CHAPTER 27

L'Hôpital's Rule

Limits of the form $\lim \frac{f(x)}{g(x)}$ can be evaluated by the following theorem in the *indeterminate cases* where $f(x)$ and $g(x)$ both approach 0 or both approach $\pm\infty$.

L'Hôpital's Rule

If $f(x)$ and $g(x)$ either both approach 0 or both approach $\pm\infty$, then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

Here, "lim" stands for any of

$$\lim_{x\to+\infty}, \quad \lim_{x\to-\infty}, \quad \lim_{x\to a}, \quad \lim_{x\to a^+}, \quad \lim_{x\to a^-}$$

For a sketch of the proof, see Problems 1, 11, and 12. It is assumed, in the case of the last three types of limits, that $g'(x) \neq 0$ for x sufficiently close to a, and in the case of the first two limits, that $g'(x) \neq 0$ for sufficiently large or sufficiently small values of x. (The corresponding statements about $g(x) \neq 0$ follow by Rolle's Theorem.)

EXAMPLE 27.1: Since $\ln x$ approaches $+\infty$ as x approaches $+\infty$, L'Hôpital's Rule implies that

$$\lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{x\to+\infty} \frac{1/x}{1} = \lim_{x\to+\infty} \frac{1}{x} = 0$$

EXAMPLE 27.2: Since e^x approaches $+\infty$ as x approaches $+\infty$, L'Hôpital's Rule implies that

$$\lim_{x\to+\infty} \frac{x}{e^x} = \lim_{x\to+\infty} \frac{1}{e^x} = 0$$

EXAMPLE 27.3: We already know from Problem 13(*a*) of Chapter 7 that

$$\lim_{x\to+\infty} \frac{3x^2+5x-8}{7x^2-2x+1} = \frac{3}{7}$$

Since both $3x^2 + 5x - 8$ and $7x^2 - 2x + 1$ approach $+\infty$ as x approaches $+\infty$, L'Hôpital's Rule tells us that

$$\lim_{x\to+\infty} \frac{3x^2+5x-8}{7x^2-2x+1} = \lim_{x\to+\infty} \frac{6x+5}{14x-2}$$

and another application of the rule tells us that

$$\lim_{x\to+\infty} \frac{6x+5}{14x-2} = \lim_{x\to+\infty} \tfrac{6}{14} = \tfrac{6}{14} = \tfrac{3}{7}$$

EXAMPLE 27.4: Since tan x approaches 0 as x approaches 0, L'Hôpital's Rule implies that

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sec^2 x}{1} = \lim_{x\to 0} \frac{1}{\cos^2 x} = \frac{1}{1^2} = 1$$

Indeterminate Type $0 \cdot \infty$

If $f(x)$ approaches 0 and $g(x)$ approaches $\pm\infty$, we do not know how to find $\lim f(x)g(x)$. Sometimes such a problem can be transformed into a problem to which L'Hôpital's Rule is applicable.

EXAMPLE 27.5: As x approaches 0 from the right, ln x approaches $-\infty$. So, we do not know how to find $\lim_{x\to 0^+} x \ln x$. But as x approaches 0 from the right, $1/x$ approaches $+\infty$. So, by L'Hôpital's Rule,

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0$$

Indeterminate Type $\infty - \infty$

If $f(x)$ and $g(x)$ both approach ∞, we do not know what happens to $\lim(f(x) - g(x))$. Sometimes we can transform the problem into a L'Hôpital's-type problem.

EXAMPLE 27.6: $\lim_{x\to 0}\left(\csc x - \frac{1}{x}\right)$ is a problem of this kind. But,

$$\lim_{x\to 0}\left(\csc x - \frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0} \frac{x - \sin x}{x \sin x}$$

Since $x - \sin x$ and $x \sin x$ both approach 0, L'Hôpital's Rule applies and we get $\lim_{x\to 0} \frac{1-\cos x}{x\cos x + \sin x}$. Here both numerator and denominator approach 0 and L'Hôpital's Rule yeilds

$$\lim_{x\to 0} \frac{\sin x}{-x\sin x + \cos x + \cos x} = \frac{0}{0+1+1} = \frac{0}{2} = 0$$

Indeterminate Types 0^0, ∞^0, and 1^∞

If lim y is of one of these types, then lim (ln y) will be of type $0 \cdot \infty$.

