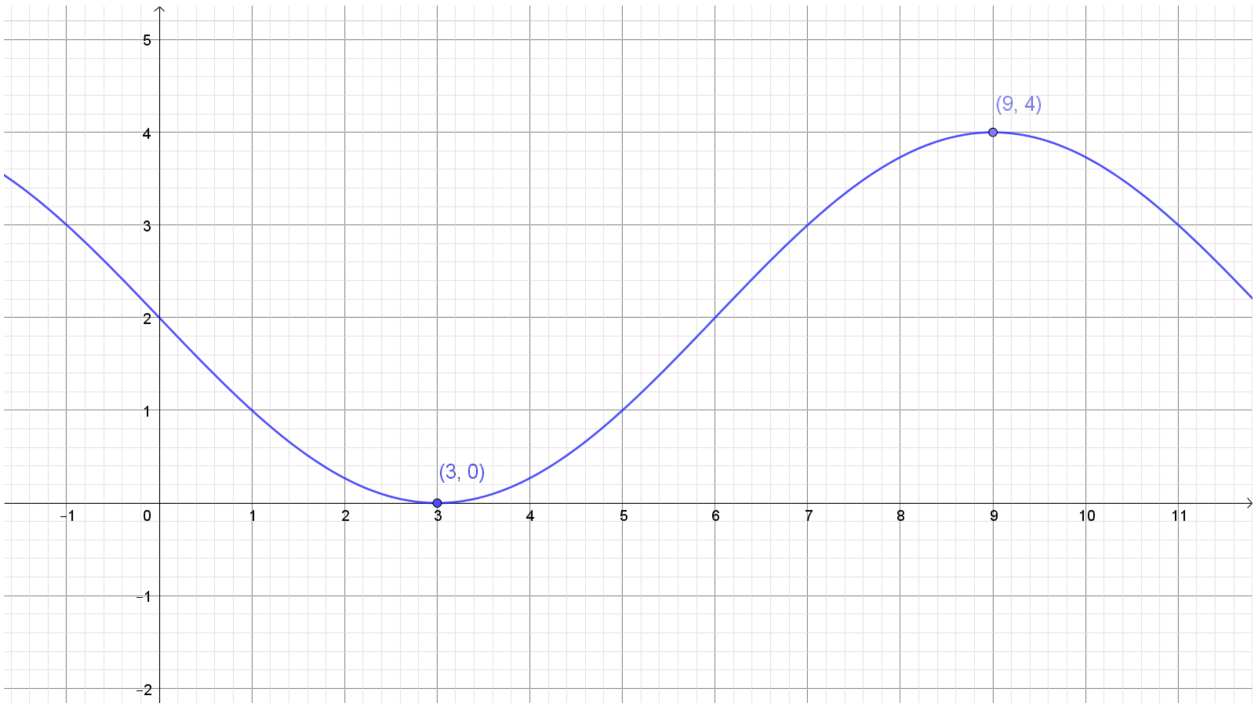
Name:

SECTION A

1. (4 points) Find the value of k so that

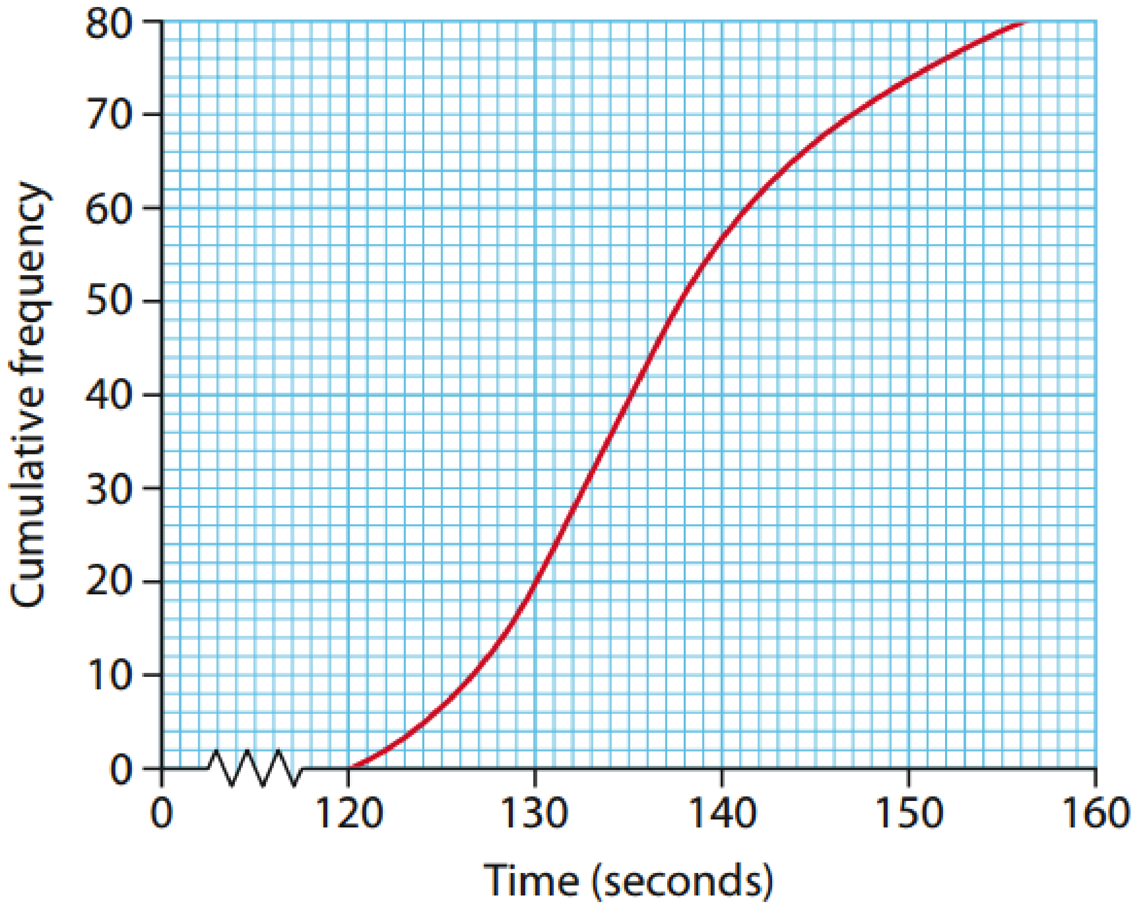
$$\sum_{i=1}^{\infty} 4k^{i-1} = 12$$

2. (4 points) The following diagram shows part of the graph of the function $f(x) = a\sin(bx) + c$



Find the values of a, b and c.

3. (5 points) The 80 applicants for a Sports Science course were required to run 800 metres and their times were recorded. The results were used to produce the following cumulative frequency graph



Estimate:

(a) the median,

(b) the interquartile range

(c) the number of applicants whose times were greater than 150 seconds.

4. (5 points) Find the value of x satisfying the equation:

$$9^x \times 2^{x+3} = 12^{x+1}$$

Give your answer in the form $\dfrac{\ln a}{\ln b}$ where $a, b \in \mathbb{Z}$.

5. (6 points) Consider the function:

$$f(x) = e^{2x-x^2}$$

(a) Find the maximum value of the function and justify that it is the maximum value.

(b) Find the x-coordinates of the points of inflexion of the graph of f and justify that these are points of inflexion.

6. (6 points) Let $f(x) = \sin x + \sqrt{3}\cos x$.

(a) Write $f(x)$ in the form $R\sin(x + \alpha)$, where R and α are constants to be found such that $R > 0$ and $0 < \alpha < \pi$.

(b) Hence describe the transformations that map the graph of $y = \sin x$ to the graph of $y = f(x)$.

(c) Solve the equation $f(x) = -1$ for $-\pi < x < 3\pi$.

7. (5 points) Find the coordinates of the point where the normal to the curve

$$x^2 - xy + y^2 = 3$$

at the point $P(-1, 1)$ intersects the curve for the second time.

8. (6 points) If $\frac{3\pi}{2} < x < 2\pi$ and $\cos x = \frac{3}{\sqrt{10}}$ find the value of:

(a) $\csc x$

(b) $\cos 2x$

(c) $\tan 3x$

9. (5 points) Consider the polynomial $P(x) = x^4 + Ax^3 + Bx^2 + 12x + 1$ with $x \in \mathbb{R}$. The remainder when $P(x)$ is divided by $x - 1$ is 8, and the remainder when it's divided by $x + 1$ is -8.

