 = \frac{1}{2}$ (e) $\log_x 2 = 4$ (f) $\log_5 x = 3$

(g) $\log_x 16 = 2$ (h) $\log_x 81 = 2$ (i) $\log_x\left(\frac{1}{3}\right) = 3$

(j) $\log_2(x - 5) = 4$ (k) $\log_3 81 = x + 1$ (l) $\log_3(x - 4) = 2$

5. Solve for x in each of the following, giving your answer to 4 d.p.

(a) $\log_e x = 4$ (b) $\log_e 4 = x$ (c) $\log_e x = \frac{1}{2}$

(d) $\log_x e = \frac{1}{2}$ (e) $\log_x e = 2$ (f) $\log_x e = -1$

(g) $\log_e(x + 2) = 4$ (h) $\log_e(x - 2) = 1$ (i) $\log_x e = -2$

7.4 THE ALGEBRA OF LOGARITHMS

The following logarithmic laws are a direct consequence of the definition of a logarithm and the index laws already established.

First law: **The logarithm of a product** $\log_a(x \times y) = \log_a x + \log_a y,\ x > 0,\ y > 0$

Proof: Let $M = \log_a x$ and $N = \log_a y$ so that $x = a^M$ and $y = a^N$.

Then, $x \times y = a^M \times a^N$

$$\Leftrightarrow x \times y = a^{M+N}$$

$$\Leftrightarrow \log_a(x \times y) = M + N$$

$$\Leftrightarrow \log_a(x \times y) = \log_a x + \log_a y$$

EXAMPLE 7.14

Simplify the expression

(a) $\log_3 x + \log_3(4x)$ (b) $\log_2 x + \log_2(4x)$

Solution

(a) $\log_3 x + \log_3(4x) = \log_3(x \times 4x)$

$$= \log_3 4x^2$$

(b) This time we note that because the base is '2' and there is a '4' in one of the logarithmic expressions, we could first try to 'remove the '4'.

$$\log_2 x + \log_2(4x) = \log_2 x + (\log_2 4 + \log_2 x)$$

$$= \log_2 x + 2 + \log_2 x$$

$$= 2\log_2 x + 2$$

EXAMPLE 7.15

Given that $\log_a p = 0.70$ and $\log_a q = 2$, evaluate the following

(a) $\log_a p^2$ (b) $\log_a(p^2 q)$ (c) $\log_a(apq)$

Solution

(a) $\log_a p^2 = \log_a(p \times p) = \log_a p + \log_a p$

$$= 2\log_a p$$

$$= 2 \times 0.70$$

$$= 1.40$$

(b) $\log_a(p^2 q) = \log_a p^2 + \log_a q$

$$= 2\log_a p + \log_a q$$

$$= 1.40 + 2$$

$$= 3.40$$

(c) $\log_a(apq) = \log_a a + \log_a p + \log_a q$
$= 1 + 0.70 + 2$
$= 3.70$

EXAMPLE 7.16 Find $\{x| \log_2 x + \log_2(x+2) = 3\}$.

Solution

$$\log_2 x + \log_2(x+2) = 3 \Leftrightarrow \log_2[x \times (x+2)] = 3$$
$$\Leftrightarrow x(x+2) = 2^3$$
$$\Leftrightarrow x^2 + 2x = 8$$
$$\Leftrightarrow x^2 + 2x - 8 = 0$$
$$\Leftrightarrow (x+4)(x-2) = 0$$
$$\Leftrightarrow x = -4 \text{ or } x = 2$$

Next, we must check our solutions.

When $x = -4$, substituting into the **original equation**, we have:

L.H.S $= \log_2(-4) + \log_2(-4+2)$ – which cannot be evaluated (as the logarithm of a negative number does not exist).

Therefore, $x = -4$, is not a possible solution.

When $x = 2$, substituting into the **original equation**, we have:

L.H.S $= \log_2(2) + \log_2(2+2)$
$= \log_2 8$
$= 3$
$=$ R.H.S

Therefore, $\{x| \log_2 x + \log_2(x+2) = 3\} = \{2\}$.

Second law: The logarithm of a quotient $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y,\ x > 0,\ y > 0$

Proof: Let $M = \log_a x$ and $N = \log_a y$ so that $x = a^M$ and $y = a^N$.

Then, $\frac{x}{y} = \frac{a^M}{a^N}$

$$\Leftrightarrow \frac{x}{y} = a^{M-N}$$
$$\Leftrightarrow \log_a\left(\frac{x}{y}\right) = M - N$$
$$\Leftrightarrow \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

EXAMPLE 7.17 Simplify

(a) $\log_{10}100x - \log_{10}xy$ (b) $\log_2 8x^3 - \log_2 x^2 + \log_2\left(\frac{y}{x}\right)$

Solution

(a) $\log_{10}100x - \log_{10}xy = \log_{10}\left(\frac{100x}{xy}\right)$

$$= \log_{10}\left(\frac{100}{y}\right)$$

Note: We could then express $\log_{10}\left(\frac{100}{y}\right)$ as $\log_{10}100 - \log_{10}y = 2 - \log_{10}y$.

(b) $\log_2 8x^3 - \log_2 x^2 + \log_2\left(\frac{y}{x}\right) = \log_2\left(\frac{8x^3}{x^2}\right) + \log_2\left(\frac{y}{x}\right)$

$$= \log_2 8x + \log_2\left(\frac{y}{x}\right)$$

$$= \log_2\left(8x \times \frac{y}{x}\right)$$

$$= \log_2 8y$$

Note: We could then express $\log_2 8y$ as $\log_2 8 + \log_2 y = 3 + \log_2 y$.

EXAMPLE 7.18 Find $\{x|\ \log_{10}(x+2) - \log_{10}(x-1) = 1\}$

Solution

$$\log_{10}(x+2) - \log_{10}(x-1) = 1 \Leftrightarrow \log_{10}\left(\frac{x+2}{x-1}\right) = 1$$

$$\Leftrightarrow \left(\frac{x+2}{x-1}\right) = 10^1$$

$$\Leftrightarrow x + 2 = 10x - 10$$

$$\Leftrightarrow 12 = 9x$$

$$\Leftrightarrow x = \frac{4}{3}$$

Next, we check our answer. Substituting into the original equation, we have:

$$\text{L.H.S} = \log_{10}\left(\frac{4}{3}+2\right) - \log_{10}\left(\frac{4}{3}-1\right) = \log_{10}\frac{10}{3} - \log_{10}\frac{1}{3} = \log_{10}\left(\frac{10}{3} \div \frac{1}{3}\right)$$

$$= \log_{10}10$$

$$= 1 = \text{R.H.S}$$

Therefore, $\{x|\ \log_{10}(x+2) - \log_{10}(x-1) = 1\} = \left\{\frac{4}{3}\right\}$