EXAMPLE 27.7: In $\lim_{x\to 0^+} x^{\sin x}$, $y = x^{\sin x}$ is of type 0^0 and we do not know what happens in the limit. But $\ln\, y = \sin x \ln x = \frac{\ln x}{\csc x}$ and ln x and csc x approach $\pm\infty$. So, by L'Hôpital's Rule,

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{1/x}{-\csc x \cot x} = \lim_{x\to 0^+} -\frac{\sin^2 x}{x\cos x} = -\lim_{x\to 0^+} \frac{\sin x}{x}\, \frac{\sin x}{\cos x}$$

$$= -\lim_{x\to 0^+} \frac{\sin x}{x} \lim_{x\to 0^+} \tan x = -(1)(0) = 0$$

Here, we used the fact that $\lim_{x\to 0}((\sin x)/x) = 1$ (Problem 1 of Chapter 17). Now, since $\lim_{x\to 0^+} \ln y = 0$,

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1$$

EXAMPLE 27.8: In $\lim_{x\to 0^+} |\ln x|^x$, $y = |\ln x|^x$ is of type ∞^0, and it is not clear what happens in the limit. But $\ln y = x \ln|\ln x| = \dfrac{\ln|\ln x|}{1/x}$ and both $\ln|\ln x|$ and $1/x$ approach $+\infty$. So L'Hôpital's Rule yields

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \left(\frac{1}{x \ln x}\right) \Big/ \left(-\frac{1}{x^2}\right) = \lim_{x\to 0^+} -\frac{x}{\ln x} = 0,$$

since

$$\lim_{x\to 0^+} \frac{1}{\ln x} = 0. \quad \text{Hence,} \quad \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1$$

EXAMPLE 27.9: In $\lim_{x\to 1} x^{1/(x-1)}$, $y = x^{1/(x-1)}$ is of type 1^∞ and we cannot see what happens in the limit. But $\ln y = \dfrac{\ln x}{x-1}$ and both the numerator and the denominator approach 0. So by L'Hôpital's Rule, we get

$$\lim_{x\to 1} \ln y = \lim_{x\to 1} \frac{1/x}{1} = 1. \quad \text{Hence,} \quad \lim_{x\to 1} y = \lim_{x\to 1} e^{\ln y} = e^1 = e$$

SOLVED PROBLEMS

1. Prove the following $\frac{0}{0}$ form of L'Hôpital's Rule. Assume $f(x)$ and $g(x)$ are differentiable and $g'(x) \neq 0$ in some open interval (a, b) and $\lim_{x\to a^+} f(x) = 0 = \lim_{x\to a^+} g(x)$. Then, if $\lim_{x\to a^+} \dfrac{f'(x)}{g'(x)}$ exists,

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$$

Since $\lim_{x\to a^+} f(x) = 0 = \lim_{x\to a^+} g(x)$, we may assume that $f(a)$ and $g(a)$ are defined and that $f(a) = g(a) = 0$. Replacing b by x in the Extended Law of the Mean (Theorem 13.5), and using the fact that $f(a) = g(a) = 0$, we obtain

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(x_0)}{g'(x_0)}$$

for some x_0 with $a < x_0 < x$. So, $x_0 \to a^+$ as $x \to a^+$. Hence,

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$$

We also can obtain the $\frac{0}{0}$ form of L'Hôpital's Rule for $\lim_{x\to a^-}$ (simply let $u = -x$), and then the results for $\lim_{x\to a^-}$ and $\lim_{x\to a^+}$ yield the $\frac{0}{0}$ form of L'Hôpital's Rule $\lim_{x\to a}$.

2. We already know by Examples 1 and 2 that $\lim_{x\to+\infty} \dfrac{\ln x}{x} = 0$ and $\lim_{x\to+\infty} \dfrac{x}{e^x} = 0$. Show further that $\lim_{x\to+\infty} \dfrac{(\ln x)^n}{x} = 0$ and $\lim_{x\to+\infty} \dfrac{x^n}{e^x} = 0$ for all positive integers n.