   (a) Find the values of A and B.

   (b) Find the coordinates of any maxima and minima of $P(x)$.

   (c) State the range of $P(x)$.

10. (6 points)

(a) A bag contains N sweets (where $N \geqslant 2$), of which a are red. Two sweets are drawn from the bag without replacement. Show that the probability that the first sweet is red is equal to the probability that the second sweet is red.

(b) There are two bags, each containing N sweets (where $N \geqslant 2$). The first bag contains a red sweets, and the second bag contains b red sweets. There is also a biased coin, showing Heads with probability p and Tails with probability q, where $p + q = 1$.

The coin is tossed. If it shows Heads then a sweet is chosen from the first bag and transferred to the second bag; if it shows Tails then a sweet is chosen from the second bag and transferred to the first bag. The coin is then tossed a second time: if it shows Heads then a sweet is chosen from the first bag, and if it shows Tails then a sweet is chosen from the second bag.Show that the probability that the first sweet is red is equal to the probability that the second sweet is red.

11. (8 points) Consider the function

$$f(x) = \frac{2x^2 + 3x}{x^2 + x - 2}$$

(a) Write down the equation of all the asymptotes of the graph of f.

(b) Find the axes intercepts of the graph of f.

(c) Find the coordinates of the point where the graph of f intersects its vertical asymptote.

(d) Find the x-coordinates of the stationary points of the graph of f.

(e) Sketch the graph of f.

SECTION B

12. (20 points) Consider the function $g(x) = 1 - x \sin x$.

(a) Evaluate $g(0)$ and $g(\frac{\pi}{2})$.

(b) Show that $g(x)$ is decreasing for $0 < x < \frac{\pi}{2}$.

(c) What can you conclude about the number of solutions to the equation

$$g(x) = 0$$

in the interval $0 < x < \frac{\pi}{2}$. Justify your answer.

Let $f(x) = e^{\cos x}$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

(d) State with a reason whether or not the function f is even.

(e) Find $f'(x)$.

(f) Given that the graph of f has a maximum point, find its coordinates.

(g) Show there is a point of inflexion on the graph of f, for $0 < x < \frac{\pi}{2}$ and find its coordinates. (You may use inverse trigonometric functions in your answer)

(e) Sketch the graph of f.

(f) A rectangle is drawn so that its lower vertices are on the x-axis and its upper vertices are on the curve $y = e^{\cos x}$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

(i) Write down an expression for the area of the rectangle.

(ii) Show that there is a positive value $x = a$ for which the area of the rectangle reaches a maximum. Hence show that its value is given by $2ae^{\frac{\sqrt{a^2-1}}{a}}$

13. (14 points) The first, second and third term of a non-zero, infinite geometric sequence are respectively the fourth, second and first terms of an arithmetic sequence.

(a) Show that r, the ratio of the geometric sequence, must satisfy the equation:

$$2r^2 - 3r + 1 = 0$$

(b) Hence find the two possible values of r.

Given further that the sum of the first three terms of the geometric sequence is $\frac{21}{4}$ and that the sum to infinity of this sequence exists (i.e. the sequence of partial sums converges):

(c) Find the first term of the sequence.

(d) The sum to infinity of this sequence.

(e) Find the smallest value of n such that the sum of first n terms of the **arithmetic** sequence exceeds the sum to infinity of the geometric sequence.

14. (16 points)
(a) Find the value of $\arctan(\sqrt{3}) + \arctan(-\frac{1}{\sqrt{3}})$.

(b) If $A = \arctan x$ and $B = \arctan y$, find $\tan(A+B)$ in terms of x and y.

(c) Show that $\arctan x + \arctan y = \arctan \frac{x+y}{1-xy}$ when $x = \frac{1}{2}$ and $y = \frac{1}{3}$, but not when $x = 2$ and $y = 3$.

(d) Explain why the above formula does not hold if $y = \frac{1}{x}$.

Consider the function $f(x) = \arctan x + \arctan(\frac{1}{x})$.

(e) State the domain of f.

(f) Show that $f(x)$ is an odd function.

(g) Find $f'(x)$.

(h) Calculate $f(1)$ and hence sketch the graph of $f(x)$.