Third law: The logarithm of a power $\log_a x^n = n\log_a x,\ x > 0$

Proof: This follows from repeated use of the First Law or it can be shown as follows:

Let $M = \log_a x \Leftrightarrow a^M = x$

$$\Leftrightarrow (a^M)^n = x^n \quad \text{[Raising both sides to the power of } n\text{]}$$
$$\Leftrightarrow a^{nM} = x^n \quad \text{[Using the index laws]}$$
$$\Leftrightarrow nM = \log_a x^n \quad \text{[Converting from exponential to log form]}$$
$$\Leftrightarrow n\log_a x = \log_a x^n$$

EXAMPLE 7.19

Given that $\log_a x = 0.2$ and $\log_a y = 0.5$, evaluate

(a) $\log_a x^3y^2$ (b) $\log_a\sqrt{\dfrac{x}{y^4}}$

Solution

(a)
$$\begin{aligned}\log_a x^3y^2 &= \log_a x^3 + \log_a y^2\\ &= 3\log_a x + 2\log_a y\\ &= 3\times 0.2 + 2\times 0.5\\ &= 1.6\end{aligned}$$

(b)
$$\begin{aligned}\log_a\sqrt{\frac{x}{y^4}} = \log_a\left(\frac{x}{y^4}\right)^{1/2} &= \frac{1}{2}\log_a\left(\frac{x}{y^4}\right)\\ &= \frac{1}{2}[\log_a(x) - \log_a y^4]\\ &= \frac{1}{2}[\log_a x - 4\log_a y]\\ &= \frac{1}{2}[0.2 - 4\times 0.5]\\ &= -0.9\end{aligned}$$

Fourth law: Change of base $\log_a b = \dfrac{\log_k b}{\log_k a},\ a, k \in \mathbb{R}^+ \backslash\{1\}$

Proof: Let $\log_a b = N$ so that $a^N = b$

Taking the logarithms to base k of both sides of the equation we have:

$$\log_k(a^N) = \log_k b \Leftrightarrow N\log_k a = \log_k b$$
$$\Leftrightarrow N = \frac{\log_k b}{\log_k a}$$

However, we have that $\log_a b = N$, therefore, $\log_a b = \dfrac{\log_k b}{\log_k a}$.

Other observations include:

1. $\log_a a = 1$
2. $\log_a 1 = 0$
3. $\log_a x^{-1} = -\log_a x,\ x > 0$
4. $\log_{\frac{1}{a}} x = -\log_a x$
5. $a^{\log_a x} = x,\ x > 0$

MISCELLANEOUS EXAMPLES

EXAMPLE 7.20

Express y in terms of x if (a) $2 + \log_{10}x = 4\log_{10}y$

(b) $\log x = \log(a - by) - \log a$

Solution

(a) Given that $2 + \log_{10}x = 4\log_{10}y$ then $2 = 4\log_{10}y - \log_{10}x$

$$\Leftrightarrow 2 = \log_{10}y^4 - \log_{10}x$$

$$\Leftrightarrow 2 = \log_{10}\left(\frac{y^4}{x}\right)$$

$$\Leftrightarrow 10^2 = \frac{y^4}{x}$$

$$\Leftrightarrow y^4 = 100x$$

$$\Leftrightarrow y = \sqrt[4]{100x} \quad (\text{ as } y > 0\ )$$

(b) Given that $\log x = \log(a - by) - \log a$ then $\log x = \log\dfrac{a - by}{a}$

$$\Leftrightarrow x = \frac{a - by}{a}$$

$$\Leftrightarrow ax = a - by$$

$$\Leftrightarrow by = a - ax$$

$$\Leftrightarrow y = \frac{a}{b}(1 - x)$$

EXAMPLE 7.21

Find x if (a) $\log_x 64 = 3$ (b) $\log_{10}x - \log_{10}(x - 2) = 1$

Solution

(a) $\log_x 64 = 3 \Leftrightarrow x^3 = 64$

$$\Leftrightarrow x^3 = 4^3$$

$$\Leftrightarrow x = 4$$

(b) $\log_{10}x - \log_{10}(x - 2) = 1 \Leftrightarrow \log_{10}\left(\dfrac{x}{x - 2}\right) = 1$

$$\Leftrightarrow \frac{x}{x-2} = 10^1$$
$$\Leftrightarrow x = 10x - 20$$
$$\Leftrightarrow -9x = -20$$
$$\Leftrightarrow x = \frac{20}{9}$$

We still need to check our answer: substituting $x = \frac{20}{9}$ into the original equation we have:

$$\text{L.H.S} = \log_{10}\frac{20}{9} - \log_{10}\left(\frac{20}{9} - 2\right) = \log_{10}\frac{20}{9} - \log_{10}\left(\frac{2}{9}\right) = \log_{10}\left(\frac{20}{9} \times \frac{9}{2}\right)$$
$$= \log_{10}10$$
$$= 1$$
$$= \text{R.H.S}$$

Therefore. solution is $x = \frac{20}{9}$.

EXAMPLE 7.22

Find $\{x | \ 5^x = 2^{x+1}\}$. Give both an exact answer and one to 2 d.p.

Solution

Taking the logarithm of base 10 of both sides $5^x = 2^{x+1}$ gives:

$$5^x = 2^{x+1} \Leftrightarrow \log_{10}5^x = \log_{10}2^{x+1}$$
$$\Leftrightarrow x\log_{10}5 = (x+1)\log_{10}2$$
$$\Leftrightarrow x\log_{10}5 - x\log_{10}2 = \log_{10}2$$
$$\Leftrightarrow x(\log_{10}5 - \log_{10}2) = \log_{10}2$$
$$\Leftrightarrow x = \frac{\log_{10}2}{\log_{10}5 - \log_{10}2}$$

And so, $x = 0.75647\ldots = 0.76$ (to 2 d.p).

Exact answer = $\left\{\frac{\log_{10}2}{\log_{10}5 - \log_{10}2}\right\}$, answer to 2 d.p = $\{0.76\}$

EXAMPLE 7.23

Find x, where $6e^{2x} - 17 \times e^x + 12 = 0$

Solution

We first note that $6e^{2x} - 17 \times e^x + 12$ can be written as $6 \times e^{2x} - 17 \times e^x + 12$.

Which in turn can be expressed as $6 \times (e^x)^2 - 17 \times e^x + 12$.

