Chapter 15

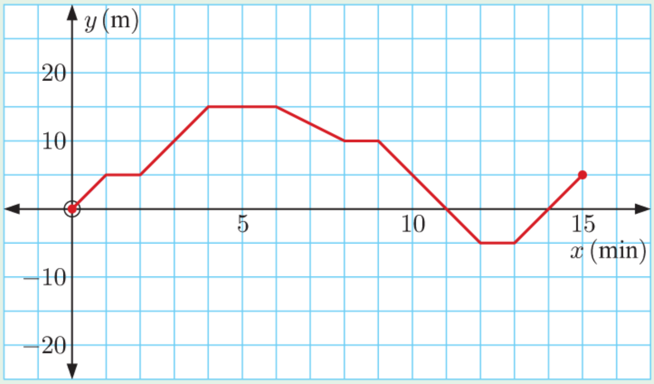
Relations and functions

Contents:

When two variables are connected, we often use a **relation** or a **function** to describe the relationship.

OPENING PROBLEM

Fernando and Gerard are competing in a cycling race. The graph alongside shows how far Fernando is ahead of Gerard after x minutes.



Things to think about:

a For what values of x are there corresponding values of y?

b For what values of y are there corresponding values of x?

c What is the value of y when $x = 3$? How can we write this using function notation?

d What are the values of x for which $y = 10$?

A RELATIONS

Sue-Ellen wants to send a parcel to a friend. The cost of posting the parcel a fixed distance is determined by the weight of the parcel, as shown in the table.

For example, it will cost \$8.00 to post a parcel weighing at least 2 kg but less than 5 kg. It will therefore cost \$8.00 to post a parcel weighing 2 kg or 3.6 kg or 4.955 kg.

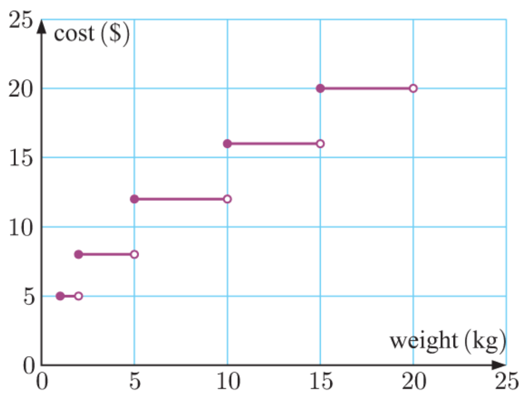
Weight w (kg)	*Cost* \$$C$
$1 \leqslant w < 2$	\$5.00
$2 \leqslant w < 5$	\$8.00
$5 \leqslant w < 10$	\$12.00
$10 \leqslant w < 15$	\$16.00
$15 \leqslant w < 20$	\$20.00

We can illustrate the postal charges on a graph.

An end point that is included has a filled in circle.

An end point that is not included has an open circle.

There is a *relationship* between the variables *weight* and *cost*, so the table of costs is an example of a **relation**.

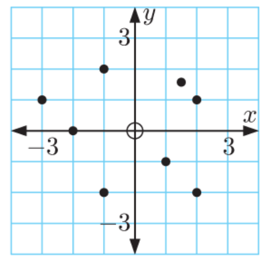


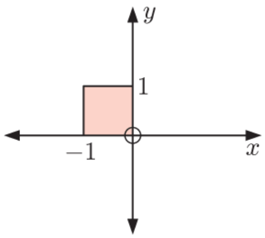
A relation may be a finite number of ordered pairs, for example $\{(2, 8), (3, 8), (4, 8), (5, 12)\}$, or an infinite number of ordered pairs, such as the relation between the variables *weight* and *cost* in the postal charges above.

A **relation** is any set of points which connects two variables.

The following are examples of relations:

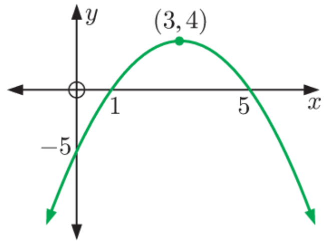
- The set of 8 points represented by the dots is a relation. There is no equation connecting the variables x and y in this case.



- 

  The set of all points on and within the illustrated square is a relation. It is the set of all points (x, y) such that $-1 \leqslant x \leqslant 0$ and $0 \leqslant y \leqslant 1$.

- The set of all points on this parabola is a relation. It is the set of all points (x, y) lying on the curve $y = -x^2 + 6x - 5$.

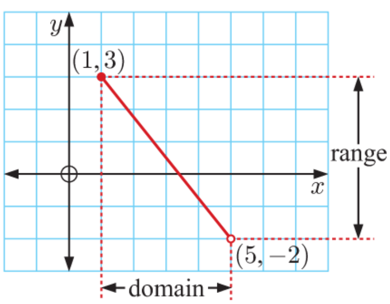


DOMAIN AND RANGE

> The **domain** of a relation is the set of possible values that x may have.
> The **range** of a relation is the set of possible values that y may have.

The domain and range of a relation are often described using **interval notation**.

Consider the following examples:

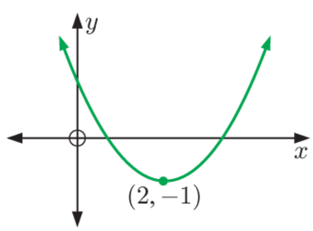
- 

  The domain consists of all real x such that $1 \leqslant x < 5$. We write this as:

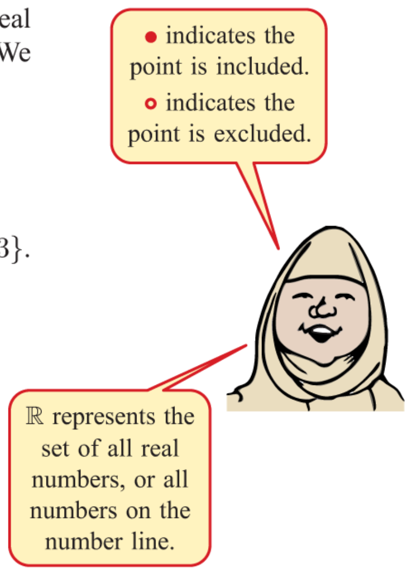
  $$\{x \mid 1 \leqslant x < 5\}.$$

  the set of all   such that

  The range is $\{y \mid -2 < y \leqslant 3\}$.

- 

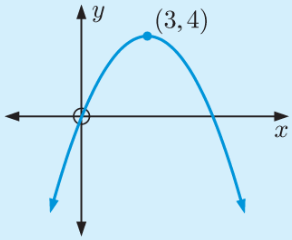
  The domain is $\{x \mid x \in \mathbb{R}\}$.
  The range is $\{y \mid y \geqslant -1\}$.



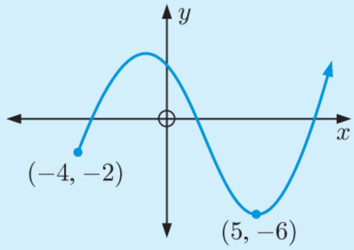
Example 1 — Self Tutor

For each of the following graphs, state the domain and range:

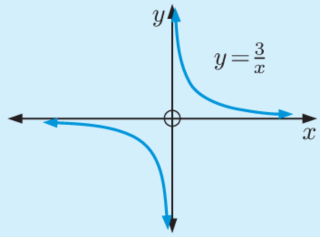
a



b



c



a Domain is $\{x \mid x \in \mathbb{R}\}$.
Range is $\{y \mid y \leqslant 4\}$.

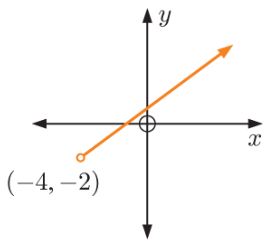
b Domain is $\{x \mid x \geqslant -4\}$.
Range is $\{y \mid y \geqslant -6\}$.

c Domain is $\{x \mid x \neq 0\}$.
Range is $\{y \mid y \neq 0\}$.

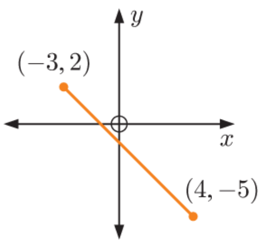
EXERCISE 15A

1 For each of the following graphs, state the domain and range:

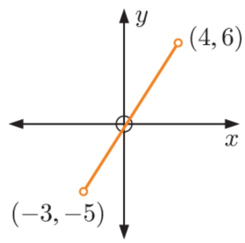
a



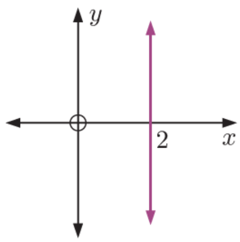
b



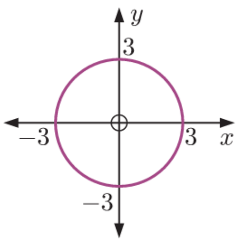
c



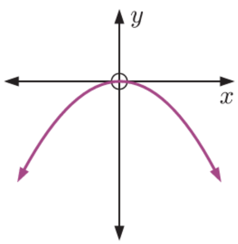
d



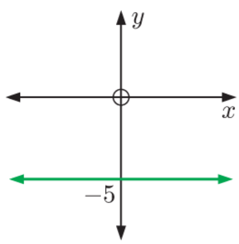
e



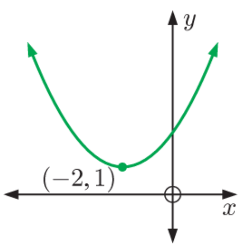
f



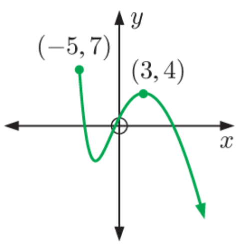
g



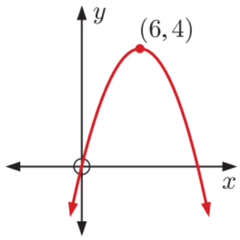
h



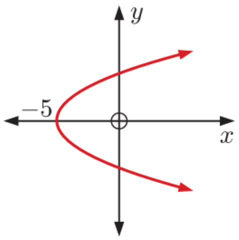
i



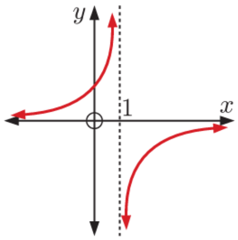
j



k

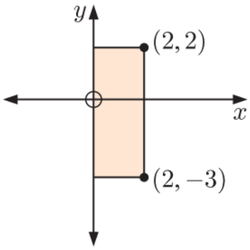


l

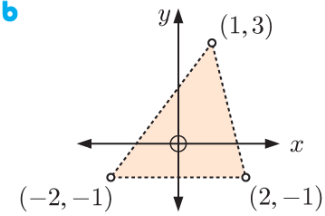


2 State the domain and range of:

a

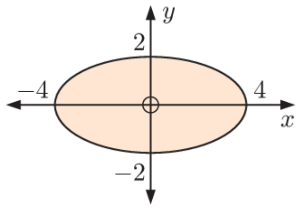


b



(boundaries not included)

