

Chapter

14

Quadratic functions

Contents:

- A** Quadratic functions
- B** Graphs of quadratic functions
- C** Using the discriminant
- D** Finding a quadratic from its graph
- E** The intersection of graphs
- F** Problem solving with quadratics
- G** Optimisation with quadratics
- H** Quadratic inequalities



OPENING PROBLEM

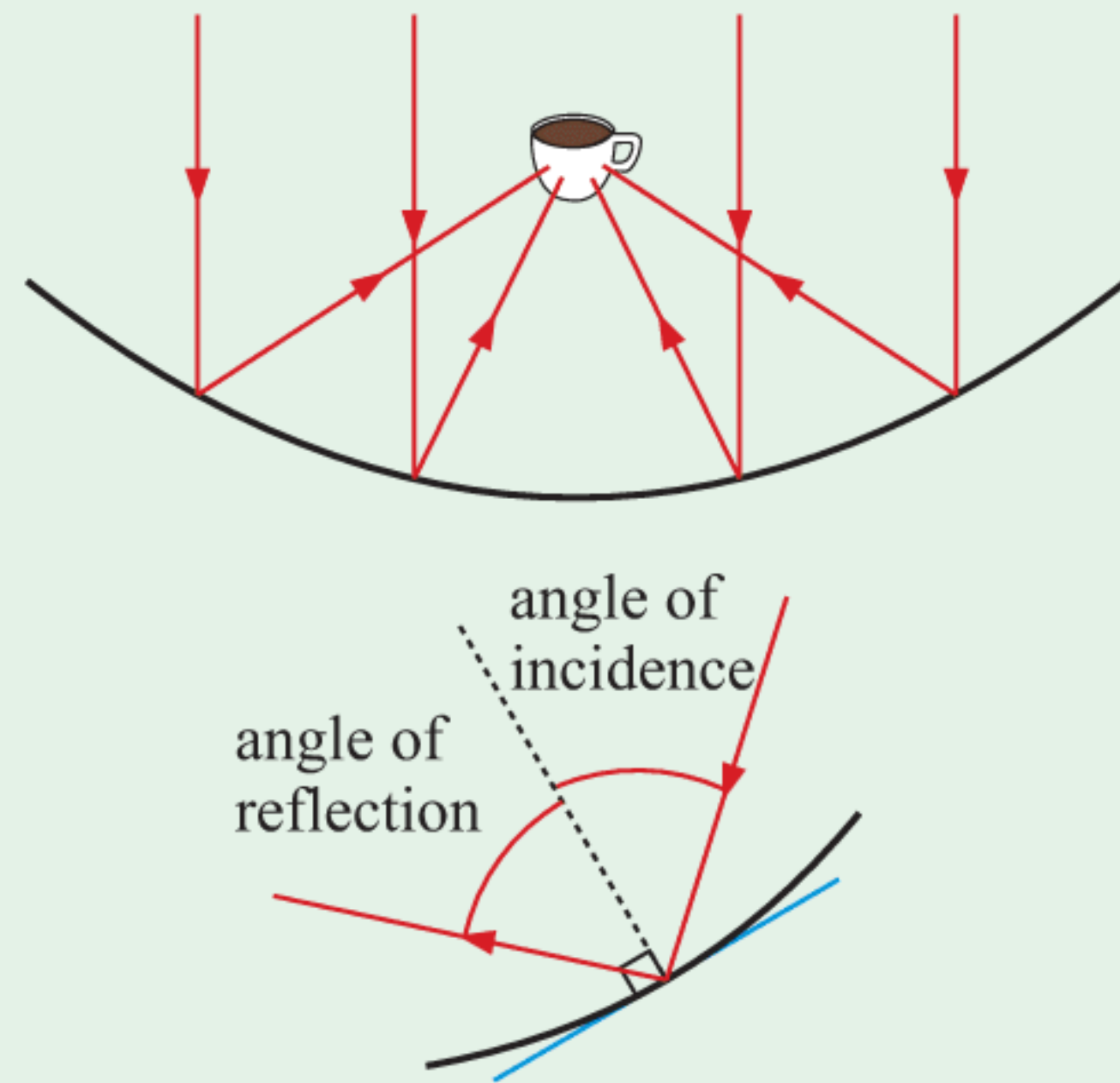
Energy-conscious Misha wants to use solar energy to heat his cup of coffee. He has decided to build a reflecting surface to focus the sun's light on the cup.

