4 Evaluate $f^{-1}(5)$ where

a $f(x) = 6 - x$ **b** $f(x) = \frac{10}{x+7}$ **c** $f(x) = \frac{2}{4x-3}$

5 If $f(x) = \frac{x+1}{x-2}$, find $f^{-1}(x)$.

Note that the image of point $(a, -b)$ after a reflection in the line $y = x$ is the point $(b, -a)$.

EXAM-STYLE QUESTION

6 **a** Draw the graph of $f(x) = 2^x$ by making a table of values and plotting several points.

b Draw the line $y = x$ on the same graph.

c Draw the graph of f^{-1} by reflecting the graph of f in the line $y = x$.

d State the domain and range of f and f^{-1}.

7 The function $f(x) = x^2$ has no inverse function. However, the square root function $g(x) = \sqrt{x}$ does have an inverse function. Find this inverse.

By comparing the range and domain explain why the inverse of $g(x) = \sqrt{x}$ is not the same as $f(x) = x^2$.

8 Prove that the graphs of a linear function and its inverse can never be perpendicular.

Extension material on CD:
Worksheet 1 - Polynomials

1.6 Transforming functions

Investigation – functions

You should use your GDC to sketch all the graphs in this investigation.

1 Sketch $y = x$, $y = x + 1$, $y = x - 4$, $y = x + 4$ on the same axes.
Compare and contrast your functions.
What effect do the constant (number) terms have on the graphs of $y = x + b$?

You will also find this standard equation of a line written as $y = mx + b$ or $y = mx + c$

2 Sketch $y = x + 3$, $y = 2x + 3$, $y = 3x + 3$, $y = -2x + 3$, $y = 0.5x + 3$ on the same axes.
Compare and contrast your functions.
What effect does changing the x-coefficient have?

The coefficient of x is the number that multiplies the x-value.

3 Sketch $y = |x|$, $y = |x + 2|$, $y = |x - 3|$ on the same axes.
Compare and contrast your functions.
What effect does changing the values of h have on the graphs of $y = |x + h|$?

$|x|$ means the modulus of x. See chapter 18 for more explanation.

4 Sketch $y = x^2$, $y = -x^2$, $y = 2x^2$, $y = 0.5x^2$ on the same axes.
Compare and contrast your functions.
What effect does the negative sign have on the graph?
What effect does changing the value of a have on the graphs of $y = ax^2$?

In the investigation you should have found that your graphs in parts 1, 2 and 3 were all the same shape but the position of the graphs changed. The graphs in part 4 should have been reflected or changed by stretching.

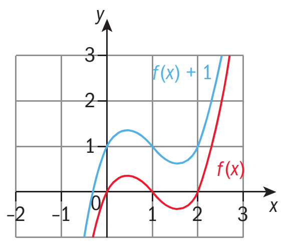
These are examples of 'transformations' of graphs. We will now look at these transformations in detail.

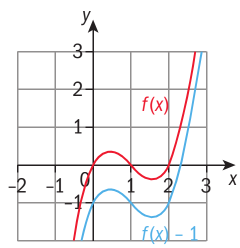
Translations

Shift upward or downward

→ $f(x) + k$ translates $f(x)$ vertically a distance of k units upward.

→ $f(x) - k$ translates $f(x)$ vertically a distance of k units downward.

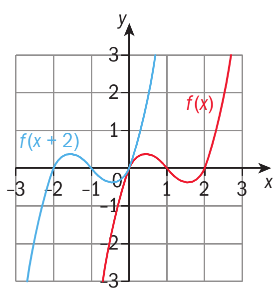


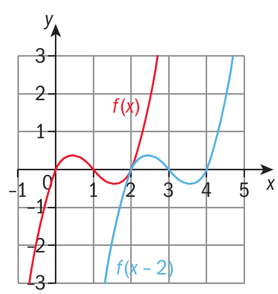


Shift to the right or left

→ $f(x + k)$ translates $f(x)$ horizontally k units to the **left**, when $k > 0$.

→ $f(x - k)$ translates $f(x)$ horizontally k units to the **right**, when $k > 0$.





Translations can be represented by vectors in the form $\begin{pmatrix} a \\ b \end{pmatrix}$ where a is the horizontal component and b is the vertical component.

$\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ is a horizontal shift of 3 units right. $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ is a vertical shift of 2 units down.

