Name:

1. (4 points) Given that $1+2i$ is a root of the polynomial

$$P(x) = 4x^4 - 24x^3 + 69x^2 - 114x + 85$$

find the other roots.

If 1+2i is a root, then so is 1-2i. Suppose that the other two roots are α and β. Using the formula for the sum of the roots we have that:

$$1 + 2i + 1 - 2i + \alpha + \beta = 6$$

and using the formula for the product we have:

$$(1+2i)(1-2i)\alpha\beta = \frac{85}{4}$$

Simplifying and rearranging we get

$$\begin{cases} \alpha + \beta = 4 \\ 4\alpha\beta = 17 \end{cases}$$

This gives:

$$4\alpha(4-\alpha) = 17$$

which gives:

$$4\alpha^2 - 16\alpha + 17 = 0$$

Solving the above we get $\alpha = 2 \pm \frac{1}{2}i$. Note that the equations are symmetric in α and β, so we get two solutions corresponding to the two roots.

Finally the roots are $1 \pm 2i$ and $2 \pm \frac{1}{2}i$.

2. (6 points)

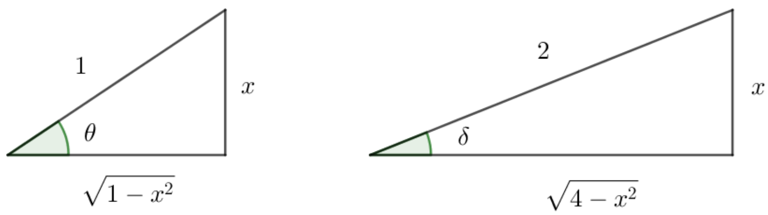
(a) Show that:

$$\cos\left(\arcsin x + \arcsin\left(\frac{x}{2}\right)\right) = \frac{\sqrt{1-x^2}\sqrt{4-x^2} - x^2}{2}$$

We let $\theta = \arcsin x$ and $\delta = \arcsin(\frac{x}{2})$. So $\sin\theta = x$ and $\sin\delta = \frac{x}{2}$. We want to calculate

$$\cos(\theta + \delta) = \cos\theta\cos\delta - \sin\theta\sin\delta$$

We can draw two triangles representing θ and δ and calculate the remaining side:



This gives:

$$\begin{aligned}\cos(\theta + \delta) &= \cos\theta\cos\delta - \sin\theta\sin\delta = \\ &= \sqrt{1-x^2} \times \frac{\sqrt{4-x^2}}{2} - x \times \frac{x}{2} = \\ &= \frac{\sqrt{1-x^2}\sqrt{4-x^2} - x^2}{2}\end{aligned}$$

□

(b) **Hence** find the value of

$$\cos\left(\arcsin\left(\frac{3}{\sqrt{21}}\right)+\arcsin\left(\frac{3}{2\sqrt{21}}\right)\right)$$

We just need to put $x=\frac{3}{\sqrt{21}}$ into the previous formula:

$$\cos\left(\arcsin\left(\frac{3}{\sqrt{21}}\right)+\arcsin\left(\frac{3}{2\sqrt{21}}\right)\right)=\frac{\sqrt{1-\frac{3}{7}}\sqrt{4-\frac{3}{7}}-\frac{3}{7}}{2}=$$
$$=\frac{\sqrt{\frac{4}{7}}\sqrt{\frac{25}{7}}-\frac{3}{7}}{2}=\frac{1}{2}$$

(c) **Hence** write down the value of

$$\arcsin\left(\frac{3}{\sqrt{21}}\right)+\arcsin\left(\frac{3}{2\sqrt{21}}\right)$$

Note that $\frac{3}{\sqrt{21}}<\frac{3}{\sqrt{18}}=\frac{1}{\sqrt{2}}$, so $\arcsin(\frac{3}{\sqrt{21}})<\frac{\pi}{4}$, of course we also have $\arcsin(\frac{3}{2\sqrt{21}})<\frac{\pi}{4}$, which means that $\arcsin(\frac{3}{\sqrt{21}})+\arcsin(\frac{3}{2\sqrt{21}})$ is in the first quadrant and since it's cosine is equal to $\frac{1}{2}$ we have

$$\arcsin\left(\frac{3}{\sqrt{21}}\right)+\arcsin\left(\frac{3}{2\sqrt{21}}\right)=\frac{\pi}{3}$$

3. (4 points) Solve:

$$3 + 3\cos x = 2\sin^2 x$$

for $0 \leqslant x \leqslant 4\pi$.