Use mathematical induction. Assume these results for a given $n \geq 1$. By L'Hôpital's Rule,

$$\lim_{x\to+\infty} \frac{(\ln x)^{n+1}}{x} = \lim_{x\to+\infty} \frac{(n+1)(\ln x)^n (1/x)}{1} = (n+1) \lim_{x\to+\infty} \frac{(\ln x)^n}{x} = (n+1)(0) = 0$$

Likewise,

$$\lim_{x\to+\infty} \frac{x^{n+1}}{e^x} = \lim_{x\to+\infty} \frac{(n+1)x^n}{e^x} = (n+1) \lim_{x\to+\infty} \frac{x^n}{e^x} = (n+1)(0) = 0$$

3. Use L'Hôpital's Rule one or more times to evaluate the following limits. Always check that the appropriate assumptions hold.

(a) $\lim_{x\to 0} \frac{x+\sin 2x}{x-\sin 2x}$.

We get $\lim_{x\to 0} \frac{1+2\cos 2x}{1-2\cos 2x} = \frac{1+2(1)}{1-2(1)} = -3$.

(b) $\lim_{x\to 0^+} \frac{e^x - 1}{x^2}$.

We get $\lim_{x\to 0^+} \frac{e^x}{2x} = \frac{1}{2}\lim_{x\to 0^+} \frac{e^x}{x} = +\infty$ by Example 2.

(c) $\lim_{x\to 0} \frac{e^x + e^{-x} - x^2 - 2}{\sin^2 x - x^2}$.

We obtain $\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{2\sin x\cos x - 2x} = \lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{\sin 2x - 2x}$.

By repeated uses of L'Hôpital's Rule, we get

$$\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{2\cos 2x - 2} = \lim_{x\to 0} \frac{e^x - e^{-x}}{-4\sin 2x} =$$

$$\lim_{x\to 0} \frac{e^x + e^{-x}}{-8\cos 2x} = \frac{1+1}{-8(1)} = -\frac{2}{8} = -\frac{1}{4}$$

(d) $\lim_{x\to \pi^+} \frac{\sin x}{\sqrt{x-\pi}}$.

We get $\lim_{x\to \pi^+} \frac{\cos x}{1/[2(x-\pi)^{1/2}]} = \lim_{x\to \pi^+} 2(x-\pi)^{1/2}\cos x = 0$.

(e) $\lim_{x\to 0^+} \frac{\ln \sin x}{\ln \tan x}$.

One obtains $\lim_{x\to 0^+} \frac{(\cos x)/(\sin x)}{(\sec^2 x)/(\tan x)} = \lim_{x\to 0^+} \cos^4 x = 1$

(f) $\lim_{x\to 0} \frac{\cot x}{\cot 2x}$.

The direct use of L'Hôpital's Rule

$$\lim_{x\to 0} \frac{-\csc^2 x}{-2\csc^2(2x)} = \frac{1}{4}\lim_{x\to 0} \frac{2\csc^2 x(\cot x)}{(\csc^2(2x))(\cot 2x)}$$

leads us to ever more complicated limits. Instead, if we change from cot to tan, we get

$$\lim_{x\to 0} \frac{\cot x}{\cot 2x} = \lim_{x\to 0} \frac{\tan 2x}{\tan x} = \lim_{x\to 0} \frac{2\sec^2(2x)}{\sec^2 x} = 2\lim_{x\to 0} \frac{\cos^2 x}{\cos^2(2x)} = 2\frac{1}{1} = 2$$

(g) $\lim_{x\to 0^+} x^2 \ln x$.

This is of type $0\cdot\infty$. Then L'Hôspiutal's Rule can be brought in as follows:

$$\lim_{x\to 0^+} \frac{\ln x}{1/x^2} = \lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+} -\tfrac{1}{2}x^2 = 0$$

(h) $\lim_{x\to \pi/4} (1-\tan x)\sec 2x$.

This is of type $0\cdot\infty$. However, it is equal to

$$\lim_{x\to \pi/4} \frac{1-\tan x}{\cos 2x} = \lim_{x\to \pi/4} \frac{-\sec^2 x}{-2\sin 2x} = \frac{-2}{-2} = 1$$

$$\left(\text{Here we used the value } \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}.\right)$$

(i) $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$.

