Name:

1. (4 points) Solve the inequality

$$|2x-3| \leqslant |x-1|$$

Squaring both sides we get:

$$(2x-3)^2 \leqslant (x-1)^2$$

which gives:

$$4x^2 - 12x + 9 \leqslant x^2 - 2x + 1$$

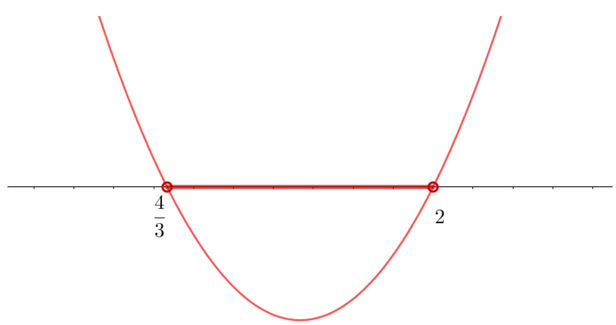
this gives:

$$3x^2 - 10x + 8 \leqslant 0$$

by factoring we get:

$$(3x-4)(x-2) \leqslant 0$$

quick sketch:



and we get that $\frac{4}{3} \leqslant x \leqslant 2$.

2. (4 points) Consider a function $f(x)$ with a domain D such that if $x \in D$, then $-x \in D$.

Let $g(x) = \dfrac{f(x) + f(-x)}{2}$

(a) Show that $g(x)$ is a an even function.

$$g(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(-x) + f(x)}{2} = g(x)$$

so $g(x)$ is even.

Let $h(x) = \dfrac{f(x) - f(-x)}{2}$

(b) Show that $h(x)$ is a an odd function.

$$h(-x) = \frac{f(-x) - f(-(-x))}{2} = \frac{f(-x) - f(x)}{2} = -\frac{f(x) - f(-x)}{2} = -h(x)$$

so $h(x)$ is odd.

Let $f(x) = \dfrac{1}{x+2}$ with the domain $D = \mathbb{R} - \{-2, 2\}$.

(c) Using parts (a) and (b) express $f(x)$ as a sum of an even and an odd function, simplify your answer.

Note that parts (a) and (b) show us how to do it since $f(x) = g(x) + h(x)$.

In our example we have

$$g(x) = \frac{\frac{1}{x+2} + \frac{1}{-x+2}}{2} = \frac{2}{4 - x^2}$$

Similarly:

$$h(x) = \frac{\frac{1}{x+2} - \frac{1}{-x+2}}{2} = \frac{x}{x^2-4}$$

So f can be written as:

$$f(x) = \frac{2}{4-x^2} + \frac{x}{x^2-4}$$

and we know that $\dfrac{2}{4-x^2}$ is even and $\dfrac{x}{x^2-4}$ is odd.

3. (6 points) Consider a polynomial

$$P(x) = 4x^3 + 4x^2 - 3x - 3$$

(a) Show that -1 is a root of $P(x)$ and hence, or otherwise, find all solutions to the equation $P(x) = 0$.

$$P(-1) = 4(-1)^3 + 4(-1)^2 - 3(-1) - 3 = 0$$

So -1 is a root.

Applying synthetic division or simply factoring we get:

$$\begin{aligned} P(x) =& 4x^3 + 4x^2 - 3x - 3 = \\ =& 4x^2(x+1) - 3(x+1) = \\ =& (x+1)(4x^2-3) = (x+1)(2x-\sqrt{3})(2x+\sqrt{3}) \end{aligned}$$

So the the solutions to $P(x) = 0$ are $-1, \dfrac{\sqrt{3}}{2}$ and $-\dfrac{\sqrt{3}}{2}$.

(b) Show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ for all θ.

$$\begin{aligned} LHS =& \sin 3\theta = \\ =& \sin(\theta + 2\theta) = \\ =& \sin\theta\cos 2\theta + \sin 2\theta\cos\theta = \\ =& \sin\theta(1 - 2\sin^2\theta) + 2\sin\theta\cos^2\theta = \\ =& \sin\theta - 2\sin^3\theta + 2\sin\theta(1-\sin^2\theta) = \\ =& 3\sin\theta - 4\sin^3\theta = RHS \end{aligned}$$

□

(c) Hence solve

$$4\sin^2\theta - 3 = \sin 3\theta$$

for $0 \leqslant \theta \leqslant 2\pi$.