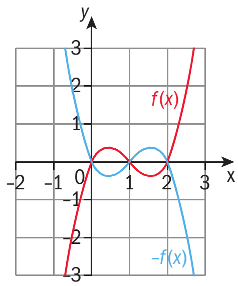
Translation by the vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ denotes a horizontal shift of 3 units to the right, and a vertical shift of 2 units down.

Try transforming some functions with different values of k on your GDC.

Reflections

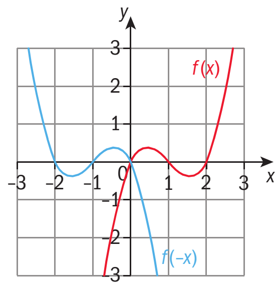
Reflection in the x-axis

> ➔ $-f(x)$ reflects $f(x)$ in the x-axis.



Reflection in the y-axis

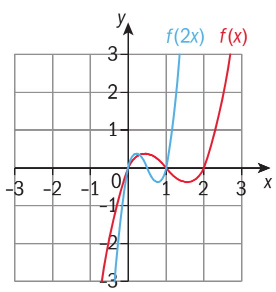
> ➔ $f(-x)$ reflects $f(x)$ in the y-axis.



Stretches

Horizontal stretch (or compress)

> ➔ $f(qx)$ stretches or compresses $f(x)$ horizontally with scale factor $\frac{1}{q}$.



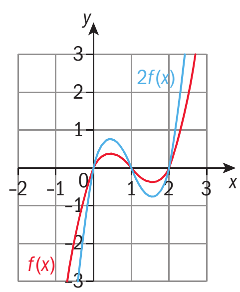
The transformation is a **horizontal stretch of scale factor** $\frac{1}{q}$.

When $q > 1$ the graph is compressed towards the y-axis

When $0 < q < 1$ the graph is stretched away from the y-axis.

Vertical stretch (or compress)

> ➔ $pf(x)$ stretches $f(x)$ vertically with scale factor p.



The transformation is a **vertical stretch of scale factor** p.

When $p > 1$ the graph stretches away from the x-axis.

When $0 < p < 1$ the graph is compressed towards the x-axis.

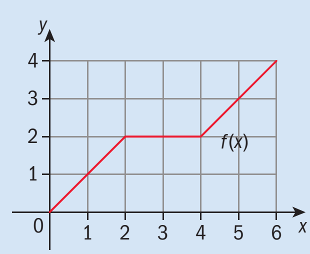
A stretch with a scale factor p where $0 < p < 1$ will actually compress the graph.

Students often make mistakes with stretches. It is important to remember the different effects of, for example, $2f(x)$ and $f(2x)$.

Example 14

1 Given the graph of the function $f(x)$ shown here, sketch the graphs of:

a $f(x + 1)$ **b** $f(x) - 2$ **c** $f(-x)$ **d** $-f(x)$ **e** $2f(x)$



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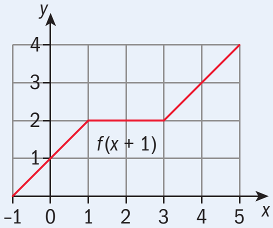
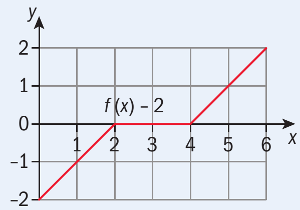
Answers

a



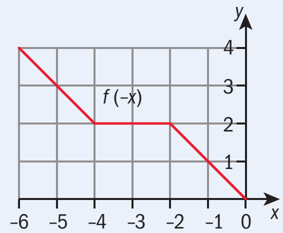
Translated one unit to the left

b



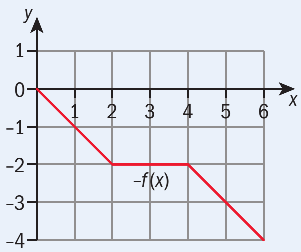
Translated two units down

c



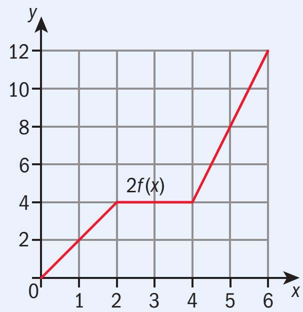
Reflected in the *y*-axis

d



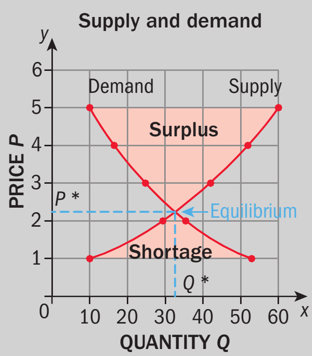
Reflected in the *x*-axis

e

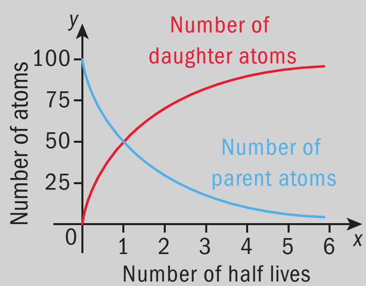


Vertical stretch of scale factor 2

Supply and demand curves in business and economics are reflections.

