Name:

1. (3 points) Solve the equation

$$2^{2x+1} + 4 = 9 \times 2^x$$

We rewrite the equation in the form:

$$2 \times (2^x)^2 - 9 \times 2^x + 4 = 0$$

Now we can set $t = 2^x$ and solve the quadratic:

$$2t^2 - 9t + 4 = 0$$

Factoring gives:

$$(2t - 1)(t - 4) = 0$$

So $t = \frac{1}{2}$ or $t = 4$.

This gives $2^x = \frac{1}{2}$ or $2^x = 4$.

So $x = -1$ or $x = 2$.

2. (7 points) Consider the function

$$f(x) = \frac{x^2}{2x+1}$$

(a) Write down the equations of all the asymptotes of the graph of $y = f(x)$.

The vertical asymptote is of course $x = -\frac{1}{2}$. To get the oblique asymptote we can use synthetic or long division to get:

$$f(x) = \frac{1}{2}x - \frac{1}{4} + \frac{\frac{1}{4}}{2x+1}$$

So the oblique asymptote is the line $y = \frac{1}{2}x - \frac{1}{4}$.

(b) Prove algebraically that the range of $y = f(x)$ is $y \in]-\infty, -1] \cup [0, \infty[$.

We want to find values of y for which the equation:

$$y = \frac{x^2}{2x+1}$$

has a solution.

Rearrange to get:

$$x^2 - 2yx - y = 0$$

This is a quadratic in x, it has a solution if the discriminant is non-negative:

$$\Delta = (-2y)^2 - 4(1)(-y) = 4y^2 + 4y$$

$$4y^2 + 4y \geq 0$$

$$4y(y+1) \geq 0$$

This gives $y \in \in]-\infty, -1] \cup [0, \infty[$ as required.

□

(c) Solve $f(x) = 0$ and $f(x) = -1$.

$f(x) = 0$ gives $x = 0$.

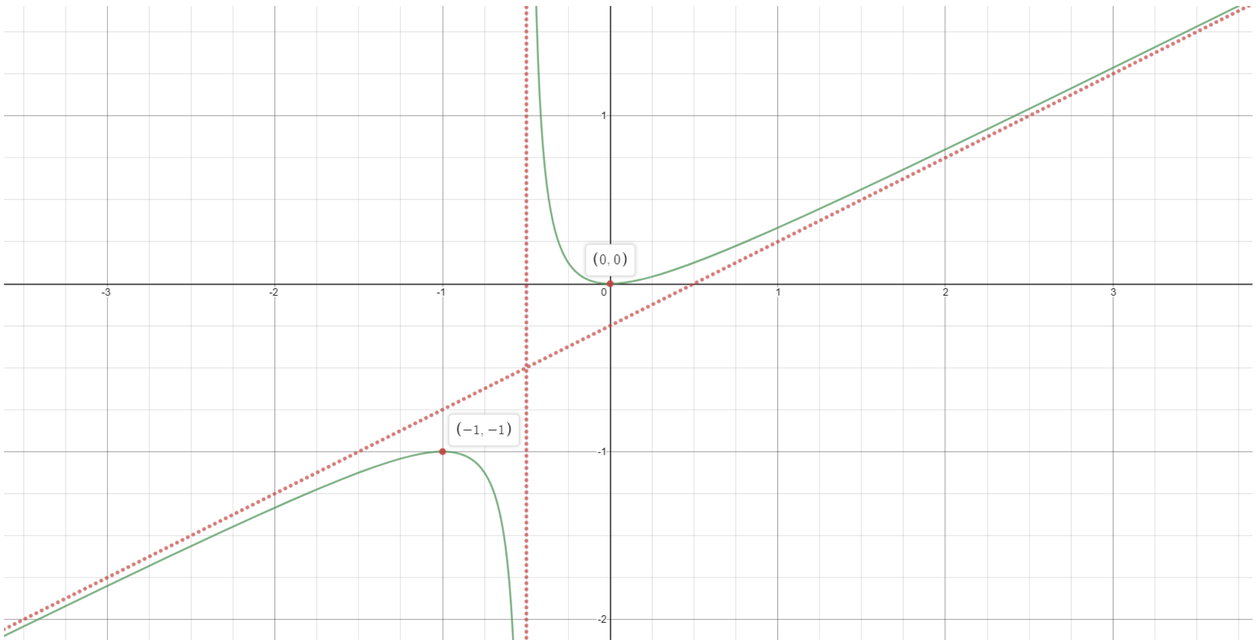
$f(x) = -1$ gives:

$$\frac{x^2}{2x+1} = -1$$

so $x^2 + 2x + 1 = 0$ and this gives $x = -1$.