c



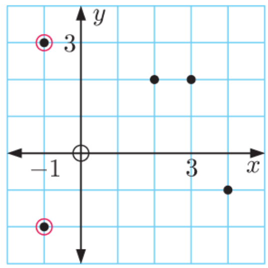
B FUNCTIONS

A **function** is a relation in which no two different ordered pairs have the same first member.

The grid alongside shows the set of points:

$\{(-1, 3), (2, 2), (-1, -2), (3, 2), (4, -1)\}$.

The two circled points $(-1, -2)$ and $(-1, 3)$ have the same first member, so the set of points is a relation but not a function.



GEOMETRIC TEST FOR FUNCTIONS: "VERTICAL LINE TEST"

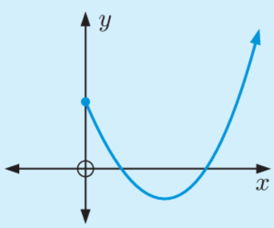
Suppose we draw all possible vertical lines on the graph of a relation.

- If each line cuts the graph at most once, then the relation is a function.
- If *any* line cuts the graph more than once, then the relation is *not* a function.

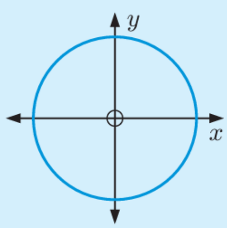
Example 2 Self Tutor

Which of these relations are functions?

a



b

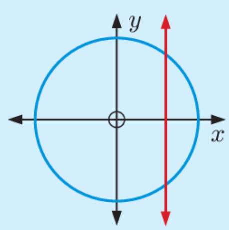


a Every vertical line we could draw cuts the graph only once.

$\therefore$ the relation is a function.

b This vertical line cuts the graph twice.

$\therefore$ the relation is not a function.



EXERCISE 15B

1 Which of the following sets of ordered pairs are functions? Give reasons for your answers.

a $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$

b $\{(-1, 2), (-3, 2), (3, 2), (1, 2)\}$

c $\{(2, 5), (-1, 4), (-3, 7), (2, -3)\}$

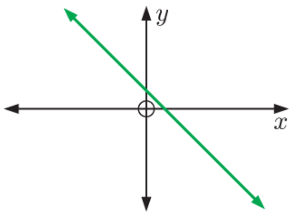
d $\{(3, -2), (3, 0), (3, 2), (3, 4)\}$

e $\{(-7, 0), (-5, 0), (-3, 0), (-1, 0)\}$

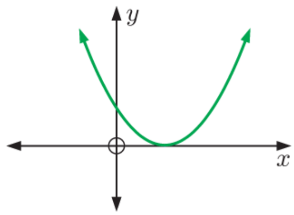
f $\{(0, 5), (0, 1), (2, 1), (2, -5)\}$

2 Use the vertical line test to determine which of the following relations are functions:

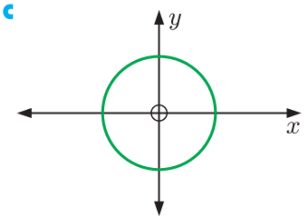
a



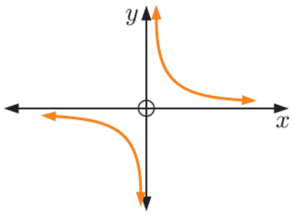
b



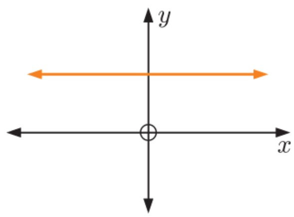
c



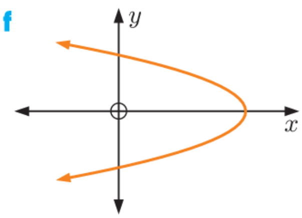
d



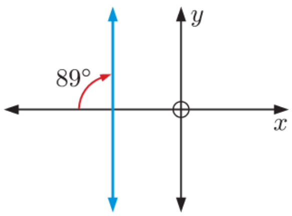
e



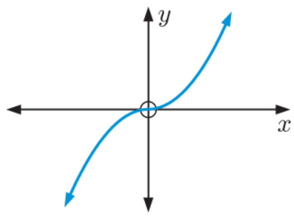
f



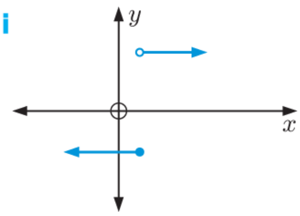
g



h



i

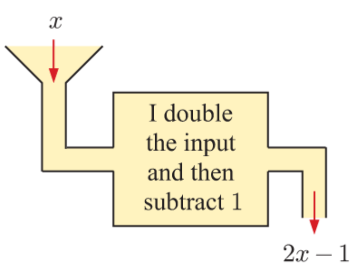


3 Will the graph of a straight line always be a function? Explain your answer.

C FUNCTION NOTATION

We sometimes use a 'function machine' to illustrate how functions behave.

For example, the machine alongside has been programmed to perform a particular function. Whatever number is fed into the machine, the machine will double the number and then subtract 1.



If f is used to represent this particular function, we can write:

f is the function that will convert x into $2x - 1$.

If 3 is fed into the machine, $2(3) - 1 = 5$ comes out.

This function can be written as: $\boldsymbol{f : x \mapsto 2x - 1}$

function f | such that | x | is converted into | $2x - 1$

We can also write this function as $f(x) = 2x - 1$

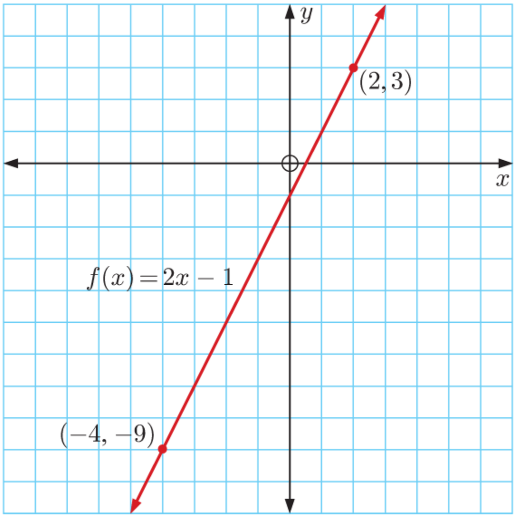
For any function f, the value of the function when $x = a$ is given by $f(a)$.

For $f(x) = 2x - 1$, $f(2) = 2(2) - 1 = 3$.

This indicates that the point (2, 3) lies on the graph of the function.

Likewise, $f(-4) = 2(-4) - 1 = -9$.

This indicates that the point $(-4, -9)$ also lies on the graph.



Note that:
- $f(x)$ is read as "f of x", and is the value of the function at any value of x.
- f is the function which converts x into $f(x)$, so $f : x \mapsto f(x)$.
- $f(x)$ is sometimes called the **image** of x.

Example 3 — Self Tutor

For $f : x \mapsto 3x^2 - 4x$, find the value of:

a $f(2)$ **b** $f(-5)$

$f(x) = 3x^2 - 4x$

a $f(2)$
$= 3(2)^2 - 4(2)$ {replacing x by (2)}
$= 3 \times 4 - 8$
$= 4$

b $f(-5)$
$= 3(-5)^2 - 4(-5)$ {replacing x by (-5)}
$= 3(25) + 20$
$= 95$

Example 4 — Self Tutor

For $f(x) = 4 - 3x - x^2$, find in simplest form:

a $f(-x)$ **b** $f(x + 2)$

a $f(-x) = 4 - 3(-x) - (-x)^2$ {replacing x by $(-x)$}
$= 4 + 3x - x^2$

b $f(x + 2) = 4 - 3(x + 2) - (x + 2)^2$ {replacing x by $(x + 2)$}
$= 4 - 3x - 6 - [x^2 + 4x + 4]$
$= -x^2 - 7x - 6$

EXERCISE 15C.1

1 For $f : x \mapsto 2x + 3$, find:

a $f(0)$ **b** $f(2)$ **c** $f(-1)$ **d** $f(-5)$ **e** $f(-\frac{1}{2})$

2 For $g(x) = -5x + 3$, find:

a $g(1)$ **b** $g(4)$ **c** $g(-2)$ **d** $g(-x)$ **e** $g(x + 4)$

3 For $f(x) = 2x^2 - 3x + 2$, find:

a $f(0)$ **b** $f(3)$ **c** $f(-4)$ **d** $f(-x)$ **e** $f(x + 1)$

4 For $P : x \mapsto 4x^2 + 4x - 3$, find:

a $P(3)$ **b** $P(-1)$ **c** $P(\frac{1}{2})$ **d** $P(x - 3)$ **e** $P(2x)$

5 Consider the function $R(x) = \dfrac{2x - 3}{x + 2}$.

a Evaluate: **i** $R(0)$ **ii** $R(1)$ **iii** $R(-\frac{1}{2})$

b Find a value of x such that $R(x)$ does not exist.

c Find $R(x - 2)$ in simplest form.