Therefore, making the substitution $y = e^x$, we have that

$6 \times (e^x)^2 - 17 \times e^x + 12 = 6y^2 - 17y + 12$ (i.e., we have a 'hidden' quadratic)

Solving for y, we have:

$$6y^2 - 17y + 12 = 0 \Leftrightarrow (2y-3)(3y-4) = 0$$

So that $$y = \frac{3}{2} \text{ or } y = \frac{4}{3}$$

However, we wish to solve for x, and so, we need to substitute back:

$$e^x = \frac{3}{2} \text{ or } e^x = \frac{4}{3}$$

$$\Leftrightarrow x = \ln\frac{3}{2} \text{ or } x = \ln\frac{4}{3}$$

EXAMPLE 7.24

Solve for x, where $8^{2x+1} = 4^{5-x}$

Solution

Taking logs of both sides of the equation $8^{2x+1} = 4^{5-x}$, we have

$$\log 8^{2x+1} = \log 4^{5-x} \Leftrightarrow (2x+1)\log 8 = (5-x)\log 4$$

$$\Leftrightarrow (2x+1)\log 2^3 = (5-x)\log 2^2$$

$$\Leftrightarrow 3(2x+1)\cancel{\log 2} = 2(5-x)\cancel{\log 2}$$

Therefore, we have that $6x + 3 = 10 - 2x \Leftrightarrow 8x = 7$

$$\therefore x = \frac{7}{8}$$

EXERCISES 7.4

1. Without using a calculator, evaluate the following.

(a) $\log_2 8 + \log_2 4$ (b) $\log_6 18 + \log_6 2$ (c) $\log_5 2 + \log_5 12.5$

(d) $\log_3 18 - \log_3 6$ (e) $\log_2 20 - \log_2 5$ (f) $\log_2 10 - \log_2 5$

2. Write down an expresion for $\log a$ in terms of $\log b$ and $\log c$ for the following.

(a) $a = bc$ (b) $a = b^2 c$ (c) $a = \frac{1}{c^2}$

(d) $a = b\sqrt{c}$ (e) $a = b^3 c^4$ (f) $a = \frac{b^2}{\sqrt{c}}$

3. Given that $\log_a x = 0.09$, find

(a) $\log_a x^2$ (b) $\log_a \sqrt{x}$ (c) $\log_a\left(\frac{1}{x}\right)$

4. Express each of the following as an equation that does not involve a logarithm.

(a) $\log_2 x = \log_2 y + \log_2 z$ (b) $\log_{10} y = 2\log_{10} x$

(c) $\log_2(x+1) = \log_2 y + \log_2 x$ (d) $\log_2 x = y + 1$

(e) $\log_2 y = \frac{1}{2}\log_2 x$ (f) $3\log_2(x+1) = 2\log_2 y$

5. Solve the following equations

(a) $\log_2(x+1) - \log_2 x = \log_2 3$

(b) $\log_{10}(x+1) - \log_{10} x = \log_{10} 3$

(c) $\log_2(x+1) - \log_2(x-1) = 4$

(d) $\log_{10}(x+3) - \log_{10} x = \log_{10} x + \log_{10} 2$

(e) $\log_{10}(x^2+1) - 2\log_{10} x = 1$

(f) $\log_2(3x^2+28) - \log_2(3x-2) = 1$

(g) $\log_{10}(x^2+1) = 1 + \log_{10}(x-2)$

(h) $\log_2(x+3) = 1 - \log_2(x-2)$

(i) $\log_6(x+5) + \log_6 x = 2$

(j) $\log_3(x-2) + \log_3(x-4) = 2$

(k) $\log_2 x - \log_2(x-1) = 3\log_2 4$

(l) $\log_{10}(x+2) - \log_{10} x = 2\log_{10} 4$

6. Simplify the following

(a) $\log_3(2x) + \log_3 w$ (b) $\log_4 x - \log_4(7y)$

(c) $2\log_a x + 3\log_a(x+1)$ (d) $5\log_a x - \frac{1}{2}\log_a(2x-3) + 3\log_a(x+1)$

(e) $\log_{10} x^3 + \frac{1}{3}\log x^3 y^6 - 5\log_{10} x$ (f) $2\log_2 x - 4\log_2\left(\frac{1}{y}\right) - 3\log_2 xy$

7. Solve the following

(a) $\log_2(x+7) + \log_2 x = 3$ (b) $\log_3(x+3) + \log_3(x+5) = 1$

(c) $\log_{10}(x+7) + \log_{10}(x-2) = 1$ (d) $\log_3 x + \log_3(x-8) = 2$

(e) $\log_2 x + \log_2 x^3 = 4$ (f) $\log_3\sqrt{x} + 3\log_3 x = 7$

8. Solve for x.

(a) $\log_2 x^2 = (\log_2 x)^2$ (b) $\log_3 x^3 = (\log_3 x)^3$

(c) $\log_4 x^4 = (\log_4 x)^4$ (d) $\log_5 x^5 = (\log_5 x)^5$

Investigate the solution to $\log_n x^n = (\log_n x)^n$

9. Solve the following, giving an exact answer and an answer to 2 d.p.

(a) $2^x = 14$ (b) $10^x = 8$ (c) $3^x = 125$

(d) $\frac{1}{1-2^x} = 12$ (e) $3^{4x+1} = 10$ (f) $0.8^{x-1} = 0.4$

(g) $10^{-2x} = 2$ (h) $2.7^{0.3x} = 9$ (i) $0.2^{-2x} = 20$

(j) $\frac{2}{1+0.4^x} = 5$ (k) $\frac{2^x}{1-2^x} = 3$ (l) $\frac{3^x}{3^x+3} = \frac{1}{3}$

10. Solve for x

(a) $(\log_2 x)^2 - \log_2 x - 2 = 0$ (b) $\log_2(2^{x+1} - 8) = x$

(c) $\log_{10}(x^2 - 3x + 6) = 1$ (d) $(\log_{10} x)^2 - 11\log_{10} x + 10 = 0$

(e) $\log_x(3x^2 + 10x) = 3$ (f) $\log_{x+2}(3x^2 + 4x - 14) = 2$

11. Solve the following simultaneous equations

(a) $\begin{aligned} x^y &= 5x - 9 \\ \log_x 11 &= y \end{aligned}$ (b) $\begin{aligned} \log_{10} x - \log_{10} y &= 1 \\ x + y^2 &= 200 \end{aligned}$ (c) $\begin{aligned} xy &= 2 \\ 2\log_2 x - \log_2 y &= 2 \end{aligned}$

12. Express each of the following as an equation that does not involve a logarithm.

(a) $\log_e x = \log_e y - \log_e z$ (b) $3\log_e x = \log_e y$ (c) $\ln x = y - 1$