He understands that the sun's rays will arrive parallel, and that each ray will bounce off the surface according to the law of reflection:

$$\text{angle of incidence} = \text{angle of reflection}$$

Things to think about:

- What *shape* should the surface have?
- Can we write a *formula* which defines the shape of the surface?



In this Chapter we will study **quadratic functions** and investigate their graphs which are called **parabolas**. There are many examples of parabolas in everyday life, including water fountains, bridges, and radio telescopes.



We will see how the curve Misha needs in the **Opening Problem** is actually a parabola, and how the **Opening Problem** relates to the geometric definition of a parabola.

ACTIVITY 1

A cone is *right-circular* if its apex is directly above the centre of the base.

Suppose we have two right-circular cones, and we place one upside-down on the first. Now suppose the cones are infinitely tall.

We call the resulting shape a **double inverted right-circular cone**.

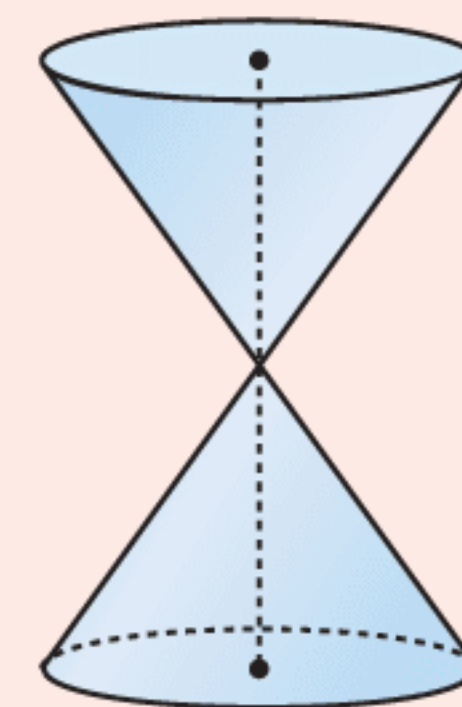
When a double inverted right-circular cone is cut by a plane, 7 possible intersections may result, called **conic sections**:

- a point
- a line
- a line-pair
- a circle
- an ellipse
- a parabola
- a hyperbola

Click on the icon to explore the conic sections.

You should observe how the parabola results when cutting the cone parallel to its slant edge.

CONIC SECTIONS



A
QUADRATIC FUNCTIONS

A **quadratic function** is a relationship between two variables x and y which can be written in the form $y = ax^2 + bx + c$ where a, b, c are constants, $a \neq 0$.

For any value of x , the corresponding value of y can be found by substitution.

Example 1
 **Self Tutor**

Determine whether the given point satisfies the quadratic function:

a $y = 3x^2 + 2x$ $(2, 16)$

b $y = -x^2 - 2x + 1$ $(-3, 1)$

a When $x = 2$,

$$y = 3(2)^2 + 2(2)$$

$$= 12 + 4$$

$$= 16$$

$\therefore (2, 16)$ satisfies the function

$$y = 3x^2 + 2x.$$

b When $x = -3$,

$$y = -(-3)^2 - 2(-3) + 1$$

$$= -9 + 6 + 1$$

$$= -2$$

$\therefore (-3, 1)$ does not satisfy the function

$$y = -x^2 - 2x + 1.$$

When we substitute a value for y into a quadratic function, we are left with a quadratic equation. Solving the quadratic equation gives us the values of x corresponding to that y -value. There may be 0, 1, or 2 solutions.