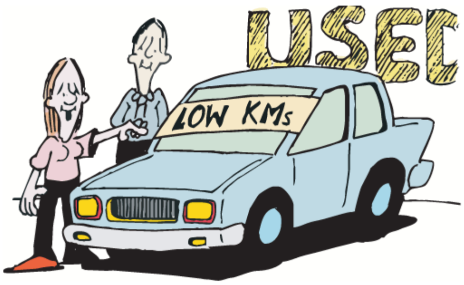
d Find x such that $R(x) = -5$.

6 The value of a car t years after purchase is given by $V(t) = 28\,000 - 4000t$ dollars.

a Find $V(4)$, and state what this value means.

b Find t when $V(t) = 8000$, and explain what this represents.

c Find the original purchase price of the car.

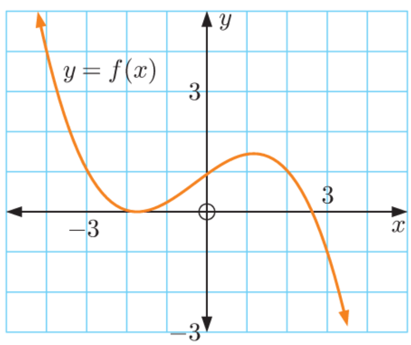


7 The graph of $y = f(x)$ is shown alongside.

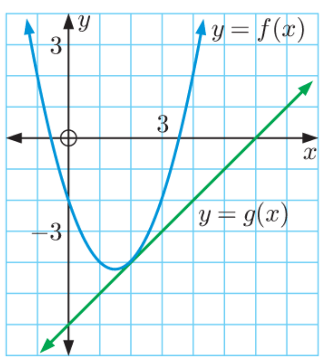
a Find:

i $f(2)$ **ii** $f(3)$

b Find the value of x such that $f(x) = 4$.



8



Consider the graphs of $y = f(x)$ and $y = g(x)$ shown.

a Find:

i $f(4)$ **ii** $g(0)$ **iii** $g(5)$

b Find the *two* values of x such that $f(x) = -2$.

c Find the value of x such that $f(x) = g(x)$.

d Show that $g(x) = x - 6$.

9 Draw a graph of $y = f(x)$ such that $f(-2) = 5$, $f(1) = 0$, and $f(4) = 3$.

10 The graph of $y = f(x)$ is a straight line passing through $(-3, -5)$ and $(1, 7)$.

a Draw the graph of $y = f(x)$. **b** Find $f(-3)$ and $f(1)$.

c Find $f(x)$.

THE DOMAIN OF A FUNCTION

To find the domain of a function, we need to consider what values of the variable make the function undefined.

For example:

- the domain of $f(x) = \sqrt{x}$ is $\{x \mid x \geqslant 0,\ x \in \mathbb{R}\}$, since $\sqrt{x}$ has meaning only when $x \geqslant 0$.
- the domain of $f(x) = \dfrac{1}{\sqrt{x-1}}$ is $\{x \mid x > 1,\ x \in \mathbb{R}\}$ since, when $x - 1 = 0$ we are 'dividing by zero', and when $x - 1 < 0$, $\sqrt{x-1}$ is undefined.

Example 5 **Self Tutor**

Find the domain of:

a $f(x) = \dfrac{2}{\sqrt{x+3}}$ **b** $f(x) = \sqrt{x} + \sqrt{5-x}$

a $f(x) = \dfrac{2}{\sqrt{x+3}}$ is defined when $x + 3 > 0$

$\therefore\ x > -3$

So, the domain of $f(x)$ is $\{x \mid x > -3,\ x \in \mathbb{R}\}$.

b $f(x) = \sqrt{x} + \sqrt{5-x}$ is defined when $x \geqslant 0$ *and* $5 - x \geqslant 0$

$\therefore\ x \geqslant 0$ and $x \leqslant 5$

So, the domain of $f(x)$ is $\{x \mid 0 \leqslant x \leqslant 5,\ x \in \mathbb{R}\}$.

EXERCISE 15C.2

1 Find the domain of:

a $f(x) = 2x$ **b** $f(x) = \dfrac{1}{x}$ **c** $f(x) = \dfrac{1}{x-3}$

d $f(x) = \dfrac{1}{(x-1)(x+2)}$ **e** $f(x) = \dfrac{x}{x^2-9}$ **f** $f(x) = \dfrac{3}{x^2-5x+4}$

2 Find the domain of:

a $f(x) = \sqrt{x-2}$ **b** $f(x) = \sqrt{3-x}$ **c** $f(x) = \sqrt{x} + \sqrt{2-x}$

d $f(x) = \dfrac{1}{\sqrt{x}}$ **e** $f(x) = \dfrac{1}{\sqrt{x}} + \dfrac{1}{\sqrt{x+2}}$ **f** $f(x) = \dfrac{1}{x\sqrt{4-x}}$

Check your answers using the graphing package.

GRAPHING PACKAGE

D COMPOSITE FUNCTIONS

Sometimes functions are built up in two or more stages.

For example, consider $f(x) = 2\sqrt{x}$.

Using the techniques studied in **Section C**, $f(x+3) = 2\sqrt{x+3}$. {replacing x by $(x+3)$}

If we let $g(x) = x + 3$, then $f(g(x)) = 2\sqrt{x+3}$.

So, the function $f(g(x)) = 2\sqrt{x+3}$ is *composed* of $f(x) = 2\sqrt{x}$ and $g(x) = x + 3$.

> Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the **composite function** of f and g will convert x into $f(g(x))$.

Example 6 — Self Tutor

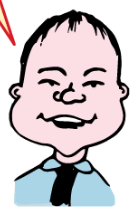
If $f(x) = 3x+2$ and $g(x) = x^2+4$, find in simplest form:

a $f(g(x))$ b $g(f(x))$

a
$f(g(x))$
$= f(x^2 + 4)$
$= 3(x^2 + 4) + 2$
$= 3x^2 + 12 + 2$
$= 3x^2 + 14$

b
$g(f(x))$
$= g(3x + 2)$
$= (3x + 2)^2 + 4$
$= 9x^2 + 12x + 4 + 4$
$= 9x^2 + 12x + 8$

To find $f(g(x))$ we look at the f function. Whenever we see x we replace it by $g(x)$ within brackets.



From the previous **Example**, we can see that in general, $f(g(x)) \neq g(f(x))$.

Example 7 — Self Tutor

If $f(x) = 2x - 1$, find:

a $f(f(x))$ b $f(f(5))$

a
$f(f(x))$
$= f(2x - 1)$
$= 2(2x - 1) - 1$
$= 4x - 2 - 1$
$= 4x - 3$

b
$f(f(5))$
$= 4(5) - 3$ {using a}
$= 17$

EXERCISE 15D

1 If $f(x) = 3x - 4$ and $g(x) = 2 - x$, find in simplest form:

a $f(g(x))$ b $g(f(x))$ c $f(f(x))$ d $g(g(x))$

2 If $f(x) = \sqrt{x}$ and $g(x) = 4x - 3$, find in simplest form:

a $f(g(x))$ b $g(f(x))$ c $f(g(7))$ d $g(f(4))$

3 Find two functions f and g such that:

a $f(g(x)) = \sqrt{x-3}$ **b** $f(g(x)) = (x+5)^3$ **c** $f(g(x)) = \dfrac{5}{x+7}$

d $g(f(x)) = \dfrac{1}{\sqrt{3-4x}}$ **e** $g(f(x)) = 3^{x^2}$ **f** $g(f(x)) = \left(\dfrac{x+1}{x-1}\right)^2$

4 Suppose $f(x) = 3x+1$ and $g(x) = x^2+2x$.

a Find $f(g(x))$. **b** Find x such that $f(g(x)) = 10$.

5 Suppose $f(x) = 4x+1$ and $g(x) = \dfrac{1}{x+2}$.

a Find $f(g(x))$. **b** Find $f(g(-1))$.

c Find x such that $f(g(x)) = 3$.

6 Suppose $f(x) = 2x+5$ and $g(x) = \dfrac{x-5}{2}$.

a Find:

i $f(4)$ **ii** $g(13)$ **iii** $g(17)$ **iv** $f(6)$.

b Find $f(g(x))$ and $g(f(x))$.

c Find:

i $f(g(3))$ **ii** $g(f(-7))$.

E INVERSE FUNCTIONS

The operations of $+$ and $-$, and $\times$ and $\div$, are **inverse operations** since one 'undoes' the other.

In the same way, some functions have **inverse functions** which 'undo' each other.

For a function f which converts x to $f(x)$, the **inverse function** f^{-1} converts $f(x)$ back to x.

The inverse function satisfies $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

For a given function f, an inverse function f^{-1} only exists if, for any value of $f(x)$, there is only *one* corresponding value of x.

INVESTIGATION 1 INVERSE FUNCTIONS

In this Investigation we explore inverse functions, how they relate to composite functions, and also how they relate to transformations.

What to do:

1 Consider $f(x) = 3x+2$ which has graph $y = 3x+2$.

a Interchange x and y and then make y the subject of this new equation. Let this function be $g(x)$.

b Hence show that $f(3) = 11$ and $g(11) = 3$.

c From **b**, notice that $g(11) = g(f(3)) = 3$. Show that $f(g(3)) = 3$ also.

d Prove that $f(g(x)) = x$ and $g(f(x)) = x$.

e Graph $y = f(x)$ and $y = g(x)$ on the same set of axes. What do you notice?