Using the identity from part (b) we get:

$$4\sin^2\theta - 3 = 3\sin\theta - 4\sin^3\theta$$

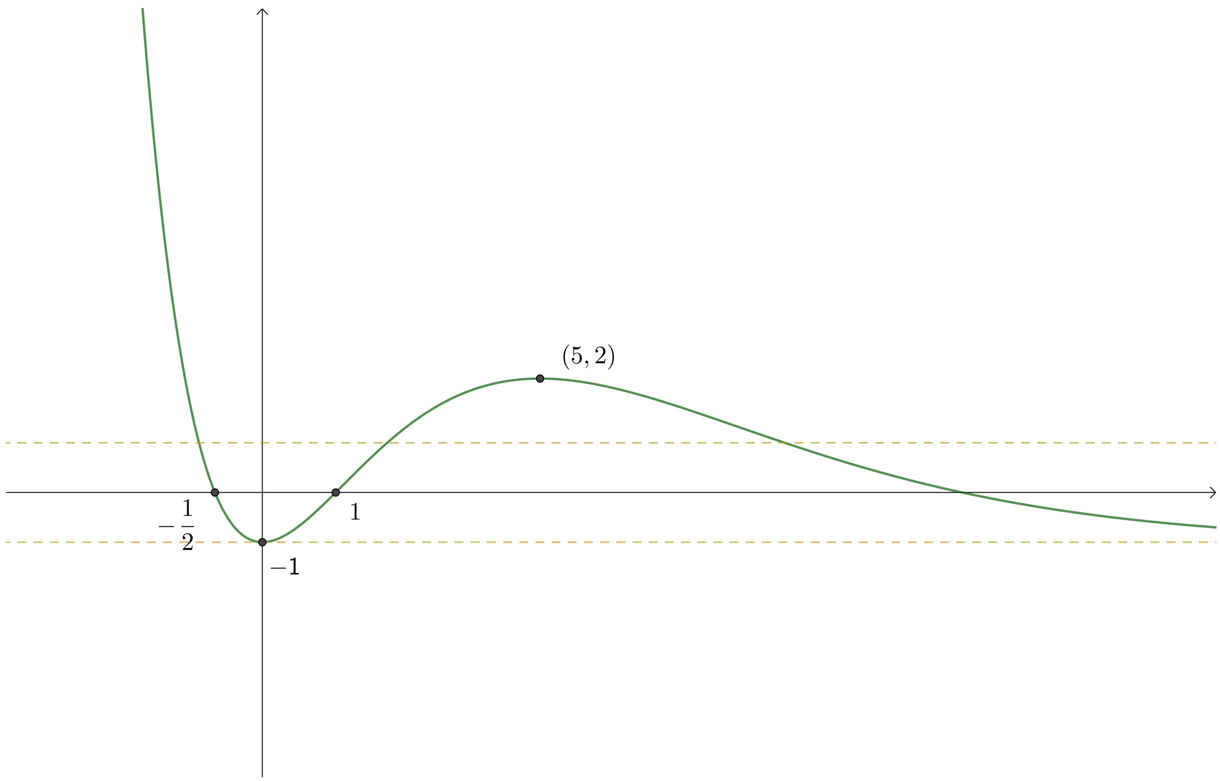
Moving all terms to one side gives:

$$4\sin^3\theta + 4\sin^2\theta - 3\sin\theta - 3 = 0$$

We know that the above equation has solutions $\sin\theta = -1$ or $\sin\theta = \frac{\sqrt{3}}{2}$ or $\sin\theta = -\frac{\sqrt{3}}{2}$ (from part (a)). Now we solve these for $0 \leqslant \theta \leqslant 2\pi$ and get:

$$\theta \in \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}\right\}$$

4. (6 points) Consider the following graph of $y = f(x)$

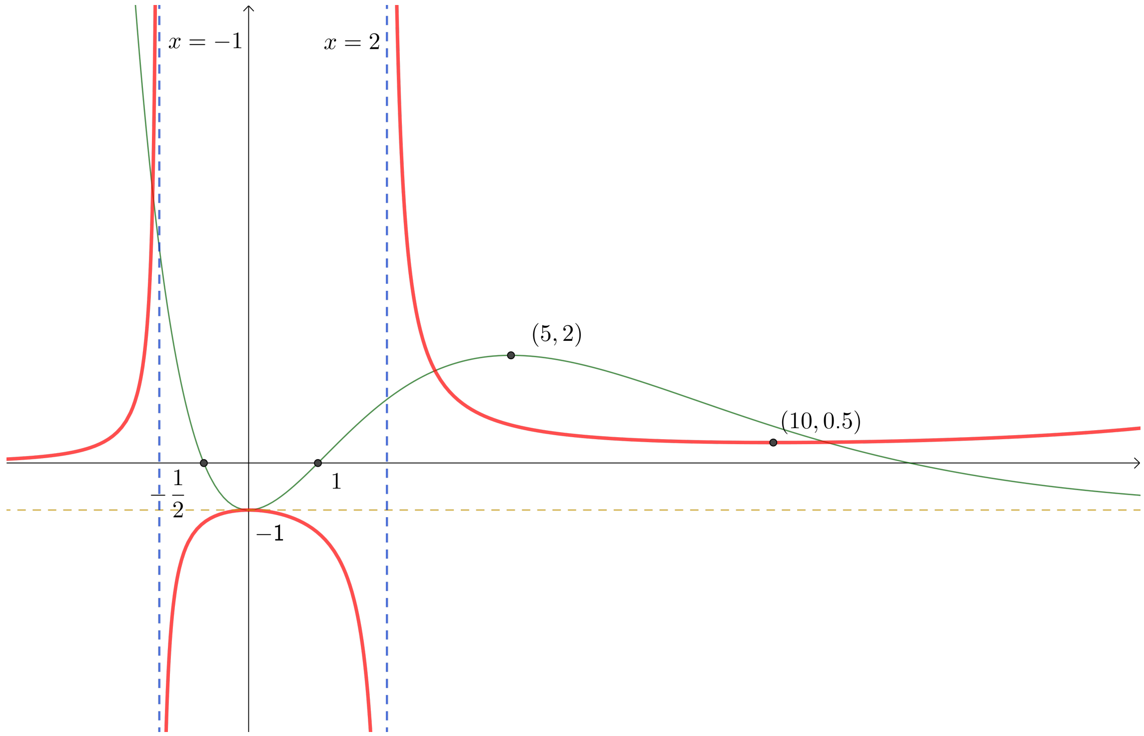


The dotted lines represent lines $y = 1$ and $y = -1$. The latter being the horizontal asymptote of the graph of $f(x)$. Use the diagrams on the next page to sketch the graphs of

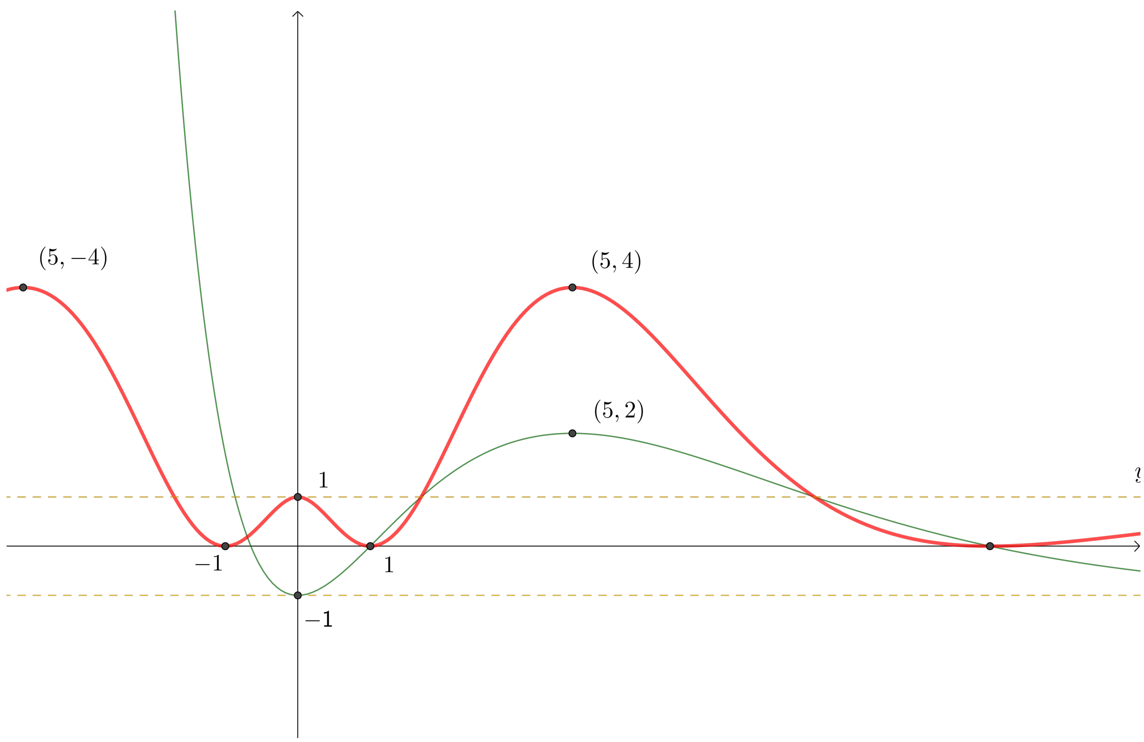
(a) $g(x) = \dfrac{1}{f(\frac{1}{2}x)}$

(b) $h(x) = (f(|x|))^2$.

Clearly indicate axes intercepts, asymptotes and maxima and minima.



$x = -1$
$x = 2$
$(5, 2)$
$(10, 0.5)$
1
$-\frac{1}{2}$
-1



$(5, -4)$
$(5, 4)$
$(5, 2)$
1
-1
1
-1