Example 2
 **Self Tutor**

If $y = x^2 - 2x + 3$, find the value(s) of x when:

a $y = 2$

b $y = 18$.

a If $y = 2$ then

$$x^2 - 2x + 3 = 2$$

$$\therefore x^2 - 2x + 1 = 0$$

$$\therefore (x - 1)^2 = 0$$

$$\therefore x = 1$$

b If $y = 18$ then

$$x^2 - 2x + 3 = 18$$

$$\therefore x^2 - 2x - 15 = 0$$

$$\therefore (x - 5)(x + 3) = 0$$

$$\therefore x = -3 \text{ or } 5$$

EXERCISE 14A

1 Copy and complete each table of values:

a $y = x^2 - 3x + 1$

| | | | | | |
|-----|----|----|---|---|---|
| x | -2 | -1 | 0 | 1 | 2 |
| y | | | | | |

c $y = 2x^2 - x + 3$

| | | | | | |
|-----|----|----|---|---|---|
| x | -4 | -2 | 0 | 2 | 4 |
| y | | | | | |

b $y = x^2 + 2x - 5$

| | | | | | |
|-----|----|----|---|---|---|
| x | -2 | -1 | 0 | 1 | 2 |
| y | | | | | |

d $y = -3x^2 + 2x + 4$

| | | | | | |
|-----|----|----|---|---|---|
| x | -4 | -2 | 0 | 2 | 4 |
| y | | | | | |

2 Determine whether the given point satisfies the quadratic function:

a $y = 2x^2 + 5$ (0, 4)

b $y = x^2 - 3x + 2$ (2, 0)

c $y = -x^2 + 2x - 5$ (-1, -8)

d $y = -2x^2 - x + 6$ (3, -15)

e $y = 3x^2 - 4x + 10$ (2, 10)

f $y = -\frac{1}{2}x^2 + 4x - 1$ (2, 5)

