Name:

1. (5 points) Consider the curve given by the equation:

$$\arcsin x + \arctan y = \frac{\pi}{2}$$

Find the gradient of the curve when $x = \frac{1}{2}$.

We start by finding the point with the first coordinate $x = \frac{1}{2}$. We need to solve:

$$\arcsin \frac{1}{2} + \arctan y = \frac{\pi}{2}$$

Which gives:

$$\arctan y = \frac{\pi}{3}$$

So $y = \sqrt{3}$.

Now we take the derivative with respect to x of both sides of the equation and we get:

$$\frac{1}{\sqrt{1-x^2}} + \frac{1}{1+y^2}\frac{dy}{dx} = 0$$

Rearranging we get:

$$\frac{dy}{dx} = -\frac{1+y^2}{\sqrt{1-x^2}}$$

Now we substitute $x = \frac{1}{2}$ and $y = \sqrt{3}$ and we get

$$\frac{dy}{dx} = -\frac{8}{\sqrt{3}}$$

2. (5 points) Consider the tangent to the graph of $y = \frac{1}{x}$ at $x = a$ for $a > 0$. Show that the area of the triangle enclosed by this tangent and the axes is independent of a and calculate this area.

The point of the curve has coordinates $P\left(a, \frac{1}{a}\right)$.

We have

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

So the gradient at P is $-\frac{1}{a^2}$. Which gives the tangent line:

$$y - \frac{1}{a} = -\frac{1}{a^2}(x - a)$$

Rearranging into gradient y-intercept form we get:

$$y = -\frac{1}{a^2}x + \frac{2}{a}$$

The y-intercept of this line is of course $(0, \frac{2}{a})$, the x-intercept is $(2a, 0)$. The area of the triangle is then:

$$Area = \frac{1}{2} \times 2a \times \frac{2}{a} = 2$$

which is of course independent of a.

□

3. (5 points)

(a) Show that

$$\sin(\arccos x) = \sqrt{1-x^2}$$

Let $\alpha = \arccos x$ with $0 \leqslant \alpha \leqslant \pi$. So we have $\cos\alpha = x$. We need to find $\sin\alpha$. We can draw an appropriate triangle or apply Pythagorean identity:

$$x^2 + \sin^2\alpha = 1$$

Which gives $\sin\alpha = \sqrt{1-x^2}$ as required (since sine is positive in the first two quadrants).

□

(b) Show that

$$\sin(2\arccos x) = 2x\sqrt{1-x^2}$$

We apply the double angle identity and use the previous part:

$$LHS = \sin(2\arccos x) = 2\sin(\arccos x)\cos(\arccos x) = 2\sqrt{1-x^2}\times x = RHS$$

(c) Hence or otherwise solve:

$$\sin(\arccos x) = \sin(2\arccos x)$$

We need to solve:

$$\sqrt{1-x^2} = 2x\sqrt{1-x^2}$$

Which gives:

$$\sqrt{1-x^2}(1-2x) = 0$$

So we get $x = 1$ or $x = -1$ or $x = \frac{1}{2}$.

4. (5 points) Consider the polynomial equation:

$$2x^3 + Ax^2 + Bx + C = 0$$

$\frac{1}{2}$ and $2+3i$ are solutions to this equation.

(a) Write down the third solution.

By the Conjugate Root Theorem the third solutions is $2-3i$.

(b) Find A, B and C.

The leading coefficient is 2 so we can write the polynomial as:

$$\begin{aligned} &(2x-1)(x-(2+3i))(x-(2-3i)) = \\ =&(2x-1)(x-2-3i)(x-2+3i) = \\ =&(2x-1)((x-2)^2+9) = \\ =&(2x-1)(x^2-4x+13) = \\ =&2x^3-9x^2+30x-13 \end{aligned}$$

So $A=-9, B=30$ and $C=-13$.

We could have also applied formulas for the sum, product, and sum of products.

(c) Find solutions to the equation:

$$2 + Ax + Bx^2 + Cx^3 = 0$$

Note that this equation is very similar to the previous one. We can get one from the other by setting $x = \frac{1}{t}$:

$$2 + \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t^3} = 0$$

Multiplying both sides by t^3 we get:

$$2t^3 + At^2 + Bt + C = 0$$

We know the solutions to the above: $t = \frac{1}{2}, \quad t = 2 + 3i, \quad t = 2 - 3i.$

So the solutions to our equation are $x = 2,\ x = \frac{1}{2+3i},\ x = \frac{1}{2-3i}.$

We should rewrite the solutions as $x = 2,\ x = \frac{2}{13} - \frac{3}{13}i,\ x = \frac{2}{13} + \frac{3}{13}i.$