

Name:

1. (25 points) This question investigates a special cases of cubic polynomials which can be solved with the aid of trigonometric identities.

(a) Show that $1 + \sqrt{2}$ is a solution of the equation:

$$2x^3 - (2\sqrt{2} + 6)x^2 + (4\sqrt{2} + 5)x - \sqrt{2} - 1 = 0$$

and hence find the other two solutions. Your answers should be exact.

(b) Solve the equation

$$2x^3 - 5x^2 - 6x + 9 = 0$$

and let $x = 3y$ to obtain the equation:

$$6y^3 - 5y^2 - 2y + 1 = 0$$

and write down its solutions.

(c) Show that $\cos 15^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}$ and find a similar expression for $\sin 15^\circ$.

(d) Express $\cos 3\alpha$ in terms of $\cos \alpha$ and hence show that $x = \cos \alpha$ is a solution to the equation

$$4x^3 - 3x - \cos 3\alpha = 0$$

and find the other two solutions in terms of $\cos \alpha$ and $\sin \alpha$.

(e) Use parts (c) and (d) and a substitution $x = ky$ for suitable value of k to solve the equation:

$$y^3 - 3y - \sqrt{2} = 0$$

Give your answers in surd form.

2. (30 points) The question investigates the hyperbolic functions and their graphs.

The hyperbolic functions are defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh x = \frac{e^x + e^{-x}}{2} \quad \tanh x = \frac{\sinh x}{\cosh x}$$

- (a) Find $\cosh x + \sinh x$ and $\cosh x - \sinh x$ and hence prove that

$$\cosh^2 x - \sinh^2 x = 1$$

- (b) Show that:

(i) $(\sinh x)' = \cosh x$,

(ii) $(\cosh x)' = \sinh x$.

- (c) Decide if $\sinh x$ and $\cosh x$ are even, odd or neither. Justify your answer.

- (d) Find the coordinates of any stationary points and inflexion points on the graphs of $\sinh x$ and $\cosh x$.

- (e) Sketch the graphs of $\sinh x$ and $\cosh x$.

- (f) Prove the following identities:

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

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(g) The hyperbolic function $\operatorname{sech} x$ is defined by $\operatorname{sech} x = \frac{1}{\cosh x}$. Show that $(\tanh x)' = \operatorname{sech}^2 x$.

(h) Using the graph of $\cosh x$ sketch the graphs of $\operatorname{sech} x$ and $\operatorname{sech}^2 x$.

(i) Calculate $\tanh(0)$.

(j) State the equation of any asymptotes of the graph of $\tanh x$.

(k) Using parts (h), (i) and (j) sketch the graph of $\tanh x$.