2 Consider $f(x) = 3 - 4x$ which has graph $y = 3 - 4x$.

a Interchange x and y and then make y the subject of this new equation. Hence find the inverse function $f^{-1}(x)$.

b Show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

c Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes. What do you notice?

From the **Investigation** you should have found that:

- $f^{-1}(x)$ can be found algebraically by interchanging x and y and then making y the subject of the resulting formula. The new y is $f^{-1}(x)$.
- $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$.

Example 8 Self Tutor

Consider $f(x) = \frac{1}{2}x - 1$.

a Find $f^{-1}(x)$. **b** Check that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

c Sketch $y = f(x)$, $y = f^{-1}(x)$, and $y = x$ on the same set of axes.

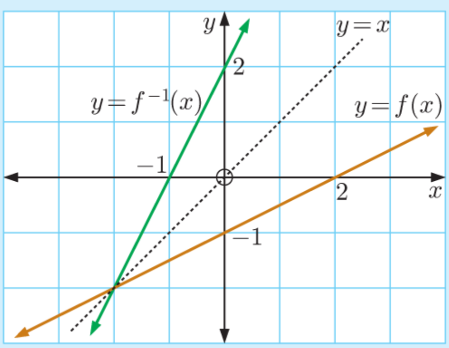
a $y = \frac{1}{2}x - 1$ has inverse function $x = \frac{1}{2}y - 1$ {interchanging x and y}

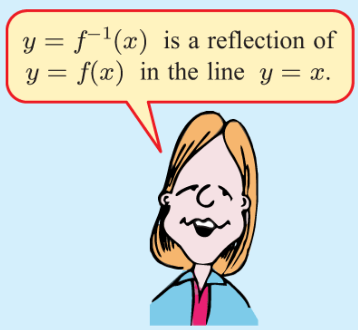
$$\therefore\ 2x = y - 2$$
$$\therefore\ y = 2x + 2$$
$$\therefore\ f^{-1}(x) = 2x + 2$$

b
$$\begin{aligned} f(f^{-1}(x)) &= f(2x+2) \\ &= \tfrac{1}{2}(2x+2) - 1 \\ &= x + 1 - 1 \\ &= x \end{aligned}$$

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(\tfrac{1}{2}x - 1) \\ &= 2(\tfrac{1}{2}x - 1) + 2 \\ &= x - 2 + 2 \\ &= x \end{aligned}$$

c





EXERCISE 15E

1 For each of the following functions:

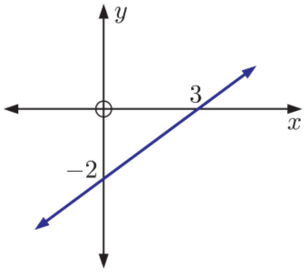
i Find $f^{-1}(x)$.

ii Sketch $y = f(x)$, $y = f^{-1}(x)$, and $y = x$ on the same set of axes.

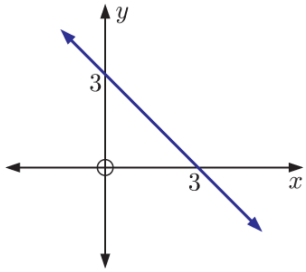
a $f(x) = x + 3$ **b** $f(x) = 2x + 5$ **c** $f(x) = \dfrac{3 - 2x}{4}$

2 Copy the following graphs and draw the graph of each inverse function:

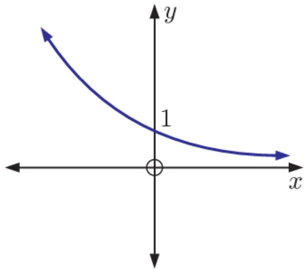
a



b



c



3 Consider $f(x) = 2x + 7$.

a Find $f^{-1}(x)$. **b** Check that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

4 Consider $f(x) = \dfrac{2x+1}{x+3}$.

a Find $f^{-1}(x)$. **b** Check that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

5 Consider $g(x) = \dfrac{x}{x-5}$.

a Find $g^{-1}(x)$. **b** Check that $g(g^{-1}(x)) = x$ and $g^{-1}(g(x)) = x$.

c Find $g(6)$ and $g^{-1}(1)$.

6 **a** Sketch the graph of $y = x^2$ and reflect it in the line $y = x$.

b Does $f(x) = x^2$ have an inverse function? Explain your answer.

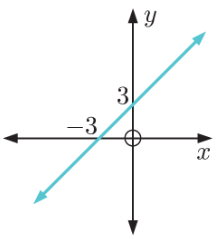
c Does $f(x) = x^2,\ x \geqslant 0$ have an inverse function? Explain your answer.

7 The **horizontal line test** says that '*for a function to have an inverse function, no horizontal line can cut it more than once*'.

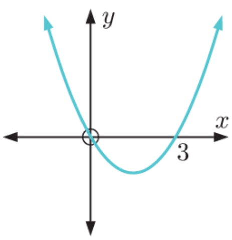
a Explain why this is a valid test for the existence of an inverse function.

b Which of the following functions have an inverse function?

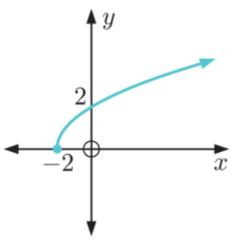
i



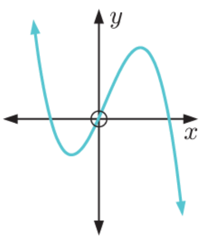
ii



iii



iv



8 **a** Explain why $f(x) = x^2 - 2x + 5$ is a function but does not have an inverse function.

b Explain why $f(x) = x^2 - 2x + 5,\ x \geqslant 1$ has an inverse function.

c Show that the inverse function of **b** is $f^{-1}(x) = 1 + \sqrt{x-4}$.

Hint: Swap x and y, then use the quadratic formula to solve for y in terms of x.

F THE MODULUS FUNCTION

MODULUS

The **modulus** or **absolute value** of a real number is its size, ignoring its sign.
We denote the modulus of x by $|x|$.

For example, the modulus of 7 is 7, and the modulus of -7 is also 7, so we write $|7| = 7$ and $|-7| = 7$.

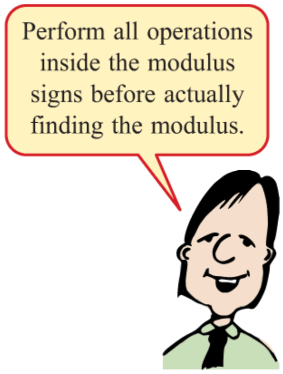
Example 9 Self Tutor

If $a = -7$ and $b = 3$, find:

a $|a+b|$ **b** $|ab|$

a $|a+b|$
$= |-7+3|$
$= |-4|$
$= 4$

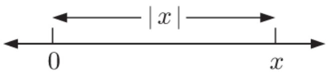
b $|ab|$
$= |-7 \times 3|$
$= |-21|$
$= 21$



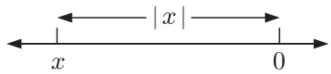
GEOMETRIC DEFINITION OF MODULUS

$|x|$ is the distance of x from 0 on the number line.
Because the modulus is a distance, it cannot be negative.

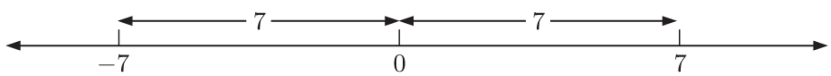
If $x > 0$:



If $x < 0$:



For example:



$|-7| = |7| = 7$

ALGEBRAIC DEFINITION OF MODULUS

INVESTIGATION 2 **ALGEBRAIC DEFINITION OF MODULUS**

What to do:

1 Suppose $y = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0. \end{cases}$

a Copy and complete the table of values:

x	-10	-8	-2	0	$\frac{1}{2}$	3	5	7
y								

b Plot the points from the table and hence graph $y = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0. \end{cases}$

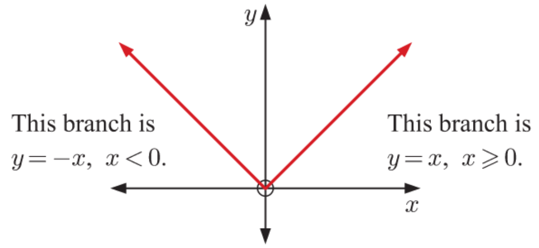
2 Copy and complete:

x	-10	-8	-2	0	$\frac{1}{2}$	3	5	7
$\sqrt{x^2}$								

3 What can you conclude from **1** and **2**?

From the **Investigation**, you should have found that: $|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$ or $|x| = \sqrt{x^2}$

$y = |x|$ has graph:

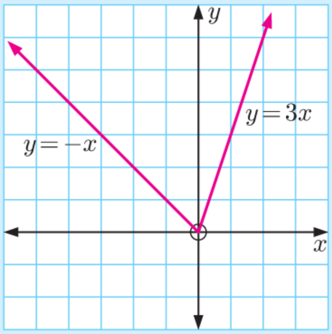


To draw graphs involving $|x|$, we must consider the cases $x \geqslant 0$ and $x < 0$ separately, so that we can write the function without the modulus sign.

Example 10 **Self Tutor**

Draw the graph of $f(x) = x + 2\,|x|$.

$$f(x) = x + 2\,|x|$$
$$= \begin{cases} x + 2(x) & \text{if } x \geqslant 0 \\ x + 2(-x) & \text{if } x < 0 \end{cases}$$
$$= \begin{cases} 3x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$



EXERCISE 15F

1 If $x = -4$, find the value of:

a $|x+6|$ **b** $|x-6|$ **c** $|2x+3|$ **d** $|7-x|$

e $|x-7|$ **f** $|x^2-6x|$ **g** $|6x-x^2|$ **h** $\dfrac{|x|}{x+2}$

2 If $a = 5$ and $b = -2$, find the value of:

a $|a+b|$ **b** $|ab|$ **c** $|b-a|$ **d** $|a|+b$

e $|3a+b|$ **f** $\dfrac{|b-8|}{a}$ **g** $\left|\dfrac{a}{b}\right|$ **h** $\dfrac{|b^2|}{|a|}$

3 **a** Copy and complete:

x	9	3	0	-3	-9
x^2					
$\|x\|^2$					

b What can you conclude from **a**?