13. Solve the following for x

(a) $\ln(x+1) - \ln x = 4$ (b) $\ln(x+1) - \ln x = \ln 4$

(c) $\log_e(x+1) + \log_e x = 0$ (d) $\log_e(x+1) - \log_e x = 0$

14. Solve the following for x

(a) $e^x = 21$ (b) $e^x - 2 = 8$ (c) $-5 + e^{-x} = 2$

(d) $200e^{-2x} = 50$ (e) $\frac{2}{1-e^{-x}} = 3$ (f) $70e^{-\frac{1}{2}x} + 15 = 60$

(g) $\ln x = 3$ (h) $2\ln(3x) = 4$ (i) $\ln(x^2) = 9$

(j) $\ln x - \ln(x+2) = 3$ (k) $\ln\sqrt{x+4} = 1$ (l) $\ln(x^3) = 9$

15. Solve the following for x

(a) $e^{2x} - 3e^x + 2 = 0$ (b) $e^{2x} - 4e^x - 5 = 0$

(c) $e^{2x} - 5e^x + 6 = 0$ (d) $e^{2x} - 2e^x + 1 = 0$

(e) $e^{2x} - 6e^x + 5 = 0$ (f) $e^{2x} - 9e^x - 10 = 0$

16. Solve each of the following

(a) $4^{x-1} = 132$ (b) $5^{5x-1} = 3^{1-2x}$

(c) $3^{2x+1} - 7 \times 3^x + 4 = 0$ (d) $2^{2x+3} - 7 \times 2^{x+1} + 5 = 0$

(e) $3 \times 4^{2x+1} - 2 \times 4^{x+2} + 5 = 0$ (f) $3^{2x} - 3^{x+2} + 8 = 0$

(g) $2\log x + \log 4 = \log(9x-2)$ (h) $2\log 2x - \log 4 = \log(2x-1)$

(i) $\log_3 2x + \log_3 81 = 9$ (j) $\log_2 x + \log_x 2 = 2$

7.5 LOGARITHMIC MODELLING

Some examples of where the logarithmic functions are used can be found in:

i. the measurement of the magnitude of an earthquake (better known as the Richter scale), where the magnitude R of an earthquake of intensity I is given by $R = \log_{10}\left(\frac{I}{I_0}\right)$, where I_0 is a certain minimum intensity.

ii. the measurement of children's weight (better known as The Ehrenberg relation) is given by $\log_{10}W = \log_{10}2.4 + 0.8h$, where W kg is the average weight for children aged 5 through to 13 years and h is the height measured in metres.

iii. the brightness of stars, given by the function $m = 6 - 2.5\log_{10}\left(\frac{L}{L_0}\right)$, where L_0 is the light flux of the faintest star visible to the naked eye (having magnitude 6), and m is the magnitude of brighter stars having a light flux L.

EXAMPLE 7.25

After working through an area of study, students in year 7 sat for a test based on this unit. Over the following two years, the same students were retested on several occasions. The average score was found to be modelled by the function

$S = 90 - 20\log_{10}(t+1), 0 \le t \le 24$ where t is measured in months.

(a) What was the average score on the first test?
(b) What was the score after i. 6 months? ii. 2 years?
(c) How long should it be before the test is re–issued, if the average score is to be 80?

Solution

(a) The first test occurs at time $t = 0$, so that

$$S = 90 - 20\log_{10}(0+1) = 90 - 20\log 1 = 90$$

That is, the average score on the first test was 90%.

(b) i. After six months we have that $t = 6$. Therefore, $S = 90 - 20\log_{10}(6+1) \approx 73$.
That is, the average score on the test after six months was 73%.

ii. After 2 years we have that $t = 24$. Therefore, $S = 90 - 20\log_{10}(24+1) \approx 62$.
That is, the average score on the test after two years was 62%.

(c) We need to find t when $S = 80$. Using the given equation we have:

$$80 = 90 - 20\log_{10}(t+1) \Leftrightarrow 20\log_{10}(t+1) = 10$$

$$\Leftrightarrow \log_{10}(t+1) = \frac{1}{2}$$

$$\Leftrightarrow t+1 = \sqrt{10}$$

$$\Leftrightarrow t = \sqrt{10} - 1$$

That is, $t \approx 2.16$

Therefore, the students should be retested in approximately 2 months time.

EXERCISES 7.5

1. The loudness of a sound, as experienced by the human ear, is based on its intensity level. This intensity level is modelled by the logarithmic function $d = 10\log_{10}\left(\frac{I}{I_0}\right)$ where d is measured in decibels and corresponds to a sound intensity I and I_0 (known as the threshold intensity) is the value of I that corresponds to be the weakest sound that can be detected by the ear under certain conditions.

(a) Find the value of d when I is 10 times as great as I_0 (i.e., $I = 10I_0$).

(b) Find the value of d when I is 1000 times as great as I_0.

(c) Find the value of d when I is 10000 times as great as I_0.

2. A model, for the relationship between the average weight W kilograms and the height h metres for children aged 5 through to 13 years has been closely approximated by the function $\log_{10}W = \log_{10}2.4 + 0.80h$

(a) Based on this model, determine the average weight of a 10-year-old child who is 1.4 metres tall.

(b) How tall would an 8 year old child weighing 50 kg be?

(c) Find an expression for the weight, W, as a function of h.

(d) Sketch the graph of W kg versus h m.

(e) Hence, or otherwise, sketch the graph of h m versus W kg .

3. A measure of the ‘energy’ of a star can be related to its brightness. To determine this ‘energy’ stars are classified into categories of brightness called magnitudes. Those considered to be the least ‘energetic’ are labelled as the faintest stars. Such stars have a light flux given by L_0, and are assigned a magnitude 6. Other brighter stars having a light flux L are assigned a magnitude m by means of the formula $m = 6 - 2.5\log_{10}\left(\frac{L}{L_0}\right)$.

(a) Find the magnitude m of a star, if relative to the faintest star, its light flux L is such that $L = 10^{0.5}L_0$.

(b) Find an equation for L in terms of m and L_0.

(c) Sketch the general shape of the function for L (as a function of m).

(d) Hence, or otherwise, sketch the graph of the function $m = 6 - 2.5\log_{10}\left(\frac{L}{L_0}\right)$.

4. For some manufacturers, it is important to consider the failure time of their computer chips. For Multi–Chips Pty Ltd, the time taken before a fraction x of their computer chips fail has been approximated by the logarithmic function $t = -\frac{1}{c}\log_{10}(1-x)$, where c is some positive constant and time t is measured in years.

(a) Define the domain for this function.

(b) Determine how long will it be before 40% of the chips fail, when

i. $c = 0.1$ ii. $c = 0.2$ iii. $c = 0.3$

(c) How does the value of c affect the reliability of a chip?

