Name:

1. (5 points) Differentiate from the first principles the following functions:

(a) $f(x) = x^2 - 3x$

$$\lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - 3(x+\Delta x) - x^2 + 3x}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x(\Delta x) + (\Delta x)^2 - 3\Delta x}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x - 3) = 2x - 3$$

So $f'(x) = 2x - 3$.

(b) $g(x) = \dfrac{2}{\sqrt{x}}$

$$\lim_{\Delta x \to 0} \frac{\frac{2}{\sqrt{x+\Delta x}} - \frac{2}{\sqrt{x}}}{\Delta x} = \lim_{\Delta x \to 0} \frac{2(\sqrt{x} - \sqrt{x+\Delta x})}{\Delta x\sqrt{x}\sqrt{x+\Delta x}} =$$

$$= \lim_{\Delta x \to 0} \frac{2(\sqrt{x} - \sqrt{x+\Delta x})(\sqrt{x} + \sqrt{x+\Delta x})}{\Delta x\sqrt{x}\sqrt{x+\Delta x}(\sqrt{x} + \sqrt{x+\Delta x})} =$$

$$= \lim_{\Delta x \to 0} \frac{-2\Delta x}{\Delta x\sqrt{x}\sqrt{x+\Delta x}(\sqrt{x} + \sqrt{x+\Delta x})} =$$

$$= \lim_{\Delta x \to 0} \frac{-2}{\sqrt{x}\sqrt{x+\Delta x}(\sqrt{x} + \sqrt{x+\Delta x})} = -\frac{1}{x\sqrt{x}}$$

So $g'(x) = -\dfrac{1}{x\sqrt{x}}$.

2. (4 points) Consider the following function:

$$f(x) = \begin{cases} x^3 & for \;\; x < 1 \\ ax + b & for \;\; x \geqslant 1 \end{cases}$$

Find the values of a and b so that f is differentiable at $x = 1$.

First of all the function needs to be continuous at $x = 1$, so we must have:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 1$$

This gives the equation

$$a + b = 1$$

Secondly the derivatives must be equal as they approach 1. For $x < 1$ we have $f'(x) = 3x^2$ and for $x > 1$ we have $f'(x) = a$.

This gives $a = 3$, so $b = -2$.

3. (4 points) Solve the following equation:

$$\cos x + \cos\frac{x}{2} + 1 = 0$$

for $0 \leqslant x \leqslant 3\pi$.

Using double angle formula for cosine we get:

$$2\cos^2\frac{x}{2} - 1 + \cos\frac{x}{2} + 1 = 0$$

Cancel the 1s and factor out the cosine:

$$\cos\frac{x}{2}\left(2\cos\frac{x}{2} + 1\right) = 0$$

Letting $\alpha = \dfrac{x}{2}$ we solve:

$$\cos\alpha = 0 \qquad \text{or} \qquad \cos\alpha = -\frac{1}{2}$$

for $0 \leqslant \alpha \leqslant \dfrac{3\pi}{2}$.

This gives

$$\alpha \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$$

So we get

$$x \in \left\{\pi, \frac{4\pi}{3}, \frac{8\pi}{3}, 3\pi\right\}$$

4. (7 points) Consider the function

$$f(x) = \frac{x^2 - 4}{x - 1}$$

(a) Write down the equations of the asymptotes of the graph of $y = f(x)$.

The vertical asymptote is of course

$$x = 1$$

For the oblique asymptote we need to perform the division (synthetic or long, both work well) and we get that the oblique asymptote is

$$y = x + 1$$

(b) Show that the range of values of $f(x)$ is all real numbers.

We start with

$$y = \frac{x^2 - 4}{x - 1}$$

Rearranging gives:

$$x^2 - yx + y - 4 = 0$$

We want to show that this equation will have solutions for x for any value of y. Treating the equation as a quadratic in x we have the discriminant:

$$\Delta = y^2 - 4y + 16 = (y - 2)^2 + 12 > 0$$

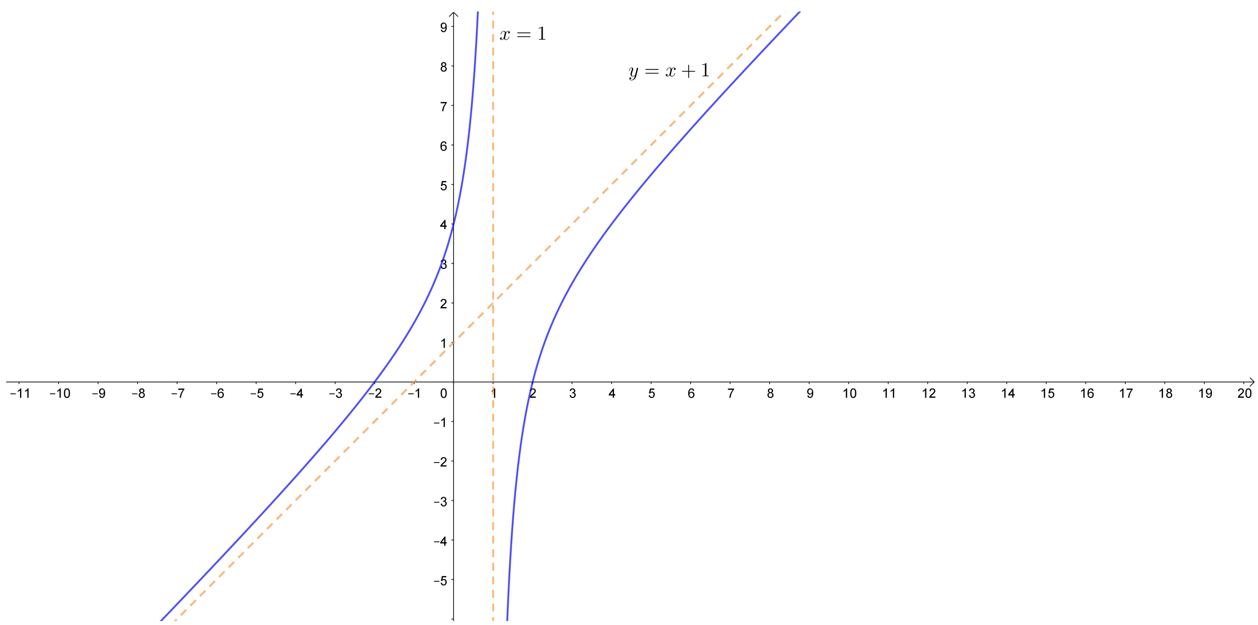
The discriminant is always positive, so the equation will always have two solutions for x, so the range of y is all real numbers. □

(c) Sketch the graph of $g(x) = f(|x|)$ and hence state the set of all possible values of parameter k, such that the equation:

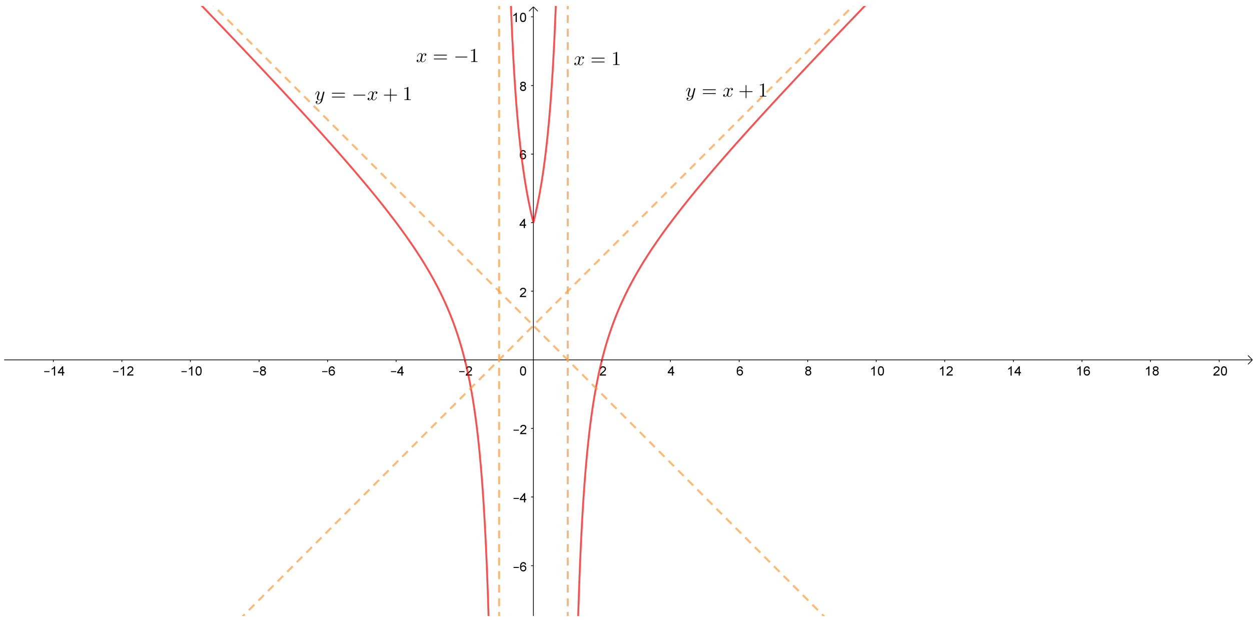
$$g(x) = k$$

has four solutions.

We will sketch $y = f(x)$ first. We already have the asymptotes. We know the range. The zeroes are of course $x = \pm 2$. We can analyse the sign of the function and we get the following graph:



Now the graph of $g(x)$ will look as follows:



Now $y = k$ is a horizontal line. So we need the value of k so that the horizontal line $y = k$ will intersect the graph of g four times. The graph of g clearly shows that we must have $k > 4$.