This is type $\infty - \infty$. But it is equal to

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x(e^x - 1)} = \lim_{x\to 0} \frac{e^x - 1}{xe^x + e^x - 1} = \lim_{x\to 0} \frac{e^x}{xe^x + 2e^x} = \frac{1}{0+2} = \frac{1}{2}$$

(j) $\lim_{x\to 0}(\csc x - \cot x)$.

This is of type $\infty - \infty$. But it is equal to

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) = \lim_{x\to 0} \frac{1-\cos x}{\sin x} = \lim_{x\to 0} \frac{\sin x}{\cos x} = 0$$

(k) $\lim_{x\to (\pi/2)^-} (\tan x)^{\cos x}$.

This if of type ∞^0. Let $y = (\tan x)^{\cos x}$. Then $\ln y = (\cos x)(\ln \tan x) = \frac{\ln \tan x}{\sec x}$.

So

$$\lim_{x\to(\pi/2)^-} \ln y = \lim_{x\to(\pi/2)^-} \frac{\ln \tan x}{\sec x} = \lim_{x\to(\pi/2)^-} (\sec^2 x/\tan x)/(\sec x \tan x) = \lim_{x\to(\pi/2)^-} \frac{\cos x}{\sin^2 x} = \frac{0}{1} = 1$$

(l) $\lim_{x\to+\infty} \frac{\sqrt{2+x^2}}{x}$.

We get $\lim_{x\to+\infty} \frac{x}{\sqrt{2+x^2}} = \lim_{x\to+\infty} \frac{\sqrt{2+x^2}}{x}$ and we are going around in a circle. So, L'Hôpital's Rule is of no use. But,

$$\lim_{x\to+\infty} \frac{\sqrt{2+x^2}}{x} = \lim_{x\to+\infty} \sqrt{\frac{2+x^2}{x^2}} == \lim_{x\to+\infty} \sqrt{\frac{2}{x^2}+1}$$

$$= \sqrt{0+1} = 1$$

4. Criticize the following use of L'Hôpital's Rule:

$$\lim_{x\to 2} \frac{x^3 - x^2 - x - 2}{x^3 - 3x^2 + 3x - 2} = \lim_{x\to 2} \frac{3x^2 - 2x - 1}{3x^2 - 6x + 3} = \lim_{x\to 2} \frac{6x - 2}{6x - 6} = \lim_{x\to 2} \frac{6}{6} = 1$$

The second equation is an incorrect use of L'Hôpital's Rule, since $\lim_{x\to 2} (3x^2 - 2x - 1) = 7$ and $\lim_{x\to 2} (3x^2 - 6x + 3) = 3$. So, the correct limit should be $\frac{7}{3}$.

5. (GC) Sketch the graph of $y = xe^{-x} = \frac{x}{e^x}$.

See Fig 27-1. By Example 2, $\lim_{x\to+\infty} y = 0$. So, the positive x axis is a horizontal asyomptote. Since $\lim_{x\to-\infty} e^{-x} = +\infty$, $\lim_{x\to-\infty} y = -\infty$. $y' = e^{-x}(1 - x)$ and $y'' = e^{-x}(x - 2)$. Then $x = 1$ is a critical number. By the second derivative test, there is a relative maximum at (1, $1/e$) since $y'' < 0$ at $x = 0$. The graph is concave downward for $x < 2$ (where $y'' < 0$) and concave upward for $x > 2$ (where $y'' > 0$). (2, $2/e^2$) is an inflection point. The graphing calculator gives us the estimates $1/e \sim 0.37$ and $2/e^2 \sim 0.27$.

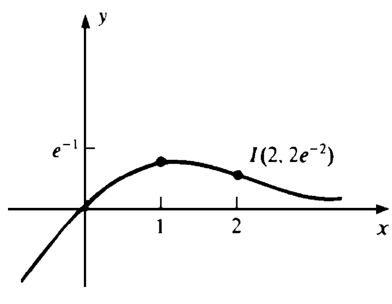

Fig. 27-1

6. (GC) Sketch the graph $y = x \ln x$.