(d) Find an expression for the fraction x of chips that will fail after t years.

(e) For the case where c = 0.10, sketch the graph of x versus t. Hence, sketch the graph of $t = -\frac{1}{c}\log_{10}(1-x)$ where $c = 0.10$.

5. Logarithms have been found useful in modelling economic situations in some countries. Pareto's law for capitalist countries states that the relationship between annual income, \$ I and the number, n, of individuals whose income exceeds \$ I is approximately modelled by the function $\log_{10}I = \log_{10}a - k\log_{10}n$ where a and k are real positive constants.

(a) Find and expression for I that does not involve logarithms.

(b) By varying the values of a and b, describe their effects on

i. the income \$ I.

ii. the number of people whose income exceeds \$ I.

6. After prolonged observations of our environment, it became obvious that the thickness of the ozone layer had being affected by the production of waste that had taken place over many years. The thickness of the ozone layer has been estimated by making use of the function $\log_{10}\lambda_0 - \log_{10}\lambda = kx$, where λ_0 is the intensity of a particular wavelength of light from the sun before it reaches the atmosphere, λ is the intensity of the same wavelength after passing through a layer of ozone x centimetres thick, and k is the absorption constant of ozone for that wavelength.
The following table has some results based on available data for one region of the Earth's atmosphere:

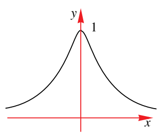
λ_0	k	$\frac{\lambda_0}{\lambda}$
3200×10^{-8}	$k \approx 0.40$	1.10

(a) Based on the above table, find the approximate thickness of the ozone layer in this region of the atmosphere, giving your answer to the nearest hundredth of a centimetre.

(b) Obtain an expression for the intensity λ, in terms of k, λ_0 and x.

(c) What would the percentage decrease in the intensity of light with a wavelength of 3200×10^{-8} cm be, if the ozone layer is 0.20 centimetre thick?

(d) For a fixed value of λ_0, how does k relate to the intensity λ?

(c) i


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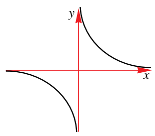

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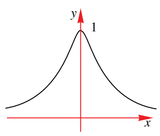

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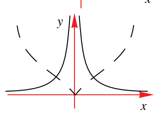

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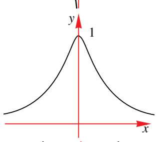

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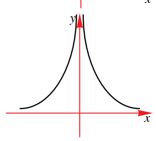
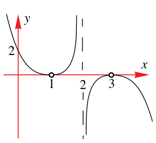


4. i


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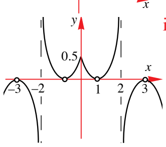

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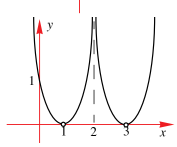

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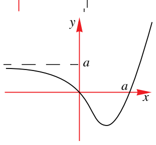

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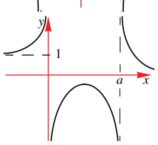

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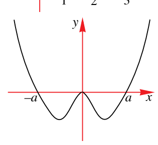

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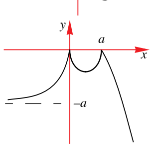

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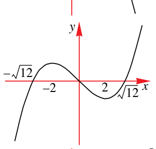
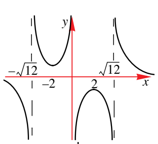


(b) i


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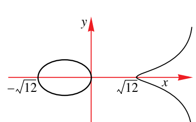
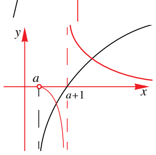
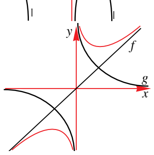
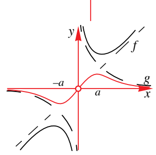


7.


8. (a)


(b)


EXERCISE 7.1.1

1. (a) $\frac{27y^{15}}{8x^3}$ (b) $\frac{91}{216a^6}$ (c) 2^n+2 (d) $\frac{8x^{11}}{27y^2}$ (e) $\frac{3x^2y^2}{8}$ (f) $3^{n+1}+3$ (g) $4^{n+1}-4$ (h) $2(4^{n+1}-4)$ (i) $\frac{1-b^6}{16b^4}$ **2.** (a) 64 (b) $\left(\frac{2}{3}\right)^x$ (c) 2^{2y+1} (d) $\frac{1}{b^{2x}}$ (e) $\left(\frac{y}{2}\right)^6$ (f) $\left(\frac{9}{2}\right)^{n+2}$ **3.** (a) $\frac{z^2}{xy}$ (b) 3^{7n-2} (c) 5^{n+1} (d) 9 (e) 2^{6n+1} (f) 2^{1-3n} (g) x^{2+4n-n^2} (h) x^{3n^2+n+1} (i) 27 **4.** $\frac{y^{2m-2}}{x^m}$

5. (a) –81 (b) $-\frac{9x^8}{8y^4}$ (c) $y-x$ (d) $\frac{2x+1}{x+1}$ (e) –1 (f) $-b$ **6.** (a) $\frac{1}{x^2y^2}$ (b) $\frac{1}{x^4}$ (c) $-\frac{1}{x(x+h)}$

(d) $\frac{1}{x-1}$ (e) $\frac{1}{(x+1)(x-1)^5}$ (f) $\frac{1}{x^2}$ **7.** (a) $118\times5^{n-2}$ (b) 1 (c) $\frac{b^7}{a^4}$ (d) a^{mn} (e) $\frac{p+q}{pq}$ (f) $\frac{2\sqrt{a}}{a-1}$ (g) $\frac{7}{8}$ (h) $a^{7/8}$ **8.** (a) $x^{11/12}$ (b) $2a^{3n-2}b^{2n-2}$ (c) 2^n (d) $-\frac{7^{m-n}}{8}$ (e) $\frac{6\times5^n}{5^n+5}$ (f) $x+1$

EXERCISE 7.1.2

1. (a) 2 (b) –2 (c) $\frac{2}{3}$ (d) 5 (e) 6 (f) –2.5 (g) 2 (h) 1.25 (i) $\frac{1}{3}$ **2.** (a) –6 (b) $-\frac{2}{3}$ (c) –3 (d) 1.5 (e) 0.25 (f) 0.25 (g) $-\frac{1}{8}$ (h) $-\frac{11}{4}$ (i) –1.25

