Name:

1. (4 points) Consider the following infinite series:

$$\ln x + (\ln x)^2 + (\ln x)^3 + \ldots$$

(a) Find the values of x for which the series converges.

We have a geometric series with $u_1 = \ln x$ and $r = \ln x$. An infinite geometric series converges it $|r| < 1$. So we must have:

$$|\ln x| < 1$$

which is equivalent to

$$-1 < \ln x < 1$$

and this gives:

$$\frac{1}{e} < x < e$$

(b) Find the value of x for which the sum of the series is 1.

For a convergent geometric series $S_\infty = \frac{u_1}{1-r}$. So in our case we have:

$$\frac{\ln x}{1 - \ln x} = 1$$

which gives $\ln x = \frac{1}{2}$, so $x = \sqrt{e}$.

2. (5 points) Let $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$ with $\tan\alpha = \frac{2}{3}$. Calculate:

(i) $\sin\alpha$

Using the identity $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$, we get that $\cos\alpha = \frac{3}{2}\sin\alpha$. Substituting this into Pythagorean identity we get:

$$\sin^2\alpha + \frac{9}{4}\sin^2\alpha = 1$$

Which gives $\sin\alpha = \pm\frac{2}{\sqrt{13}}$ and since we're in III quadrant sine is negative so $\sin\alpha = -\frac{2}{\sqrt{13}}$

(ii) $\sin 2\alpha$

Using the above we calculate that $\cos\alpha = -\frac{3}{\sqrt{13}}$, so we have:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = \frac{12}{13}$$

(iii) $\sin 3\alpha$

$$\sin 3\alpha = \sin 2\alpha\cos\alpha + \sin\alpha\cos 2\alpha$$

So we need to calculate $\cos 2\alpha$.

$$\cos 2\alpha = 1 - 2\sin^2\alpha = \frac{5}{13}$$

So now we have:

$$\sin 3\alpha = \sin 2\alpha\cos\alpha + \sin\alpha\cos 2\alpha = -\frac{46}{13\sqrt{13}}$$

3. (5 points) Find the following limits:

(a) $\lim\limits_{x\to\infty} \dfrac{2x^2+2x+1}{1-x^2}$

$$\lim_{x\to\infty} \frac{2x^2+2x+1}{1-x^2} = \lim_{x\to\infty} \frac{x^2(2+\frac{2}{x}+\frac{1}{x^2})}{x^2(\frac{1}{x^2}-1)} = \frac{2}{-1} = -2$$

(b) $\lim\limits_{x\to 3} \dfrac{x^2-4x+3}{x^2-5x+6}$

$$\lim_{x\to 3} \frac{x^2-4x+3}{x^2-5x+6} = \lim_{x\to 3} \frac{(x-3)(x-1)}{(x-3)(x-2)} = \frac{2}{1} = 2$$

(c) $\lim\limits_{x\to 0} \dfrac{\sqrt{4+x}-2}{x}$

$$\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} = \lim_{x\to 0} \frac{(\sqrt{4+x}-2)(\sqrt{4+x}+2)}{x(\sqrt{4+x}+2)} = \lim_{x\to 0} \frac{x}{x(\sqrt{4+x}+2)} = \frac{1}{4}$$

(d) $\lim\limits_{x\to -\infty} \dfrac{\sqrt{x^2+3x}-3x}{x+1}$

$$\lim_{x\to -\infty} \frac{\sqrt{x^2+3x}-3x}{x+1} = \lim_{x\to -\infty} \frac{|x|\sqrt{1+\frac{3}{x}}-3x}{x(1+\frac{1}{x})} = \lim_{x\to -\infty} \frac{-x(\sqrt{1+\frac{3}{x}}+3)}{x(1+\frac{1}{x})} = \frac{-4}{1} = -4$$

(e) $\lim\limits_{x\to 0} \dfrac{\sin^2(2x)}{x^2} = \lim\limits_{x\to 0} \left(\dfrac{\sin(2x)}{x}\right)^2 = \lim\limits_{x\to 0} \left(2 \times \dfrac{\sin(2x)}{2x}\right)^2 = 2^2 = 4$

4. (6 points) Consider the function

$$f(x) = \ln x + \ln(x-1) - \ln(x^2-1)$$

(a) Find the domain of $f(x)$.

We need to have $x > 0$ and $x - 1 > 0$ and $x^2 - 1 > 0$. So in the end we get that the domain is $x > 1$.

(b) Write $f(x)$ in the form $\ln\left(\frac{x}{x+a}\right)$, where a is a constant to be found.

Combining logarithms we get:

$$f(x) = \ln x + \ln(x-1) - \ln(x^2-1) = \ln\left(\frac{x(x-1)}{x^2-1}\right) = \ln\left(\frac{x(x-1)}{(x+1)(x-1)}\right) = \ln\left(\frac{x}{x+1}\right)$$

so $a = 1$.

(c) Find the inverse of $f(x)$.

We have:

$$y = \ln\left(\frac{x}{x+1}\right)$$

which gives:

$$e^y = \frac{x}{x+1}$$

Solving for x gives $x = \frac{e^y}{1-e^y}$. So $f^{-1}(x) = \frac{e^x}{1-e^x}$.