4 By replacing $|x|$ with x for $x \geqslant 0$ and $(-x)$ for $x < 0$, write the following functions without the modulus sign. Hence, graph each function:

a $f(x) = -|x|$ **b** $f(x) = |x| + x$ **c** $f(x) = |x| + 2$

d $f(x) = x - 2|x|$ **e** $f(x) = 3|x| + 1$ **f** $f(x) = 5 - |x|$

g $f(x) = |x|^2 - 4$ **h** $f(x) = \frac{|x|}{x}$ **i** $f(x) = \sqrt{|x|}$

5 **a** Use technology to graph:

i $y = |(x-2)(x-4)|$ **ii** $y = |x(x-3)|$

b Explain how these functions are related to $y = (x-2)(x-4)$ and $y = x(x-3)$.

GRAPHING PACKAGE

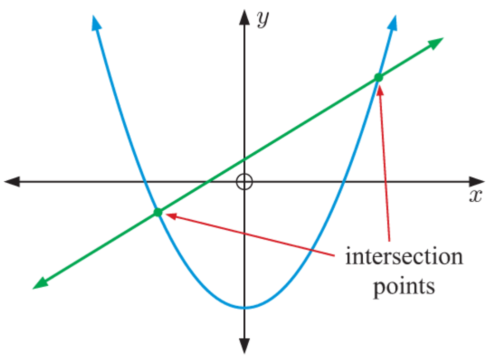
G WHERE FUNCTIONS MEET

We often draw the graphs of two functions on the same set of axes, and are interested in finding where the functions meet.

To find the points of intersection of the graphs of $y = f(x)$ and $y = g(x)$, we solve the equation $f(x) = g(x)$.

The solutions of this equation give us the x-coordinates of the intersection points.

The y-coordinates can then be found by substituting the x-coordinates into one of the functions.



Example 11 Self Tutor

Find the coordinates of the points of intersection of the graphs with equations $y = x^2 - x + 3$ and $y = 2x + 7$.

The graphs meet when $x^2 - x + 3 = 2x + 7$

$\therefore\ x^2 - 3x - 4 = 0$

$\therefore\ (x+1)(x-4) = 0$

$\therefore\ x = -1$ or 4

Substituting into $y = 2x + 7$: when $x = -1$, $y = 2(-1) + 7 = 5$

when $x = 4$, $y = 2(4) + 7 = 15$

$\therefore$ the graphs meet at $(-1, 5)$ and $(4, 15)$.

EXERCISE 15G

1 Find the coordinates of the point of intersection of the graphs with equations:

a $y = 5x - 1$ and $y = 2x + 5$

b $y = 12 - x$ and $y = 3x + 7$

c $y = \dfrac{1}{x}$ and $y = \dfrac{2}{x+5}$

d $y = x^2 - x + 3$ and $y = x^2 + 5x - 3$.

2 Find the coordinates of the point(s) of intersection of the graphs with equations:

a $y = x^2 + 2x - 1$ and $y = x + 5$

b $y = \dfrac{2}{x}$ and $y = x - 1$

c $y = 3x^2 + 4x - 1$ and $y = x^2 - 3x - 4$

d $y = \dfrac{1}{x}$ and $y = 5x - 4$.

3 Use a **graphing package** or a **graphics calculator** to find the coordinates, correct to 2 decimal places, of the points of intersection of the graphs with equations:

a $y = x^2 + 3x + 1$ and $y = 2x + 2$

b $y = x^2 - 5x + 2$ and $y = \dfrac{3}{x}$

c $y = -x^2 - 2x + 5$ and $y = x^2 + 7$

d $y = x^2 - 1$ and $y = x^3$.

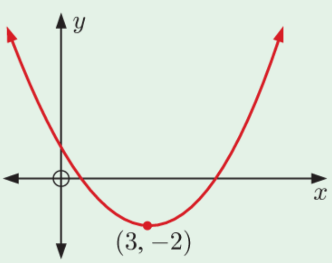
GRAPHING PACKAGE

GRAPHICS CALCULATOR INSTRUCTIONS

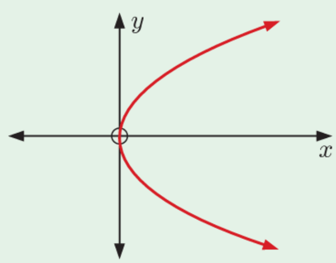
REVIEW SET 15A

1 Find the domain and range of the following relations:

a



b



2 For $f(x) = 3x - x^2$, find:

a $f(2)$ **b** $f(-1)$ **c** $f(x-3)$

3 Determine whether the following sets of ordered pairs are functions:

a $\{(-3, 5), (1, 7), (-1, 7), (2, 5)\}$

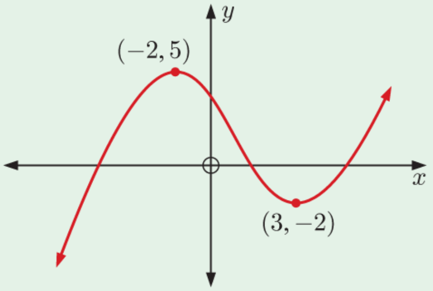
b $\{(-4, -5), (-1, 3), (5, 4), (0, 3), (-1, 2)\}$

4 Consider the function $g(x) = x^2 + 2x$. Find:

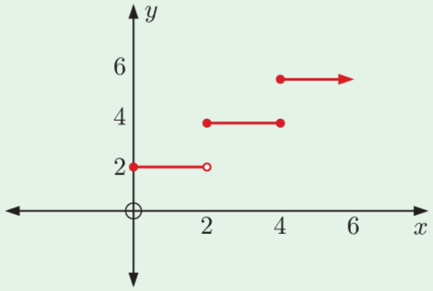
a $g(2)$ **b** $g(3x)$ **c** x such that $g(x) = 15$.

5 Determine whether the following relations are functions:

a



b



6 Draw a graph of $y = f(x)$ such that $f(-3) = 2$, $f(1) = -4$, and $f(4) = 5$.

7 Answer the **Opening Problem** on page **324**.

8 If $f(x) = \sqrt{x}$ and $g(x) = 5x - 3$, find in simplest form:

a $f(g(x))$ **b** $g(f(x))$ **c** $g(g(x))$

9 Consider the function $f(x) = \dfrac{x+2}{7}$.

a Find $f^{-1}(x)$.

b Check that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

c Sketch $y = f(x)$, $y = f^{-1}(x)$, and $y = x$ on the same set of axes.

10 Suppose $g(x) = \dfrac{x+4}{x}$.

a Find $g^{-1}(x)$.

b Check that $g(g^{-1}(x)) = g^{-1}(g(x)) = x$.

c Find $g(2)$ and $g^{-1}(3)$.

11 If $x = -3$, find the value of:

a $|x - 4|$ **b** $|x| - 4$ **c** $|x^2 + 3x|$

12 Draw the graph of $y = f(x)$ for:

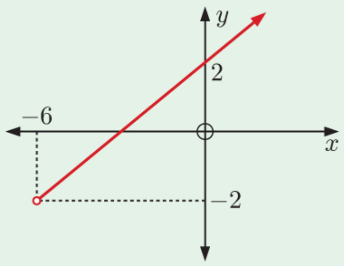
a $f(x) = |x| + 3x$ **b** $f(x) = 2|x| - 4$

13 Find the coordinates of the points of intersection of the graphs with equations $y = x^2 + 4x - 2$ and $y = 2x + 1$.

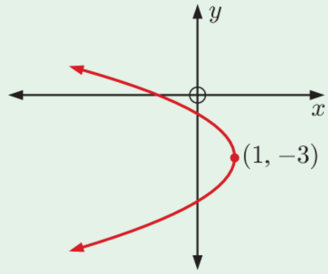
REVIEW SET 15B

1 Find the domain and range of the following relations:

a

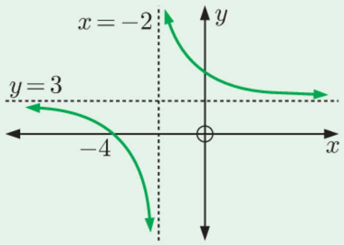


b

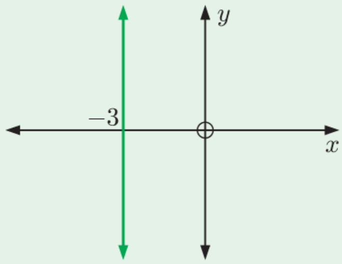


2 Determine whether the following relations are functions:

a



b



3 For $f(x) = 5x - x^2$, find:

a $f(-3)$ **b** $f(-x)$ **c** $f(x+1)$

4 Find the domain of:

a $f(x) = \dfrac{1}{x+4}$ **b** $f(x) = \dfrac{x}{x^2+4x-5}$ **c** $f(x) = \sqrt{x} + \sqrt{6-x}$

5 The graph of $y = f(x)$ is a straight line passing through $(-1, 5)$ and $(3, -3)$.

a Draw the graph of $y = f(x)$. **b** Find $f(-1)$ and $f(3)$.

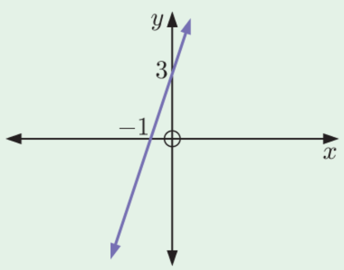
c Find $f(x)$. **d** Find $f^{-1}(x)$.