EXERCISE 7.1.3

1. (a) 3.5 (b) 3.5 (c) –3 (d) 1.5 (e) 3.5 (f) 1.5 (g) 1.8 (h) $-\frac{4}{7}$ (i) 0 **2.** (a) –0.75 (b) –1,4 (c) 0,1 (d) 3,4 (e) –1,4 (f) 0,2 **3.** (a) –1,1,2 (b) –3,1,3,4 (c) $\frac{4}{3},\frac{5}{3}$, 2 (d) –1,1,2 (e) 3,7,$\frac{-1\pm\sqrt{233}}{2}$, $\frac{1}{3}$

EXERCISE 7.1.4

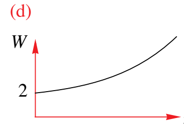
1. (a) i. 5.32 ii. 9.99 iii. 2.58 (b) i. 2.26 ii. 3.99 iii. 5.66 (c) i. 3.32 ii. –4.32 iii. –6.32 (d) i. –1.43 ii. 1.68 iii. –2.86 **2.** (a) 0 (b) 0.54 (c) –0.21 (d) –0.75, 0 (e) 1.13 (f) 0, 0.16

EXERCISE 7.1.5

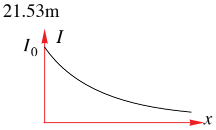
1. (a) 2 (b) –1 (c) 0.5 (d) 0.5 **2.** (a) 1 (b) 0.6 (c) 0 **3.** (a) 0 (b) $\frac{2}{3}$ **4.** (a) –1,2 (b) –2,3 (c) –1 (d) –6,1 (e) 0,1 (f) 1 **5.** (a) 1.3863 (b) 2.1972 (c) 3.2189 (d) Ø **6.** (a) 0.4236 (b) 0.4055 (c) 0.3054 (d) –0.4176 **7.** (a) 0 (b) –0.6733 (c) 0 **9.** 36 **10.** $a = \sqrt{2}e, k = \ln(\sqrt{2})$

EXERCISE 7.2

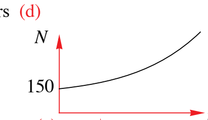
1. (a) 1000 (b) 1516 (c) 2000 (d) 10 days **2.** (a) 0.0013 (b) 2.061 kg (c) 231.56 yrs (d)



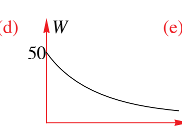
3. (a) 0.01398 (b) 52.53% (c) 51.53m (d) 21.53m



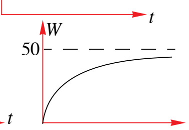
4. (a) i. 157 ii. 165 iii. 191 (b) 14.2 yrs (c) 20.1 yrs (d)



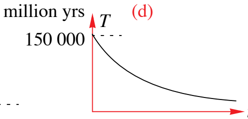
5. (a) 50 (b) 0.0222 (c) 17.99 kg (d)



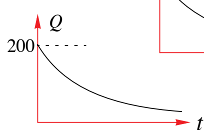
(e)



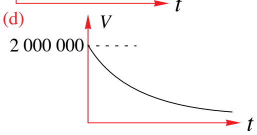
6. (a) 15 000°C (b) i. 11 900°C ii. 1500°C (c) 3.01 million yrs (d)



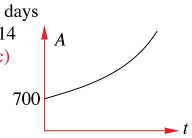
7. (a) 0.0151 (b) 12.50gm (c) 20 years (d)



8. (a) \$2 million (b) \$1.589 mil (c) 30.1 years (d)

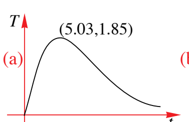


9. (b) 0.01761 (c) 199 230 (d) 22.6 years
10. (a) 20 cm^2 (b) 19.72 cm^2 (c) 100 days (d) 332 days
11. (a) 1 (b) i. 512170 ii. 517217 (c) 54.1 early 2014
12. (a) i. \$933.55 ii. \$935.50 (b) 11.95 years (c)



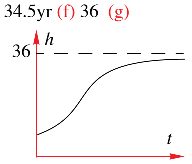
13. (a) 99 (b) $99 \times 2^{0.1394t}$ (c) 684

14. (a)

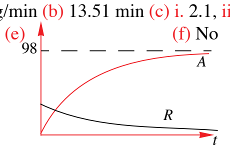


(b) 38.85°C at ~ midnight

15. (a) 19 (b) 2.63 (c) 100
16. (a) 18 cm (b) 4 cm (c) 1.28 m (d) 36 m (e) i. 21.7yr ii. 27.6yr iii. 34.5yr (f) 36 (g)

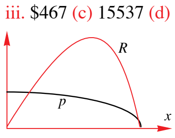


17. (a) 5 mg/min (b) 13.51 min (c) i. 2.1, ii. 13.9 iii. 68 min
(d) 19.6 mg (e)



(f) No

18. (a) i. \$499 ii. \$496 iii. \$467 (c) 15537 (d) i. \$499k ii. \$2.48mil iii. \$4.67mil (f) 12358
(g) \$5.14mil (b), (e)



EXERCISE 7.3

1. (a) 2 (b) 2 (c) 5 (d) 3 (e) –3 (f) –2 (g) 0 (h) 0 (i) –1 (j) –2 (k) 0.5 (l) –2 **2.** (a) $\log_{10}10000 = 4$ (b) $\log_{10}0.001 = -3$ (c) $\log_{10}(x+1) = y$ (d) $\log_{10}p = 7$ (e) $\log_2(x-1) = y$ (f) $\log_2(y-2) = 4x$ **3.** (a) $2^9 = x$ (b) $b^x = y$ (c) $b^{ax} = t$ (d) $10^{x^2} = z$ (e) $10^{1-x} = y$ (f) $2^y = ax - b$ **4.** (a) 16 (b) 2 (c) 2 (d) 9 (e) $\sqrt[4]{2}$ (f) 125 (g) 4 (h) 9 (i) $\sqrt[3]{\frac{1}{3}}$ (j) 21 (k) 3 (l) 13 **5.** (a) 54.5982 (b) 1.3863 (c) 1.6487 (d) 7.3891 (e) 1.6487 (f) 0.3679 (g) 52.5982 (h) 4.7183 (i) 0.6065