See Fig. 27-2. The graph is defined only for $x > 0$. Clearly, $\lim_{x\to+\infty} y = +\infty$. By Example 5, $\lim_{x\to 0^+} y = 0$. Since $y' = 1 + \ln x$ and $y'' = 1/x > 0$, the critical number at $x = 1/e$ (where $y' = 0$) yields, by the second derivative test, a relative minimum at ($1/e$, $-1/e$). The graph is concave upward everywhere.

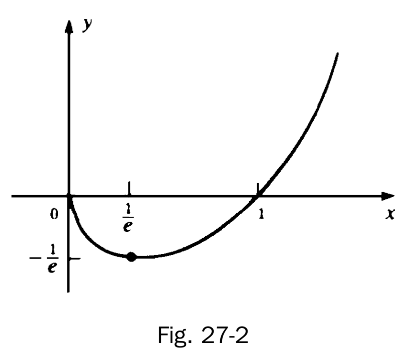

Fig. 27-2

SUPPLEMENTARY PROBLEMS

7. Show that $\lim_{x\to-\infty} x^n e^x = 0$ for all positive integers x.

8. Find $\lim_{x\to+\infty} x\sin\frac{\pi}{x}$.

Ans. π

9. Sketch the graphs of the following functions: (a) $y = x - \ln x$; (b) $y = \frac{\ln x}{x}$; (c) $y = x^2 e^x$

Ans. See Fig. 27-3.