6 If $f(x) = 2x + 1$ and $g(x) = 7 - x$, find in simplest form:

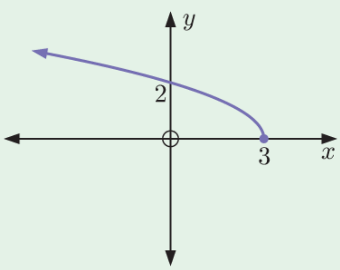
a $f(g(x))$ **b** $g(f(x))$ **c** $f(g(-2))$

7 Copy the following graphs, and draw the graph of each inverse function:

a



b



8 If $a = -4$ and $b = 9$, find the value of:

a $|ab|$ **b** $|2a - b| + a$ **c** $\dfrac{|a^2 - b|}{|a|}$

9 Suppose $f(x) = 3x - 1$ and $g(x) = \dfrac{1}{x-4}$.

a Find $f(g(x))$.

b Find $f(g(5))$.

c Find x such that $f(g(x)) = -\frac{1}{2}$.

10 Consider $f(x) = \dfrac{x+1}{x-2}$.

a Find $f^{-1}(x)$.

b Check that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

c Find $f^{-1}(4)$.

11 Draw the graph of:

a $f(x) = 7 - |x|$

b $f(x) = \dfrac{x}{|x|} + 3$

12 Find the coordinates of the points of intersection of the graphs with equations $y = \dfrac{3}{x}$ and $y = 2x - 1$.

3 **a** $M = 37$ **b** $r = 8$ **4** $V = 4x^2y$

5 **a** $a = \dfrac{B+f}{d}$ **b** $a = \dfrac{9Q^2}{t^2}$ **6** $h = \dfrac{5 - G^2}{G^2}$

7 **a** $b = \dfrac{a}{a-1}$ **b** $b = \frac{3}{2}$; $3 \times \frac{3}{2} = 3 + \frac{3}{2} = 4\frac{1}{2}$ ✓

8 $x = \dfrac{2y+3}{4-3y}$ **9** 8, 13, 18, ... $\therefore$ $M = 5n + 3$

10 **a** **i** 6 **ii** 12 **iii** 20 **iv** 30
b $S_1 = 2 = 1 \times 2$
$S_2 = 6 = 2 \times 3$
$S_3 = 12 = 3 \times 4$
$S_4 = 20 = 4 \times 5$
$S_5 = 30 = 5 \times 6$ $\therefore$ $S_n = n(n+1)$

11 **a** $E = 1000$ joules **b** $v = \sqrt{\dfrac{2E}{m}}$ **c** 8 m/s

EXERCISE 15A

1 **a** Domain is $\{x \mid x > -4\}$. Range is $\{y \mid y > -2\}$.
b Domain is $\{x \mid -3 \leqslant x \leqslant 4\}$. Range is $\{y \mid -5 \leqslant y \leqslant 2\}$.
c Domain is $\{x \mid -3 < x < 4\}$. Range is $\{y \mid -5 < y < 6\}$.
d Domain is $\{x \mid x = 2\}$. Range is $\{y \mid y \in \mathbb{R}\}$.
e Domain is $\{x \mid -3 \leqslant x \leqslant 3\}$. Range is $\{y \mid -3 \leqslant y \leqslant 3\}$.
f Domain is $\{x \mid x \in \mathbb{R}\}$. Range is $\{y \mid y \leqslant 0\}$.
g Domain is $\{x \mid x \in \mathbb{R}\}$. Range is $\{y \mid y = -5\}$.
h Domain is $\{x \mid x \in \mathbb{R}\}$. Range is $\{y \mid y \geqslant 1\}$.
i Domain is $\{x \mid x \geqslant -5\}$. Range is $\{y \mid y \leqslant 7\}$.
j Domain is $\{x \mid x \in \mathbb{R}\}$. Range is $\{y \mid y \leqslant 4\}$.
k Domain is $\{x \mid x \geqslant -5\}$. Range is $\{y \mid y \in \mathbb{R}\}$.
l Domain is $\{x \mid x \in \mathbb{R},\ x \neq 1\}$.
Range is $\{y \mid y \in \mathbb{R},\ y \neq 0\}$.

2 **a** Domain is $\{x \mid 0 \leqslant x \leqslant 2\}$. Range is $\{y \mid -3 \leqslant y \leqslant 2\}$.
b Domain is $\{x \mid -2 < x < 2\}$. Range is $\{y \mid -1 < y < 3\}$.
c Domain is $\{x \mid -4 \leqslant x \leqslant 4\}$. Range is $\{y \mid -2 \leqslant y \leqslant 2\}$.

EXERCISE 15B

1 **a**, **b**, and **e** are functions as no two ordered pairs have the same x-coordinate.

2 **a**, **b**, **d**, **e**, **g**, **h**, and **i** are functions.

3 No, a vertical line is not a function as it does not satisfy the vertical line test.

EXERCISE 15C.1

1 **a** 3 **b** 7 **c** 1 **d** -7 **e** 2

2 **a** -2 **b** -17 **c** 13 **d** $5x + 3$ **e** $-5x - 17$

3 **a** 2 **b** 11 **c** 46 **d** $2x^2 + 3x + 2$
e $2x^2 + x + 1$

4 **a** 45 **b** -3 **c** 0 **d** $4x^2 - 20x + 21$
e $16x^2 + 8x - 3$

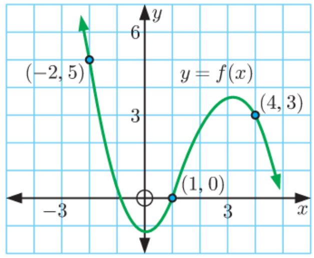
5 **a** **i** $-\frac{3}{2}$ **ii** $-\frac{1}{3}$ **iii** $-\frac{8}{3}$ **b** $x = -2$
c $2 - \dfrac{7}{x}$ **d** $x = -1$

6 **a** $V(4) = 12\,000$. The value of the car after 4 years is \$12 000.
b $V(t) = 8000$ when $t = 5$. 5 years after purchase the value of the car is \$8000.
c \$28 000

7 **a** **i** $f(2) = 1$ **ii** $f(3) = -1$ **b** $x = -4$

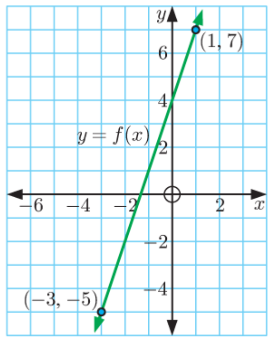
8 **a** **i** $f(4) = 2$ **ii** $g(0) = -6$ **iii** $g(5) = -1$
b $x = 0$ and $x = 3$ **c** $x = 2$
d g has gradient 1 and y-intercept -6.

9



Other graphs are possible.

10 **a**



b $f(-3) = -5$,
$f(1) = 7$
c $f(x) = 3x + 4$

EXERCISE 15C.2

1 **a** $\{x \mid x \in \mathbb{R}\}$ **b** $\{x \mid x \neq 0\}$
c $\{x \mid x \neq 3\}$ **d** $\{x \mid x \neq -2$ and $x \neq 1\}$
e $\{x \mid x \neq 3$ and $x \neq -3\}$ **f** $\{x \mid x \neq 1$ and $x \neq 4\}$

2 **a** $\{x \mid x \geqslant 2\}$ **b** $\{x \mid x \leqslant 3\}$
c $\{x \mid 0 \leqslant x \leqslant 2\}$ **d** $\{x \mid x > 0\}$
e $\{x \mid x > 0\}$ **f** $\{x \mid x < 4$ and $x \neq 0\}$

EXERCISE 15D

1 **a** $2 - 3x$ **b** $6 - 3x$ **c** $9x - 16$ **d** x

2 **a** $\sqrt{4x - 3}$ **b** $4\sqrt{x} - 3$ **c** 5 **d** 5

3 **a** $f(x) = \sqrt{x}$, $g(x) = x - 3$ **b** $f(x) = x^3$, $g(x) = x + 5$
c $f(x) = \dfrac{5}{x}$, $g(x) = x + 7$ **d** $f(x) = 3 - 4x$, $g(x) = \dfrac{1}{\sqrt{x}}$
e $f(x) = x^2$, $g(x) = 3^x$ **f** $f(x) = \dfrac{x+1}{x-1}$, $g(x) = x^2$
(**Note:** There may be other answers.)