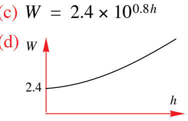
EXERCISE 7.4

1. (a) 5 (b) 2 (c) 2 (d) 1 (e) 2 (f) 1 **2.** (a) $\log a = \log b + \log c$ (b) $\log a = 2\log b + \log c$ (c) $\log a = -2\log c$ (d) $\log a = \log b + 0.5\log c$ (e) $\log a = 3\log b + 4\log c$ (f) $\log a = 2\log b - 0.5\log c$ **3.** (a) 0.18 (b) 0.045 (c) –0.09 **4.** (a) $x = yz$ (b) $y = x^2$ (c) $y = \frac{x+1}{x}$ (d) $x = 2^{y+1}$ (e) $y = \sqrt{x}$ (f) $y^2 = (x+1)^3$ **5.** (a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{17}{15}$ (d) $\frac{3}{2}$ (e) $\frac{1}{3}$ (f) no real sol'n (g) 3,7 (h) $\frac{\sqrt{33}-1}{2}$ (i) 4 (j) $\sqrt{10}+3$ (k) $\frac{64}{63}$ (l) $\frac{2}{15}$ **6.** (a) $\log_3 2wx$ (b) $\log_4 \frac{x}{7y}$ (c) $\log_a[x^2(x+1)^3]$ (d) $\log_a\left[\frac{(x^5)(x+1)^3}{\sqrt{2x-3}}\right]$ (e) $\log_{10}\left(\frac{y^2}{x}\right)$ (f) $\log_2\left(\frac{y}{x}\right)$ **7.** (a) 1 (b) –2 (c) 3 (d) 9 (e) 2 (f) 9 **8.** (a) 1,4 (b) $1, 3^{\pm\sqrt{3}}$ (c) $1, 4^{\sqrt[3]{4}}$ (d) $1, 5^{\pm\sqrt[4]{5}}$ **9.** (a) $\frac{\log 14}{\log 2} = 3.81$ (b) $\frac{\log 8}{\log 10} = 0.90$ (c) $\frac{\log 125}{\log 3} = 4.39$ (d) $\frac{1}{\log 2} \times \log\left(\frac{11}{3}\right) - 2 = -0.13$ (e) $\frac{\log 10 - \log 3}{4\log 3} = 0.27$ (f) 5.11 (g) $\frac{-\log 2}{2\log 10} = -0.15$ (h) 7.37 (i) 0.93 (j) no real solution (k) $\frac{\log 3}{\log 2} - 2 = -0.42$ (l) $\frac{\log 1.5}{\log 3} = 0.37$ **10.** (a) 0.5,4 (b) 3 (c) –1,4 (d) $10, 10^{10}$ (e) 5 (f) 3 **11.** (a) $(4, \log_4 11)$ (b) (100,10) (c) (2,1) **12.** (a) $y = xz$ (b) $y = x^3$ (c) $x = e^{y-1}$ **13.** (a) $\frac{1}{e^4 - 1}$ (b) $\frac{1}{3}$ (c) $\frac{\sqrt{5}-1}{2}$ (d) Ø **14.** (a) $\ln 21 = 3.0445$ (b) $\ln 10 = 2.3026$ (c) $-\ln 7 = -1.9459$ (d) $\ln 2 = 0.6931$ (e) $\ln 3 = 1.0986$ (f) $2\ln\left(\frac{14}{9}\right) = 0.8837$ (g) $e^3 = 20.0855$ (h) $\frac{1}{3}e^2 = 2.4630$ (i) $\pm\sqrt{e^9} = \pm 90.0171$ (j) Ø (k) $e^2 - 4 = 3.3891$ (l) $\sqrt[3]{e^9} = 20.0855$ **15.** (a) 0, ln2 (b) ln5 (c) ln2, ln3 (d) 0 (e) 0, ln5 (f) ln10 **16.** (a) 4.5222 (b) 0.2643 (c) 0,0.2619 (d) –1,0.3219 (e) –1.2925,0.6610 (f) 0,1.8928 (g) 0.25,2 (h) 1 (i) 121.5 (j) 2

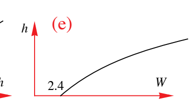
EXERCISE 7.5

1. (a) 10 (b) 30 (c) 40 **2.** (a) 31.64 kg (b) 1.65 (c) $W = 2.4 \times 10^{0.8h}$

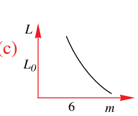
(d)



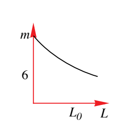
(e)



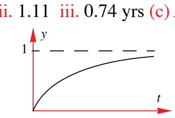
3. (a) 4.75 (b) $L = L_0 \times 10^{\left(\frac{6-m}{2.5}\right)}$ (c)

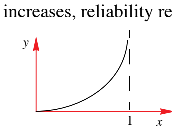


(d)



4. (a) [0,1[(b) i. 2.22 ii. 1.11 iii. 0.74 yrs (c) As c increases, reliability reduces.
(d) $x = 1 - 10^{-ct}$ (e)





5. (a) $I = \frac{a}{n^k}$ **6.** (a) 0.10 (b) $\lambda = \lambda_0 \times 10^{-kx}$ (c) 16.82% (d) $k = -\frac{1}{x}\log\left(\frac{\lambda}{\lambda_0}\right)$

EXERCISE 8.1.1

1. i. (b) 4 (c) $t_n = 4n - 2$ ii. (b) –3 (c) $t_n = -3n + 23$ iii. (b) –5 (c) $t_n = -5n + 6$ iv. (b) 0.5 (c) $t_n = 0.5n$ v. (b) 2 (c) $t_n = y + 2n - 1$ vi. (b) –2 (c) $t_n = x - 2n + 4$ **2.** –28 **3.** 9,17 **4.** –43 **5.** 7 **6.** 7 **7.** –5 **8.** 0 **9.** (a) 41 (b) 31st **10.** 2, $\sqrt{3}$ **11.** (a) i. 2 ii. –3 (b) i. 4 ii. 11 **12.** $x - 8y$ **13.** $t_n = 5 + \frac{10}{3}(n-1)$ **14.** (a) –1 (b) 0

EXERCISE 8.1.2

1. (a) 145 (b) 300 (c) –170 **2.** (a) –18 (b) 690 (c) 70.4 **3.** (a) –105 (b) 507 (c) 224 **4.** (a) 126 (b) 3900 (c) 14th week **5.** 855 **6.** (a) 420 (b) –210 **7.** $a = 9, b = 7$

EXERCISE 8.1.3

1. 123 **2.** –3, –0.5, 2, 4.5, 7, 9.5, 12 **3.** 3.25 **4.** $a = 3$ $d = -0.05$ **5.** 10 000 **6.** 330 **7.** –20 **8.** 328 **9.** \$725, 37wks **10.** i. \$55 ii. 2750 **11.** (a) (i) 8m (ii) 40m (b) 84m (c) Dist = $2n^2 - 2n = 2n(n-1)$ (d) 8 (e) 26 players, 1300m **12.** (a) 5050 (b) 10200 (c) 4233 **13.** (a) 145 (b) 390 (c) –1845 **14.** (b) $3n - 2$