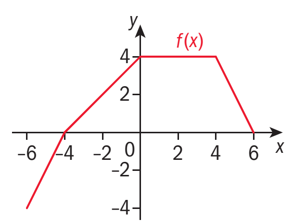


Radioactive decay curves are reflections.



Exercise 1I

EXAM-STYLE QUESTION

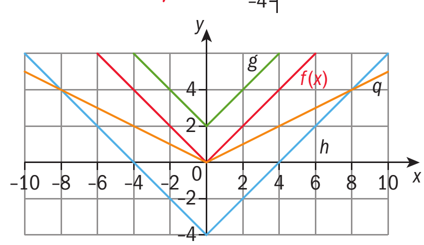


1 Copy the graph. Draw these functions on the same axes.

a $f(x) + 4$ **b** $f(x) - 2$ **c** $-f(x)$

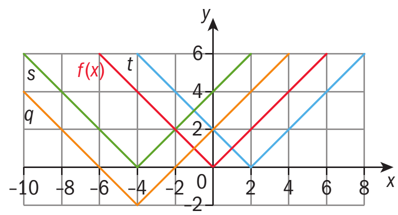
d $f(x + 3)$ **e** $f(x - 4)$ **f** $2f(x)$

g $f(2x)$



2 Functions g, h and q are transformations of $f(x)$. Write each transformation in terms of $f(x)$.

3 Functions q, s and t are transformations of $f(x)$. Write each transformation in terms of $f(x)$.

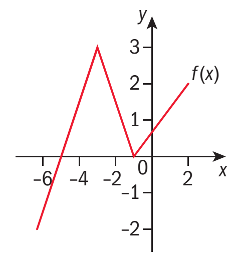


EXAM-STYLE QUESTION

4 Copy the graph of $f(x)$. Sketch the graph of each of these functions, and state the domain and range for each.

a $2f(x-5)$

b $-f(2x)+3$

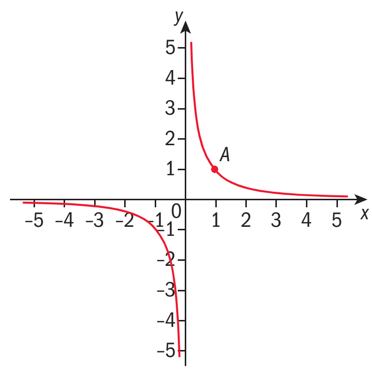


5 The graph of $f(x)$ is shown. A is the point $(1, 1)$. Make separate copies of the graph and draw the function after each transformation. On each graph, label the new position of A as A_1.

a $f(x+1)$ **b** $f(x)+1$

c $f(-x)$ **d** $2f(x)$

e $f(x-2)+3$



6 In each case, describe the transformation that would change the graph of $f(x)$ into the graph of $g(x)$.

a $f(x) = x^3$, $g(x) = -(x^3)$

b $f(x) = x^2$, $g(x) = (x-3)^2$

c $f(x) = x$, $g(x) = -2x + 5$

EXAM-STYLE QUESTION

7 Let $f(x) = 2x + 1$.

a Draw the graph of $f(x)$ for $0 \le x \le 2$.

b Let $g(x) = f(x+3) - 2$. On the same graph draw $g(x)$ for $-3 \le x \le -1$.

If a domain is given in the question, you must only draw the function for that domain.

Review exercise

1 **a** If $g(a) = 4a - 5$, find $g(a-2)$.

b If $h(x) = \frac{1+x}{1-x}$, find $h(1-x)$.

2 **a** Evaluate $f(x-3)$ when $f(x) = 2x^2 - 3x + 1$.

b For $f(x) = 2x + 7$ and $g(x) = 1 - x^2$, find the composite function defined by $(f \circ g)(x)$.