(d) Sketch the graph of $y = f(x)$.

We need to draw the asymptotes and the points found in part (c). Remember that the function is not defined for $-1 < y < 0$. You should also analyse the sign of the function. In the end we get the following graph:



3. (10 points)

(a) Use the formula

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

to solve the equation

$$\sin 4x - \cos x = 0$$

for $0 \le x \le \frac{\pi}{2}$.

We need to change $\cos x$ into $\sin\left(\frac{\pi}{2} - x\right)$ in order to apply the formula. We get:

$$\sin 4x - \sin\left(\frac{\pi}{2} - x\right) = 0$$

which gives:

$$2\sin\left(\frac{5x - \frac{\pi}{2}}{2}\right)\cos\left(\frac{3x + \frac{\pi}{2}}{2}\right) = 0$$

If we substitute $\alpha = \frac{5x - \frac{\pi}{2}}{2}$ and $\beta = \frac{3x + \frac{\pi}{2}}{2}$.

We know want to solve

$$2\sin\alpha\cos\beta = 0$$

for $-\frac{\pi}{4} \le \alpha \le \pi$ and $\frac{\pi}{4} \le \beta \le \pi$. We get three possibilities:

$\alpha = 0$ or $\alpha = \pi$ or $\beta = \frac{\pi}{2}$.

This gives the following values of x:

$x = \frac{\pi}{10}$ or $x = \frac{\pi}{2}$ or $x = \frac{\pi}{6}$.

(b) Show that

$$\sin 4x = 4\sin x\cos x - 8\sin^3 x\cos x$$

for all x.

We use double angle formula three times:

$$\begin{aligned} LHS = \sin 4x &= 2\sin 2x\cos 2x = \\ &= 4\sin x\cos x(1-2\sin^2 x) = \\ &= 4\sin x\cos x - 8\sin^3 x\cos x = RHS \end{aligned}$$

□

(c) Show that $x = \frac{1}{2}$ is a solution to the equation

$$-8x^3 + 4x - 1 = 0$$

and find the other solutions.

$$-8\left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right) - 1 = -1 + 2 - 1 = 0$$

so $x = \frac{1}{2}$ is a solution.

Using synthetic division we can rewrite the above equation as:

$$\left(x - \frac{1}{2}\right)(-8x^2 - 4x + 2) = 0$$

or even better:

$$(1-2x)(4x^2 + 2x - 1) = 0$$

We need to solve the quadratics. We get $x = \frac{-1 \pm \sqrt{5}}{4}$.

(d) Using all the previous parts, or otherwise, find the exact value of $\sin\left(\frac{\pi}{10}\right)$. Justify your answer.

Now the fun begins.

Going back to the original equation:

$$\sin 4x - \cos x = 0$$

We can use part (b) to write it as:

$$4\sin x\cos x - 8\sin^3 x\cos x - \cos x = 0$$

We can now factor $\cos x$ to get:

$$\cos x(-8\sin^3 x + 4\sin x - 1) = 0$$

One possibility is $\cos x = 0$, the other is that the expression in brackets is 0.

Now this expression in brackets looks very much like the polynomial in part (c).

So we know (using our answers to part (c)) that the solutions to this polynomial are $\sin x = \frac{1}{2}$ or $\sin x = \frac{-1\pm\sqrt{5}}{4}$.

But from part (a) we know that the solutions to our original equation (for $0 \leq x \leq \frac{\pi}{2}$) are $x = \frac{\pi}{10}$ or $x = \frac{\pi}{2}$ or $x = \frac{\pi}{6}$.

Given that $\frac{\pi}{10}$ is in the I quadrant so it's sine is positive we get that

$$\sin\left(\frac{\pi}{10}\right) = \frac{-1+\sqrt{5}}{4}$$