EXERCISE 8.2.1

1. (a) $r = 2, u_5 = 48, u_n = 3 \times 2^{n-1}$ (b) $r = \frac{1}{3}, u_5 = \frac{1}{27}, u_n = 3 \times \left(\frac{1}{3}\right)^{n-1}$ (c) $r = \frac{1}{5}, u_5 = \frac{2}{625}, u_n = 2 \times \left(\frac{1}{5}\right)^{n-1}$ (d) $r = -4, u_5 = -256, u_n = -1 \times (-4)^{n-1}$ (e) $r = \frac{1}{b}, u_5 = \frac{a}{b^3}, u_n = ab \times \left(\frac{1}{b}\right)^{n-1}$ (f) $r = \frac{b}{a}, u_5 = \frac{b^4}{a^2}, u_n = a^2 \times \left(\frac{b}{a}\right)^{n-1}$ **2.** (a) ±12 (b) $\frac{\pm\sqrt{5}}{2}$ **3.** (a) ±96 (b) 15th **4.** (a) $u_n = 10 \times \left(\frac{5}{6}\right)^{n-1}$ (b) $\frac{15625}{3888} \cong 4.02$ (c) $n = 5$ (4 times) **5.** –2, $\frac{4}{3}$ **6.** (a) i. \$4096 ii. \$2097.15 (b) 6.2 yrs **7.** $\left(u_n = \frac{1000}{169} \times \left(\frac{12}{5}\right)^{n-1}\right), \frac{1990656}{4225} \cong 471.16$ **8.** 2.5,5,10 or 10,5,2.5 **9.** 53757 **10.** 108 952 **11.** (a) \$56 156 (b) \$299 284

EXERCISE 8.2.2

1. (a) 3 (b) $\frac{1}{3}$ (c) –1 (d) $-\frac{1}{3}$ (e) 1.25 (f) $-\frac{2}{3}$ **2.** (a) 216513 (b) 1.6384×10^{-10} (c) $\frac{256}{729}$ (d) $\frac{729}{2401}$ (e) $-\frac{81}{1024}$ **3.** (a) 11; 354292 (b) 7; 473 (c) 8; 90.90909 (d) 8; 172.778 (e) 5; 2.256 (f) 13; 111.1111111111 **4.** (a) $\frac{127}{128}$ (b) $\frac{63}{8}$ (c) $\frac{130}{81}$ (d) 60 (e) $\frac{63}{64}$ **5.** 4; 118096 **6.** \$2109.50 **7.** 9.28cm **8.** (a) $V_n = V_0 \times 0.7^n$ (b) 7 **9.** 54 **10.** 53.5gms; 50 weeks. **11.** 7 **12.** 9 **13.** –0.5, –0.7797 **14.** $r = 5$, 1.8×10^{10} **15.** \$8407.35 **16.** 1.8×10^{19} or about 200 billion tonnes.

EXERCISE 8.2.3

1. Term 9 AP = 180, GP = 256. Sum to 11 terms AP = 1650, GP = 2047. **2.** 18. **3.** 12 **4.** 7, 12 **5.** 8 weeks (Ken \$220 & Bo-Youn \$255) **6.** (a) week 8 (b) week 12 **7.** (a) 1.618 (b) 121379 [~121400, depends on rounding errors]

EXERCISE 8.2.4

1. (i) $\frac{81}{2}$ (ii) $\frac{10}{13}$ (iii) 5000 (iv) $\frac{30}{11}$ **2.** $23\frac{23}{99}$ **3.** 6667 fish. [Nb: $t_{43} < 1$. If we use $n = 43$ then ans is 6660 fish]; 20 000 fish. Overfishing means that fewer fish are caught in the long run. [*An alternate estimate for the total catch is* 1665 *fish*.] **4.** 27 **5.** 48,12,3 or 16,12,9 **6.** (a) $\frac{11}{30}$ (b) $\frac{37}{99}$ (c) $\frac{191}{90}$ **7.** 128 cm **8.** $\frac{121}{9}$ **9.** $2 + \frac{4}{3}\sqrt{3}$ **10.** $\frac{1-(-t)^n}{1+t}$ $\frac{1}{1+t}$ **11.** $\frac{1-(-t^2)^n}{1+t^2}$ $\frac{1}{1+t^2}$

EXERCISE 8.2.5

1. 3, –0.2 **2.** $\frac{2560}{93}$ **3.** $\frac{10}{3}$ **4.** (a) $\frac{43}{18}$ (b) $\frac{458}{99}$ (c) $\frac{413}{990}$ **5.** 9900 **6.** 3275 **7.** 3 **8.** $t_n = 6n - 14$ **9.** 6 **10.** $-\frac{1}{6}$ **11.** i. 12 ii. 26 **12.** 9, 12 **13.** ±2 **14.** (5, 5, 5), (5, –10, 20) **15.** (a) 2, 7 (b) 2, 5, 8 (c) $3n - 1$ **16.** (a) 5 (b) 2 m

EXERCISE 8.3

1. \$2773.08 **2.** \$4377.63 **3.** \$1781.94 **4.** \$12216 **5.** \$35816.95 **6.** \$40349.37 **7.** \$64006.80 **8.** \$276971.93, \$281325.41 **9.** \$63762.25 **10.** \$98.62, \$9467.14, interest \$4467.14. Flat interest = \$6000 **11.** \$134.41, \$3790.44, 0.602% /month (or 7.22% p.a)

EXERCISE 9.1

1.	*a* cm	*b* cm	*c* cm	*A*	*B*	*C*
1	3.8	4.1	1.6	67°	90°	23°
2	81.5	98.3	55.0	56°	90°	34°
3	32.7	47.1	33.9	44°	90°	46°
4	1.61	30.7	30.7	3°	90°	87°
5	2.3	2.74	1.49	57°	90°	33°
6	48.5	77	59.8	39°	90°	51°
7	44.4	81.6	68.4	33°	90°	57°
8	2.93	13.0	12.7	13°	90°	77°
9	74.4	94.4	58.1	52°	90°	38°
10	71.8	96.5	64.6	48°	90°	42°
11	23.3	34.1	24.9	43°	90°	47°
12	43.1	43.2	2.3	87°	90°	3°
13	71.5	80.2	36.4	63°	90°	27°
14	33.5	34.1	6.5	79°	90°	11°
15	6.1	7.2	3.82	58°	90°	32°
16	29.1	30	7.3	76°	90°	14°
17	29.0	29.1	2.0	86°	90°	4°
18	34.5	88.2	81.2	23°	90°	67°
19	24.0	29.7	17.5	54°	90°	36°
20	41.2	46.2	21.0	63°	90°	27°
21	59.6	72.9	41.8	55°	90°	35°
22	5.43	6.8	4.09	53°	90°	37°