3 Find the inverses of these functions.

a $f(x) = \frac{3x+17}{2}$

b $g(x) = 2x^3 + 3$

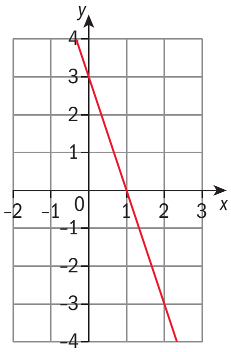
4 Find the inverse of $f(x) = -\frac{1}{5}x - 1$. Then graph the function and its inverse.

5 Find the inverse functions for

a $f(x) = 3x + 5$ b $f(x) = \sqrt[3]{x+2}$

6 Copy each graph and draw the inverse of each function.

a



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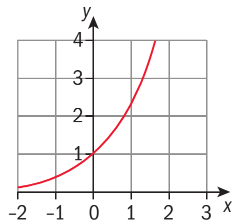
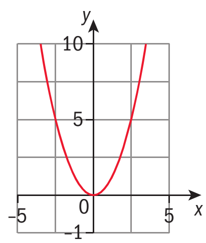


7 Find the domain and range for each of these graphs.

a



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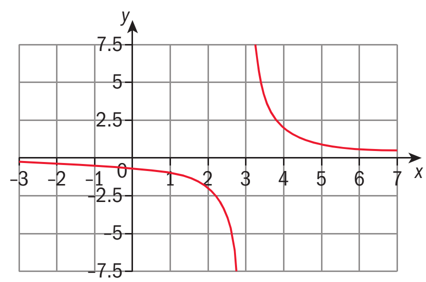


EXAM-STYLE QUESTION

8 For each function, write a single equation to represent the given combination of transformations.

a $f(x) = x$, reflected in the y-axis, stretched vertically by a factor of 2, horizontally by a factor of $\frac{1}{3}$ and translated 3 units left and 2 units up.

b $f(x) = x^2$, reflected in the x-axis, stretched vertically by a factor of $\frac{1}{4}$, horizontally by a factor of 3, translated 5 units right and 1 unit down.

9 a Explain how to draw the inverse of a function from its graph.

b Graph the inverse of $f(x) = 2x + 3$.

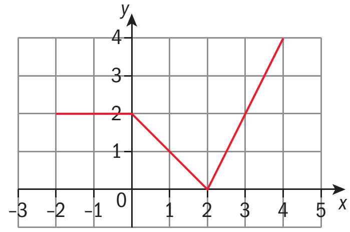
EXAM-STYLE QUESTION

10 Let $f(x) = 2x^3 + 3$ and $g(x) = 3x - 2$.

a Find $g(0)$. b Find $(f \circ g)(0)$. c Find $f^{-1}(x)$.

EXAM-STYLE QUESTIONS

11 The graph shows the function $f(x)$, for $-2 \le x \le 4$.



a Let $h(x) = f(-x)$. Sketch the graph of $h(x)$.

b Let $g(x) = \frac{1}{2} f(x-1)$. The point $A(3, 2)$ on the graph of f is transformed to the point P on the graph of g. Find the coordinates of P.

12 The functions f and g are defined as $f(x) = 3x$ and $g(x) = x + 2$.

a Find an expression for $(f \circ g)(x)$.

b Show that $f^{-1}(12) + g^{-1}(12) = 14$.

13 Let $g(x) = 2x - 1$, $h(x) = \frac{3x}{x-2}$, $x \ne 2$

a Find an expression for $(h \circ g)(x)$. Simplify your answer.

b Solve the equation $(h \circ g)(x) = 0$.

The instruction 'Show that…' means 'Obtain the required result (possibly using information given) without the formality of proof'.
For 'Show that' questions you do not usually need to use a calculator.
A good method is to cover up the right-hand side of the equation and then work out the left-hand side until your answer is the same as the right-hand side.

Review exercise

1 Use your GDC to sketch the function and state the domain and range of $f(x) = \sqrt{x+2}$.

2 Sketch the function $y = (x + 1)(x - 3)$ and state its domain and range.

3 Sketch the function $y = \frac{1}{x+2}$ and state its domain and range.

EXAM-STYLE QUESTIONS

4 The function $f(x)$ is defined as $f(x) = 2 + \frac{1}{x+1}$, $x \ne -1$.

a Sketch the curve $f(x)$ for $-3 \le x \le 2$.

b Use your GDC to help you write down the value of the x-intercept and the y-intercept.

5 **a** Sketch the graph of $f(x) = \frac{1}{x^2}$

b For what value of x is $f(x)$ undefined?

c State the domain and range of $f(x)$.