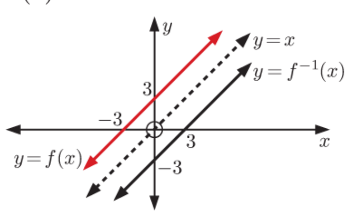
4 **a** $f(g(x)) = 3x^2 + 6x + 1$ **b** $x = -3$ or 1

5 **a** $f(g(x)) = \dfrac{4}{x+2} + 1$ $\left(\text{or } \dfrac{x+6}{x+2}\right)$ **b** 5 **c** $x = 0$

6 **a** **i** 13 **ii** 4 **iii** 6 **iv** 17
b $f(g(x)) = x$ and $g(f(x)) = x$ **c** **i** 3 **ii** -7

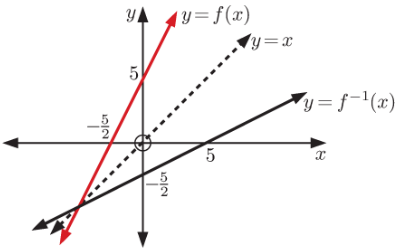
EXERCISE 15E

1 **a** **i** $f^{-1}(x) = x - 3$
ii



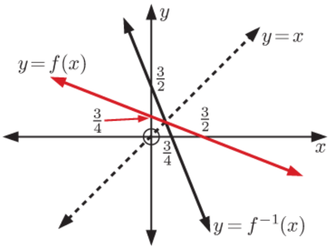
b **i** $f^{-1}(x) = \dfrac{x-5}{2}$

ii

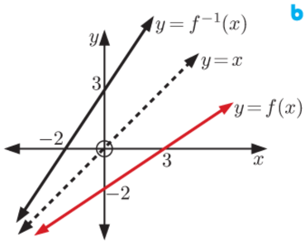


c **i** $f^{-1}(x) = -2x + \frac{3}{2}$

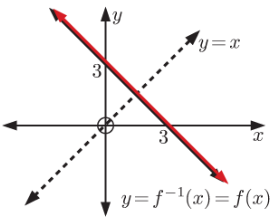
ii



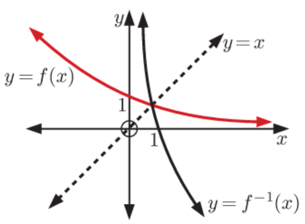
2 **a**



b



c

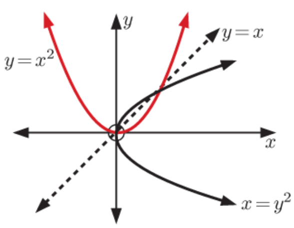


3 **a** $f^{-1}(x) = \dfrac{x-7}{2}$

4 **a** $f^{-1}(x) = \dfrac{1-3x}{x-2}$

5 **a** $g^{-1}(x) = \dfrac{5x}{x-1}$ **c** $g(6) = 6$, $g^{-1}(1)$ is undefined

6 **a**



b No, as the vertical line test fails.

c Yes, it is $y = \sqrt{x}$ (not $y = \pm\sqrt{x}$).

7 **b** **i** and **iii** have inverse functions.

8 **a** Is a function as it passes the vertical line test, but does not have an inverse as it fails the horizontal line test.

b It passes both the vertical line and horizontal line tests.

EXERCISE 15F

1 **a** 2 **b** 10 **c** 5 **d** 11 **e** 11 **f** 40 **g** 40 **h** −2

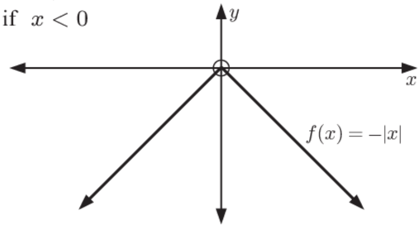
2 **a** 3 **b** 10 **c** 7 **d** 3 **e** 13 **f** 2 **g** $\frac{5}{2}$ **h** $\frac{4}{5}$

3 **a**

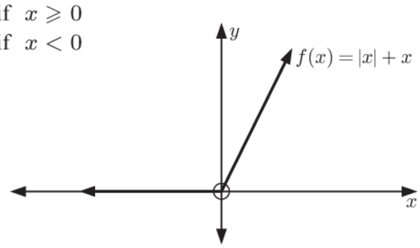
x	9	3	0	−3	−9
x^2	81	9	0	9	81
$\lvert x\rvert^2$	81	9	0	9	81

b $x^2 = |x|^2$

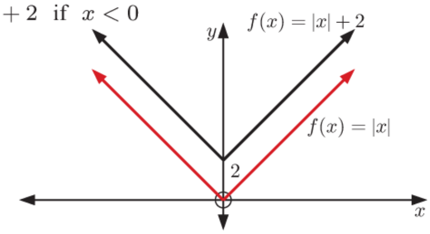
4 **a** $f(x) = \begin{cases} -x & \text{if } x \geqslant 0 \\ x & \text{if } x < 0 \end{cases}$



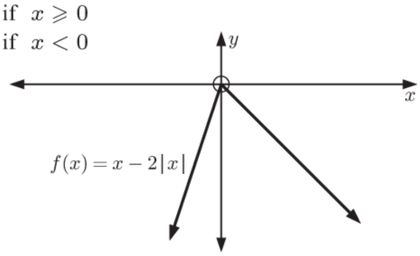
b $f(x) = \begin{cases} 2x & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases}$



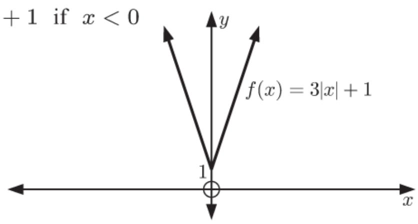
c $f(x) = \begin{cases} x+2 & \text{if } x \geqslant 0 \\ -x+2 & \text{if } x < 0 \end{cases}$



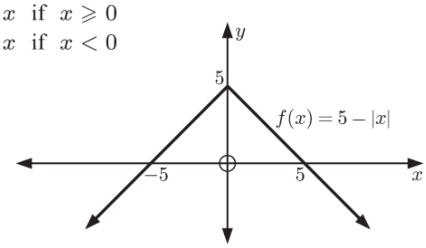
d $f(x) = \begin{cases} -x & \text{if } x \geqslant 0 \\ 3x & \text{if } x < 0 \end{cases}$



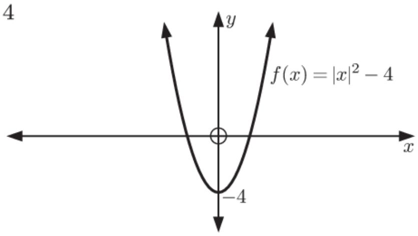
e $f(x) = \begin{cases} 3x+1 & \text{if } x \geqslant 0 \\ -3x+1 & \text{if } x < 0 \end{cases}$



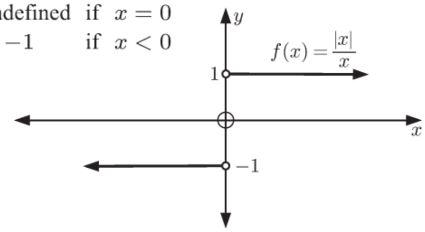
f $f(x) = \begin{cases} 5-x & \text{if } x \geqslant 0 \\ 5+x & \text{if } x < 0 \end{cases}$



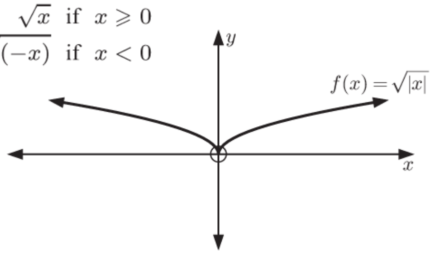
g $f(x) = x^2 - 4$



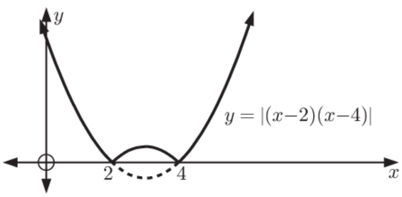
h $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$



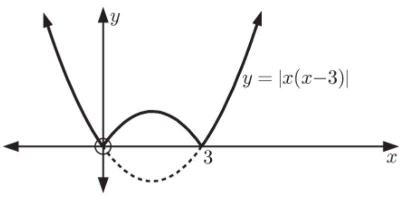
i $f(x) = \begin{cases} \sqrt{x} & \text{if } x \geqslant 0 \\ \sqrt{(-x)} & \text{if } x < 0 \end{cases}$



5 **a** **i**



ii



b The part of the graph $y = (x-2)(x-4)$ or $y = x(x-3)$ below the x-axis is reflected in the x-axis.

EXERCISE 15G

1 **a** $(2, 9)$ **b** $(\frac{5}{4}, \frac{43}{4})$ **c** $(5, \frac{1}{5})$ **d** $(1, 3)$

2 **a** at $(-3, 2)$ and $(2, 7)$ **b** at $(2, 1)$ and $(-1, -2)$
c at $(-\frac{1}{2}, -\frac{9}{4})$ and $(-3, 14)$ **d** at $(1, 1)$ and $(-\frac{1}{5}, -5)$

3 **a** at $(-1.62, -1.24)$ and $(0.62, 3.24)$ **b** at $(4.71, 0.64)$
c They do not intersect, $\therefore$ no solutions exist.
d at $(-0.75, -0.43)$

REVIEW SET 15A

1 **a** Domain is $\{x \mid x \in \mathbb{R}\}$. Range is $\{y \mid y \geqslant -2\}$.
b Domain is $\{x \mid x \geqslant 0\}$. Range is $\{y \mid y \in \mathbb{R}\}$.

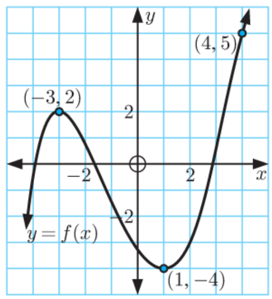
2 **a** 2 **b** -4 **c** $-x^2 + 9x - 18$

3 **a** function **b** not a function

4 **a** 8 **b** $9x^2 + 6x$ **c** $x = -5$ or 3

5 **a** function **b** not a function

6

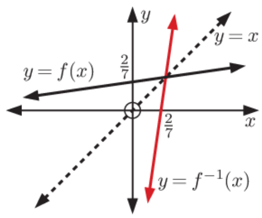


Note: There may be other answers.

7 **a** $0 \leqslant x \leqslant 15$ **b** $-5 \leqslant y \leqslant 15$
c When $x = 3$, $y = 10$, or $f(3) = 10$.
d When $y = 10$, $x = 3$, or $8 \leqslant x \leqslant 9$.