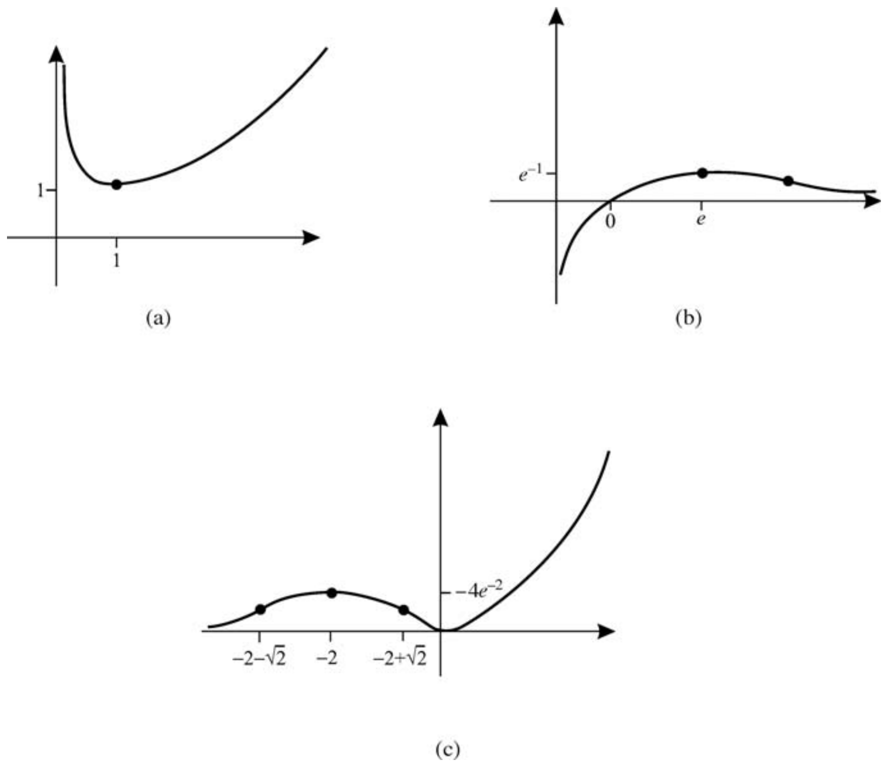

Fig. 27-3

10. Evaluate the following limits:

(a) $\lim_{x\to4} \frac{x^4-256}{x-4} = 256$

(b) $\lim_{x\to4} \frac{x^4-256}{x^2-16} = 32$

(c) $\lim_{x\to3} \frac{x^2-3x}{x^2-9} = \frac{1}{2}$

(d) $\lim_{x\to2} \frac{e^x-e^2}{x-2} = e^2$

(e) $\lim_{x\to0} \frac{xe^x}{1-e^x} = -1$

(f) $\lim_{x\to0} \frac{e^x-1}{\tan 2x} = \frac{1}{2}$

(g) $\lim_{x\to-1} \frac{\ln(2+x)}{x+1} = 1$

(h) $\lim_{x\to0} \frac{\cos x-1}{\cos 2x-1} = \frac{1}{4}$

(i) $\lim_{x\to0} \frac{e^{2x}-e^{-2x}}{\sin x} = 4$

(j) $\lim_{x\to0} \frac{8^x-2^x}{4x} = \frac{1}{2}\ln 2$

(k) $\lim_{x\to0} \frac{2\tan^{-1}x - x}{2x-\sin^{-1}x} = 1$

(l) $\lim_{x\to0} \frac{\ln\sec 2x}{\ln\sec x} = 4$

(m) $\lim_{x\to0} \frac{\ln\cos x}{x^2} = -\frac{1}{2}$

(n) $\lim_{x\to0} \frac{\cos 2x-\cos x}{\sin^2 x} = -\frac{3}{2}$

(o) $\lim_{x\to+\infty} \frac{\ln x}{\sqrt{x}} = 0$

(p) $\lim_{x\to\frac{1}{2}\pi} \frac{\csc 6x}{\csc 2x} = \frac{1}{3}$

(q) $\lim_{x\to+\infty} \frac{5x+2\ln x}{x+3\ln x} = 5$

(r) $\lim_{x\to+\infty} \frac{x^4+x^2}{e^x+1} = 0$

(s) $\lim_{x\to0^+} \frac{\ln\cot x}{e^{\csc^2 x}} = 0$

(t) $\lim_{x\to0^+} \frac{e^x+3x^3}{4e^x+2x^2} = \frac{1}{4}$

(u) $\lim_{x\to0} (e^x-1)\cos x = 1$

(v) $\lim_{x\to-\infty} x^2e^x = 0$

(w) $\lim_{x\to0} x\csc x = 1$

(x) $\lim_{x\to1} \csc\pi x \ln x = -1/\pi$

(y) $\lim_{x\to\frac{1}{2}\pi^-} e^{-\tan x}\sec^2 x = 0$ (z) $\lim_{x\to 0}(x-\sin^{-1}x)\csc^3 x = -\frac{1}{6}$ (a′) $\lim_{x\to 2}\left(\frac{4}{x^2-4}-\frac{1}{x-2}\right) = -\frac{1}{4}$

(b′) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right) = 0$ (c′) $\lim_{x\to\frac{1}{2}\pi}(\sec^3 x-\tan^3 x) = \infty$ (d′) $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{x-1}\right) = -\frac{1}{2}$

(e′) $\lim_{x\to 0}\left(\frac{4}{x^2}-\frac{2}{1-\cos x}\right) = -\frac{1}{3}$ (f′) $\lim_{x\to+\infty}\left(\frac{\ln x}{x}-\frac{1}{\sqrt{x}}\right) = 0$ (g′) $\lim_{x\to 0^+} x^x = 1$

(h′) $\lim_{x\to 0}(\cos x)^{1/x} = 1$ (i′) $\lim_{x\to 0}(e^x+3x)^{1/x} = e^4$ (j′) $\lim_{x\to+\infty}(1-e^{-x})^{e^x} = 1/e$

(k′) $\lim_{x\to\frac{1}{2}\pi}(\sin x-\cos x)^{\tan x} = 1/e$ (l′) $\lim_{x\to\frac{1}{2}\pi^-}(\tan x)^{\cos x} = 1$ (m′) $\lim_{x\to 1} x^{\tan\frac{1}{2}\pi x} = e^{-2/\pi}$

(n′) $\lim_{x\to+\infty}(1+1/x)^x = e$ (o′) $\lim_{x\to+\infty}\frac{2^x}{3^{x^2}} = 0$ (p′) $\lim_{x\to+0^+}\frac{e^{-3/x}}{x^2} = 0$

(q′) $\lim_{x\to+\infty}\frac{\ln^5 x}{x^2} = 0$ (r′) $\lim_{x\to+\infty}\frac{\ln^{1000}}{x^5} = 0$