6 Given the function $f(x) = \frac{2x-5}{x+2}$

a write down the equations of the asymptotes

b sketch the function

c write down the coordinates of the intercepts with both axes.

7 Let $f(x) = 2 - x^2$ and $g(x) = x^2 - 2$.

a Sketch both functions on one graph with $-3 \le x \le 3$.

b Solve $f(x) = g(x)$.

EXAM-STYLE QUESTIONS

8 Let $f(x) = x^3 - 3$.

a Find the inverse function $f^{-1}(x)$.

b Sketch both $f(x)$ and $f^{-1}(x)$ on the same axes.

c Solve $f(x) = f^{-1}(x)$.

9 $f(x) = e^{2x-1} + \frac{2}{x+1}$, $x \neq 1$.

Sketch the curve of $f(x)$ for $-5 \leq x \leq 2$, including any asymptotes.

10 Consider the functions f and g where $f(x) = 3x - 2$ and $g(x) = x - 3$.

a Find the inverse function, f^{-1}.

b Given that $g^{-1}(x) = x + 3$, find $(g^{-1} \circ f)(x)$.

c Show that $(f^{-1} \circ g)(x) = \frac{x-1}{3}$.

d Solve $(f^{-1} \circ g)(x) = (g^{-1} \circ f)(x)$

Let $h(x) = \frac{f(x)}{g(x)}$, $x \neq 2$.

d **Sketch** the graph of h for $-6 \leq x \leq 10$ and $-4 \leq y \leq 10$, including any asymptotes.

e Write down the **equations** of the asymptotes.

> When IB exams have words in **bold** script, it means that you must do exactly what is required. For example the equation could be given as $x = 3$ but not just as 3.

CHAPTER 1 SUMMARY

Introducing functions

- A **relation** is a set of ordered pairs.
- The **domain** is the set of all the first numbers (x-values) of the ordered pairs.
- The **range** is the set of the second numbers (y-values) in each pair.
- A **function** is a relation where every x-value is related to a unique y-value.
- A relation is a function if any vertical line drawn will not intersect the graph more than once. This is called the **vertical line test**.

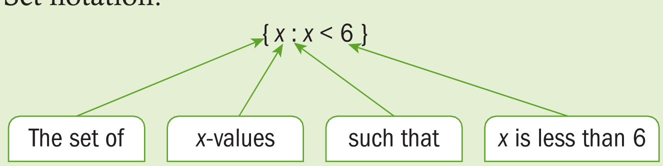
The domain and range of a relation on a Cartesian plane

Interval notation:

Use round brackets (,) if the value is not included in the graph or when the graph is undefined at that point (a hole or **asymptote**, or a jump).

Use square brackets [,] if the value is included in the graph.

- Set notation:



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Function notation

- $f(x)$ is read as 'f of x' and means 'the value of function f at x'.

Composite functions

- The composition of the function f with the function g is written as $f(g(x))$, which is read as 'f of g of x', or $(f \circ g)(x)$, which is read as 'f composed with g of x'.
- A **composite function** applies one function to the result of another and is defined by $(f \circ g)(x) = f(g(x))$.

Inverse functions

- The **inverse** of a function $f(x)$ is $f^{-1}(x)$. It reverses the action of the function.
- Functions $f(x)$ and $g(x)$ are inverses of one another if:
  $(f \circ g)(x) = x$ for all of the x-values in the domain of g and
  $(g \circ f)(x) = x$ for all of the x-values in the domain of f.
- You can use the **horizontal line test** to identify inverse functions. If a horizontal line crosses a function more than once, there is no inverse function.

The graphs of inverse functions

- The graph of the inverse of a function is a reflection of that function in the line $y = x$.
- To find the inverse function algebraically, replace $f(x)$ with y and solve for y.
- The function $I(x) = x$ is called the identity function. It leaves x unchanged. So $f \circ f^{-1} = I$.

Transformations of functions

- $f(x) + k$ translates $f(x)$ vertically a distance of k units upward.
- $f(x) - k$ translates $f(x)$ vertically a distance of k units downward.
- $f(x + k)$ translates $f(x)$ horizontally k units to the left, where $k > 0$.
- $f(x - k)$ translates $f(x)$ horizontally k units to the right, where $k > 0$.
- $-f(x)$ reflects $f(x)$ in the x-axis.
- $f(-x)$ reflects $f(x)$ in the y-axis.
- $f(qx)$ stretches $f(x)$ horizontally with scale factor $\frac{1}{q}$.
- $pf(x)$ stretches $f(x)$ vertically with scale factor p.