8 **a** $f(g(x)) = \sqrt{5x - 3}$ **b** $g(f(x)) = 5\sqrt{x} - 3$
c $g(g(x)) = 25x - 18$

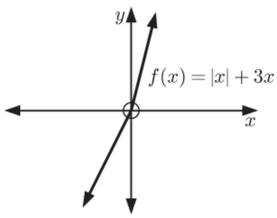
9 **a** $f^{-1}(x) = 7x - 2$ **c**

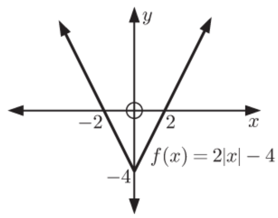


10 **a** $g^{-1}(x) = \dfrac{4}{x - 1}$ **c** $g(2) = 3$, $g^{-1}(3) = 2$

11 **a** 7 **b** -1 **c** 0

12 **a** $f(x) = \begin{cases} 4x & \text{if } x \geqslant 0 \\ 2x & \text{if } x < 0 \end{cases}$ **b** $f(x) = \begin{cases} 2x - 4 & \text{if } x \geqslant 0 \\ -2x - 4 & \text{if } x < 0 \end{cases}$





13 $(-3, -5)$ and $(1, 3)$

REVIEW SET 15B

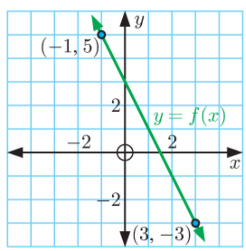
1 **a** Domain is $\{x \mid x > -6\}$. Range is $\{y \mid y > -2\}$.
b Domain is $\{x \mid x \leqslant 1\}$. Range is $\{y \mid y \in \mathbb{R}\}$.

2 **a** function **b** not a function

3 **a** -24 **b** $-5x - x^2$ **c** $-x^2 + 3x + 4$

4 **a** Domain is $\{x \mid x \neq -4\}$.
b Domain is $\{x \mid x \neq -5$ and $x \neq 1\}$.
c Domain is $\{x \mid 0 \leqslant x \leqslant 6\}$.

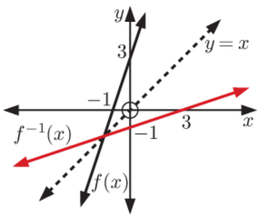
5 **a**



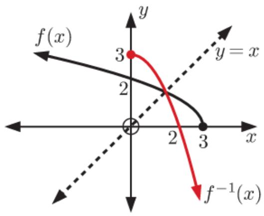
b $f(-1) = 5$, $f(3) = -3$
c $f(x) = -2x + 3$
d $f^{-1}(x) = \dfrac{3 - x}{2}$

6 **a** $f(g(x)) = 15 - 2x$ **b** $g(f(x)) = 6 - 2x$
c $f(g(-2)) = 19$

7 **a**



b

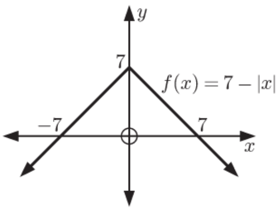


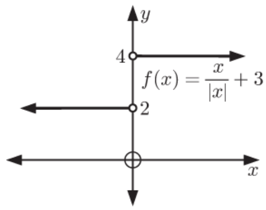
8 **a** 36 **b** 13 **c** $\frac{7}{4}$

9 **a** $f(g(x)) = \dfrac{7-x}{x-4}$ **b** 2 **c** $x = 10$

10 **a** $f^{-1}(x) = \dfrac{2x+1}{x-1}$ **c** 3

11 **a** $f(x) = \begin{cases} 7 - x & \text{if } x \geqslant 0 \\ 7 + x & \text{if } x < 0 \end{cases}$ **b** $f(x) = \begin{cases} 4 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \\ \text{undef.} & \text{if } x = 0 \end{cases}$





12 $(-1, -3)$ and $(\frac{3}{2}, 2)$

EXERCISE 16A

1 **a** 4 **b** 16 **c** 13

2 **a** 3, 7, 11, 15, 19, ... **b** 40, 35, 30, 25, 20, ...
c 2, 3, 5, 7, 11, ...

3 **a** The sequence starts at 5 and increases by 3 each time.
b $u_1 = 5, \quad u_n = u_{n-1} + 3 \quad \text{for} \quad n \geqslant 2$

4 **a** The nth term of the sequence is n squared. **b** $u_n = n^2$

5 **a** 19 **b** 19 **c** 96 **d** 27

6 **a** $u_6 = 14$ **b** 136 **c** $u_8 = -14$

7 **a** 7, 10, 13, 16 **b** 25, 21, 17, 13 **c** 5, 15, 45, 135
d 100, 50, 25, 12.5 **e** 3, 5, 9, 17
f 4, 6, 4, 6 **g** 3, 4, 12, 48

EXERCISE 16B

1 **a** arithmetic **b** not arithmetic
c not arithmetic **d** arithmetic

2 **a** $\square = 16$ **b** $\square = 34$ **c** $\square = 15, \ \triangle = 3$
d $\square = 13, \ \triangle = -5$

3 **a** $u_1 = 41, \ d = 1$ **b** $u_1 = 1, \ d = 11$
c $u_1 = 98, \ d = -10$ **d** $u_1 = 91, \ d = -9$

4 **a** common difference is 7 **b** $u_n = 7n - 3$
c $u_{30} = 207$ **d** yes, the 49th term **e** no

5 **a** $d = -4$ **b** $u_n = 71 - 4n$ **c** $u_{60} = -169$
d no **e** no

6 **a** $u_1 = 4$ and $d = 11$ **b** $u_{37} = 400$ **c** $u_{24} = 257$

7 **a** $k = 22$ **b** $k = 2\frac{1}{2}$ **c** $k = \frac{13}{3}$

8 **a** $k = 6$ **b** 29, 37

9 **a** $u_n = 5n + 17$ **b** $u_n = 10 - 4n$
c $u_n = 3n - 16$ **d** $u_n = -\frac{17}{2} - \frac{3}{4}n$

10 **a** $u_n = 6n - 8$ **b** $u_{30} = 172$

EXERCISE 16C

1 **a** geometric **b** not geometric **c** geometric
d not geometric

2 **a** 5 **b** $\frac{1}{2}$ **c** -2 **d** $-\frac{1}{10}$

3 **a** $b = 12, \ c = 24$ **b** $b = \frac{1}{2}, \ c = \frac{1}{8}$ **c** $b = \frac{5}{3}, \ c = -\frac{5}{9}$

4 **a** $r = 3$ **b** $u_n = 3^{n-1}$ **c** $u_{10} = 19\,683$

5 **a** $r = -\frac{1}{2}$ **b** $u_n = 40 \times (-\frac{1}{2})^{n-1}$ **c** $u_{12} = -\frac{5}{256}$

6 **a** $r = -\frac{1}{4}$ **b** $-0.000\,976\,562\,5$

7 $u_n = 3 \times 2^{\frac{n-1}{2}}$ or $3\left(\sqrt{2}\right)^{n-1}$

8 **a** $k = \frac{2}{3}$ **b** $k = 9$ **c** $k = 2$ **d** $k = \pm 9$
e $k = -7$ **f** $k = -\frac{8}{7}$ or 2

9 **a** $u_n = \frac{16}{9} \times 3^{n-1}$ **b** $u_n = 128 \times (-\frac{1}{2})^{n-1}$
c $u_n = 2 \times 5^{n-1}$ or $u_n = (-2) \times (-5)^{n-1}$
d $u_n = 6 \times (\pm\frac{1}{\sqrt{2}})^{n-1}$

10 $\pm\frac{3}{32}$

EXERCISE 16D

1 **a** $u_3 = 8$ **b** $S_3 = 16$ **c** $u_5 = 19$ **d** $S_5 = 46$

2 **a** **i** 5, 7, 9, 11 **ii** $S_4 = 32$ **3** $u_6 = 12$
b **i** $-5, -2, 3, 10$ **ii** $S_4 = 6$
c **i** 7, 14, 28, 56 **ii** $S_4 = 105$
d **i** 3, 5, 11, 29 **ii** $S_4 = 48$

4 **a** $S_1 = \frac{1}{2}, \ S_2 = \frac{2}{3}, \ S_3 = \frac{3}{4}, \ S_4 = \frac{4}{5}$ **b** $\frac{100}{101}$

EXERCISE 16E

1 **a, b, c** 75 **2** $S_{12} = 450$ **3** $S_{20} = 220$

4 **a** 210 **b** 75 **c** 4075 **d** 6780
e -2280 **f** -2400 **g** 275 **h** 387.5

5 2500 **6** **a** 24 **b** 1476

7 **a** 775 **b** 1705 **c** 969 **d** 1040 **e** -345 **f** 306

8 **a** $S_n = \frac{n}{2}(2u_1 + (n-1)d)$ where $u_1 = 4, \ d = 4$
$= \frac{n}{2}(8 + 4(n-1))$ etc.
b 840

9 **a** €65 **b** €1350

10 **a** $\dfrac{n(n^2+1)}{2}$ **b** $\dfrac{3(3^2+1)}{2} = \dfrac{3 \times 10}{2} = 15$ ✓ **c** 260

EXERCISE 16F.1

1 **a, b** 315

2 **a** 3280 **b** 4 882 812 **c** $\frac{1533}{32}$ **d** $63 + 63\sqrt{2}$
e -1364 **f** $\frac{1640}{27}$ **g** ≈ 52.2 **h** ≈ 12.8

3 **a** $u_1 = 2, \ r = 3$ **b** 59 048

5 **a** **i** ≈ 158 mL **ii** ≈ 37.5 mL **iii** 8.91 mL
b The amount of water Doug drinks is a geometric sequence with $r = \frac{3}{4}$.
c **i** ≈ 1887.4 mL **ii** ≈ 1993.7 mL **iii** ≈ 1999.6 mL
d If Doug followed the formula, the amount would eventually become too small to measure, and too small to sustain him. (In theory Doug would never run out of water.)

EXERCISE 16F.2

1 **a** diverge **b** converge **c** diverge **d** converge

2 **a** **i** ≈ 23.44 **ii** ≈ 26.53 **iii** ≈ 26.99
b 27 **c** $S = \dfrac{9}{1 - \frac{2}{3}} = 27$