3 For each of the following quadratic functions, find the value(s) of x for the given value of y :

a $y = x^2 + 3x + 6$ when $y = 4$

b $y = x^2 - 4x + 7$ when $y = 3$

c $y = x^2 - 6x + 1$ when $y = -4$

d $y = 2x^2 + 5x + 1$ when $y = 4$

e $y = \frac{1}{2}x^2 + \frac{5}{2}x - 2$ when $y = 1$

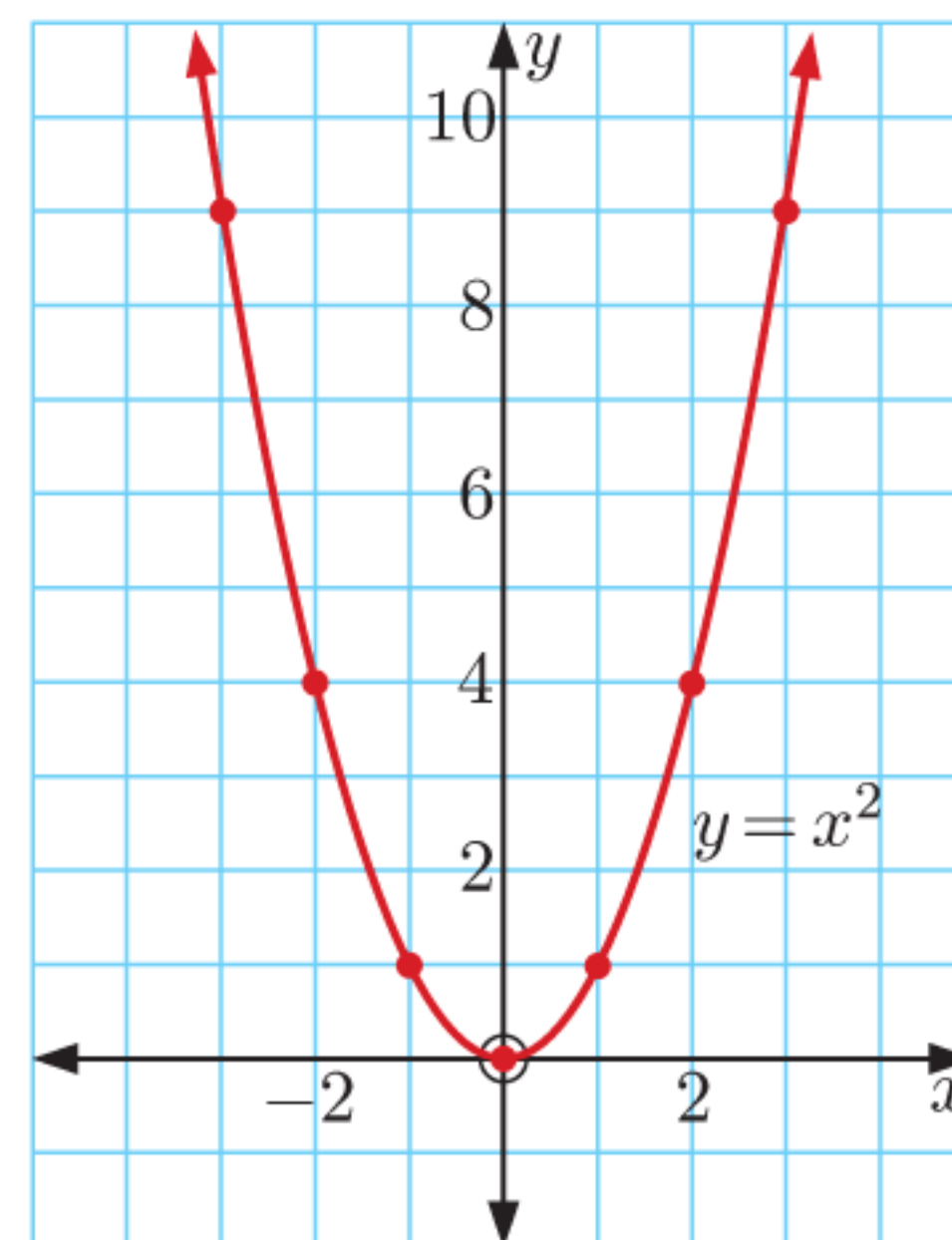
f $y = -\frac{1}{2}x^2 + 2x - 1$ when $y = 2$

B

GRAPHS OF QUADRATIC FUNCTIONS

The simplest quadratic function is $y = x^2$. Its graph can be drawn from a table of values.

| | | | | | | | |
|-----|----|----|----|---|---|---|---|
| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
| y | 9 | 4 | 1 | 0 | 1 | 4 | 9 |