(s′) $\lim_{x\to 0}\frac{e^x(1-e^x)}{(1+x)\ln(1-x)} = \lim_{x\to 0}\frac{e^x}{1+x}\lim_{x\to 0}\frac{1-e^x}{(1-x)} = 1$

11. Verify the sketch of the proof of the following $\frac{0}{0}$ form of L'Hôpital's Rule at $+\infty$. Assume $f(x)$ and $g(x)$ are differentiable and $g'(x)\neq 0$ for all $x\geq c$, and $\lim_{x\to+\infty} f(x) = 0 = \lim_{x\to+\infty} g(x)$. Then,

$$\text{if} \quad \lim_{x\to+\infty}\frac{f'(x)}{g'(x)} \quad \text{exists,} \quad \lim_{x\to+\infty}\frac{f(x)}{g(x)} = \lim_{x\to+\infty}\frac{f'(x)}{g'(x)}$$

Proof: Let $F(u) = f(1/u)$ and $G(u) = g(1/u)$. Then, by Problem 1 for $a\to 0^+$, and with F and G instead of f and g,

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)} = \lim_{u\to 0^+}\frac{F(u)}{G(u)} = \lim_{u\to 0^+}\frac{F'(u)}{G'(u)}$$
$$= \lim_{u\to 0^+}\frac{(f'(1/u)\cdot(-1/u^2))}{(g'(1/u)\cdot(-1/u^2))} = \lim_{u\to 0^+}\frac{f'(1/u)}{g'(1/u)} = \lim_{x\to+\infty}\frac{f'(x)}{g'(x)}$$

12. Fill in the gaps in the proof of the following $\frac{\infty}{\infty}$ form of L'Hôpital's Rule in the $\lim_{x\to a^+}$ case. (The other cases follow easy as in the $\frac{0}{0}$ form.) Assume $f(x)$ and $g(x)$ are differentiable and $g'(x)\neq 0$ in some open interval (a, b) and $\lim_{x\to a^+} f(x) = \pm\infty = \lim_{x\to a^+} g(x)$. Then,

$$\text{if} \quad K = \lim_{x\to a^+}\frac{f'(x)}{g'(x)} \quad \text{exists,} \quad \lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{x\to a^+}\frac{f'(x)}{g'(x)}$$

Proof: Assume $\in > 0$ and choose c so that $|K-(f'(x)/g'(x))| < \in/2$ for $a < x < c$. Fix d in (a, c). Let $a < y < d$. By the extended mean value theorem, there exists x^* such that

$$y < x^* < d \quad \text{and} \quad \frac{f(d)-f(y)}{g(d)-g(y)} = \frac{f'(x^*)}{g'(x^*)}$$

Then

$$\left|K-\frac{f(d)-f(y)}{g(d)-g(y)}\right| < \frac{\in}{2} \quad \text{and so} \quad \left|K-\left[\left(\frac{f(y)}{g(y)}-\frac{f(d)}{g(y)}\right)\Big/\left(1-\frac{g(d)}{g(y)}\right)\right]\right| < \frac{\in}{2}$$

Now we let $y\to a^+$. Since $g(y)\to\pm\infty$ and $f(d)$ and $g(d)$ are constant, $f(d)/g(y)\to 0$ and $1-g(d)/g(y)\to 1$. So, for y close to a,

$$\left|K-\frac{f(y)}{g(y)}\right| < \in. \qquad \text{Hence,} \qquad \lim_{y\to a^+}\frac{f(y)}{g(y)} = K$$

13. (GC) In the following cases, try to find the limit by analytic methods, and then check by estimating the limit on a graphing calculator: (a) $\lim_{x \to 0^+} x^{1/x}$; (b) $\lim_{x \to +\infty} x^{1/x}$; (c) $\lim_{x \to 0} (1 - \cos x)^x$; (d) $\lim_{x \to +\infty} \left(\sqrt{x^2 + 3x} - x\right)$.

Ans. (a) 0; (b) 1; (c) 1; (d) $\frac{3}{2}$

14. The current in a coil containing a resistance R, an inductance, L, and a constant electromotive force, E, at time t is given by $i = \frac{E}{R}(1 - e^{-Rt/L})$. Obtain a formula for estimating i when R is very close to 0.

Ans. $\frac{Et}{L}$