We use Pythagorean identity to get:

$$3 + 3\cos x = 2(1 - \cos^2 x)$$

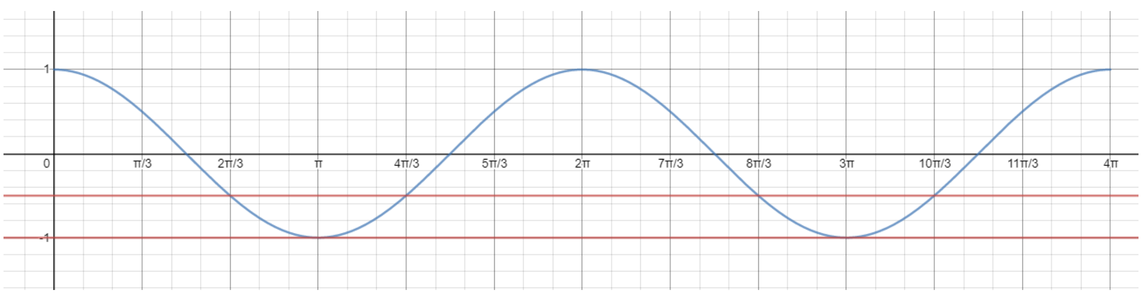
which gives:

$$2\cos^2 x + 3\cos x + 1 = 0$$

Factorize to get:

$$(2\cos x + 1)(\cos x + 1) = 0$$

So $\cos x == -\frac{1}{2}$ or $\cos x = -1$. We draw the graph of $\cos x$ for $0 \leqslant x \leqslant 4\pi$.



We get a total of six solutions:

$$x \in \left\{\frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{8\pi}{3}, 3\pi, \frac{10\pi}{3}\right\}$$

4. (6 points)

(a) Show that $\dfrac{1}{\sin^2 x} + \dfrac{1}{\cos^2 x} = \dfrac{4}{\sin^2 2x}$.

Starting from left hand side:

$$\begin{aligned} LHS &= \frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \\ &= \frac{\cos^2 x + \sin^2 x}{\sin^2 x \cos^2 x} = \\ &= \frac{1}{\sin^2 x \cos^2 x} = \\ &= \frac{4}{4\sin^2 x \cos^2 x} = \\ &= \frac{4}{(2\sin x \cos x)^2} = \\ &= \frac{4}{\sin^2 2x} = RHS \end{aligned}$$

□

(b) Hence find the exact solutions to the equation

$$\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \frac{16}{3}$$

for $-\pi < x < \pi$.

Replacing the left hand side using the above identity we get:

$$\frac{4}{\sin^2 2x} = \frac{16}{3}$$

solving this results in:

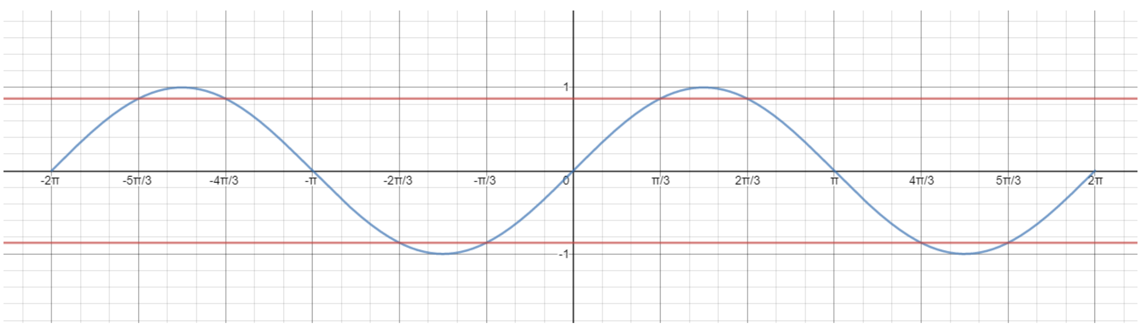
$$\sin 2x = \pm\frac{\sqrt{3}}{2}$$

Now letting $\alpha = 2x$ we want to solve:

$$\sin \alpha = \pm\frac{\sqrt{3}}{2}$$

for $-2\pi \leqslant \alpha \leqslant 2\pi$.

We draw the graph of $\sin \alpha$ in the given interval:



And the solutions we get for α are:

$$\alpha \in \left\{-\frac{5\pi}{3}, -\frac{4\pi}{3}, -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$$

which gives the following solutions for x $(x = \frac{\alpha}{2})$

$$x \in \left\{-\frac{5\pi}{6}, -\frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}\right\}$$