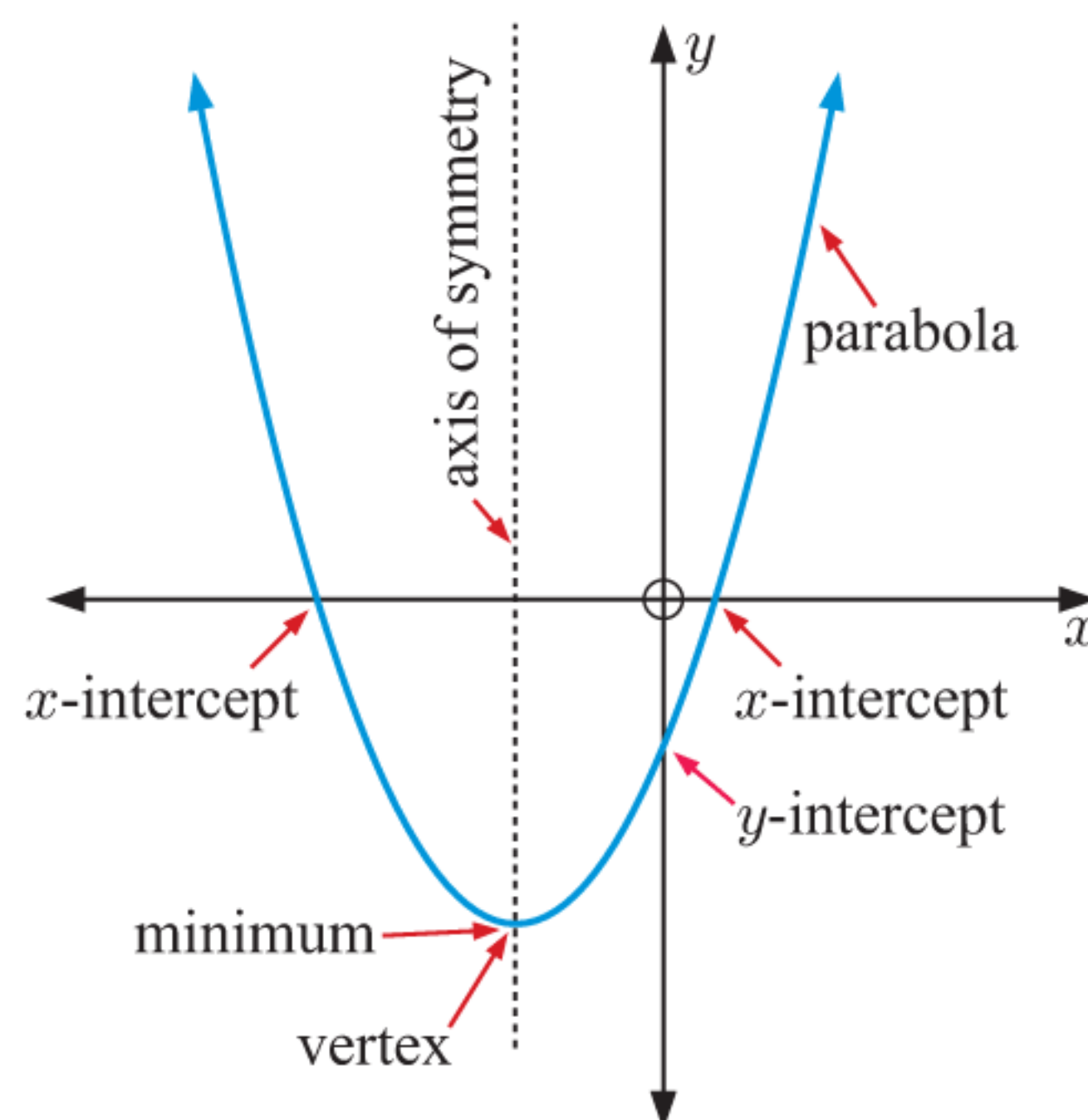
The graph of a quadratic function is called a **parabola**.

The point where the graph “turns” is called the **vertex**.

If the graph opens upwards, the vertex is the **minimum** or **minimum turning point**, and the graph is **concave upwards**.

If the graph opens downwards, the vertex is the **maximum** or **maximum turning point**, and the graph is **concave downwards**.

The vertical line that passes through the vertex is called the **axis of symmetry**. Every parabola is symmetrical about its axis of symmetry.



The value of y where the graph crosses the y -axis is the **y -intercept**.

The values of x (if they exist) where the graph crosses the x -axis are called the **x -intercepts**. They correspond to the **roots** of the quadratic equation $ax^2 + bx + c = 0$.

INVESTIGATION 1
GRAPHING $y = a(x - p)(x - q)$

In this Investigation we consider the properties of the graph of a quadratic stated in factored form. It is best done using a **graphing package** or **graphics calculator**.

What to do:

- 1
 - a Use technology to help you to sketch:
 $y = (x - 1)(x - 3)$, $y = 2(x - 1)(x - 3)$, $y = -(x - 1)(x - 3)$,
 $y = -3(x - 1)(x - 3)$, and $y = -\frac{1}{2}(x - 1)(x - 3)$
 - b Find the x -intercepts for each function in **a**.
 - c What is the geometrical significance of a in $y = a(x - 1)(x - 3)$?
- 2
 - a Use technology to help you to sketch:
 $y = 2(x - 1)(x - 4)$, $y = 2(x - 3)(x - 5)$, $y = 2(x + 1)(x - 2)$,
 $y = 2x(x + 5)$, and $y = 2(x + 2)(x + 4)$
 - b Find the x -intercepts for each function in **a**.
 - c What is the geometrical significance of p and q in $y = 2(x - p)(x - q)$?
- 3
 - a Use technology to help you to sketch:
 $y = 2(x - 1)^2$, $y = 2(x - 3)^2$, $y = 2(x + 2)^2$, and $y = 2x^2$
 - b Find the x -intercepts for each function in **a**.
 - c What is the geometrical significance of p in $y = 2(x - p)^2$?
- 4 Copy and complete:
 - If a quadratic has the form $y = a(x - p)(x - q)$ then it the x -axis at
 - If a quadratic has the form $y = a(x - p)^2$ then it the x -axis at

 GRAPHING
PACKAGE

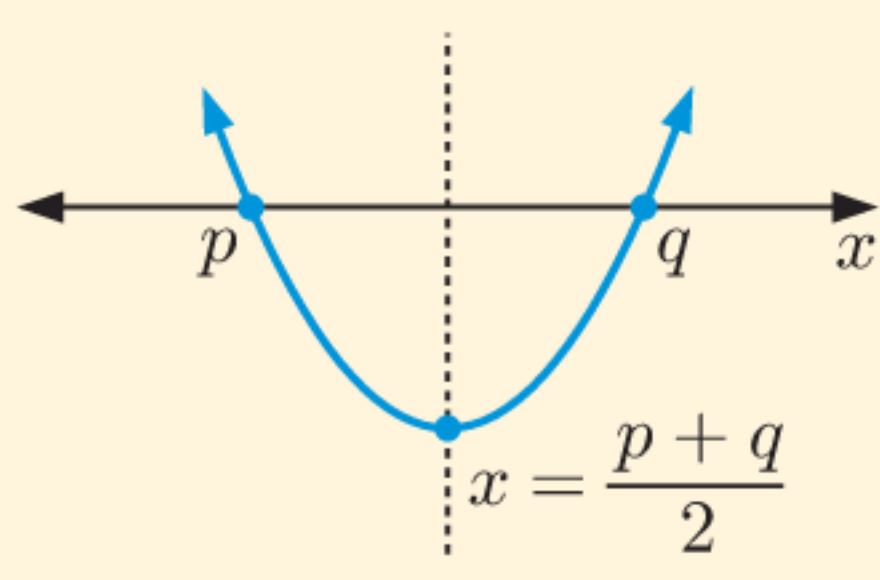
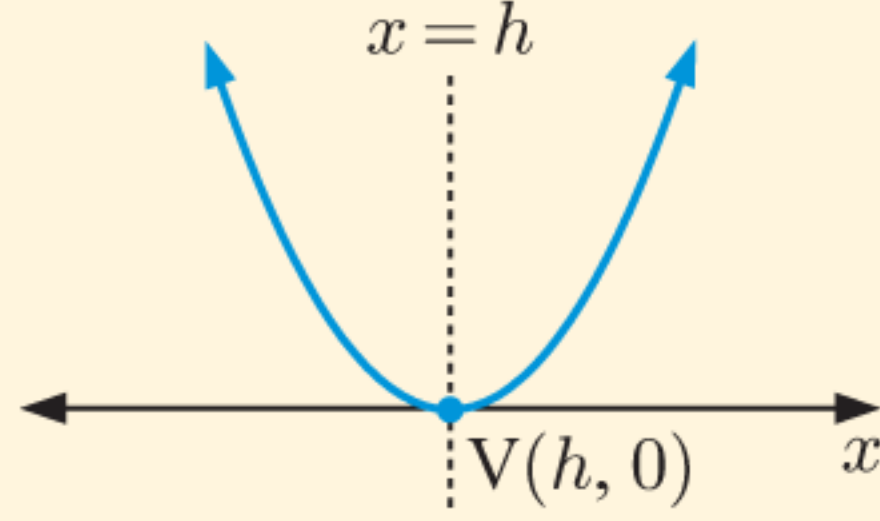
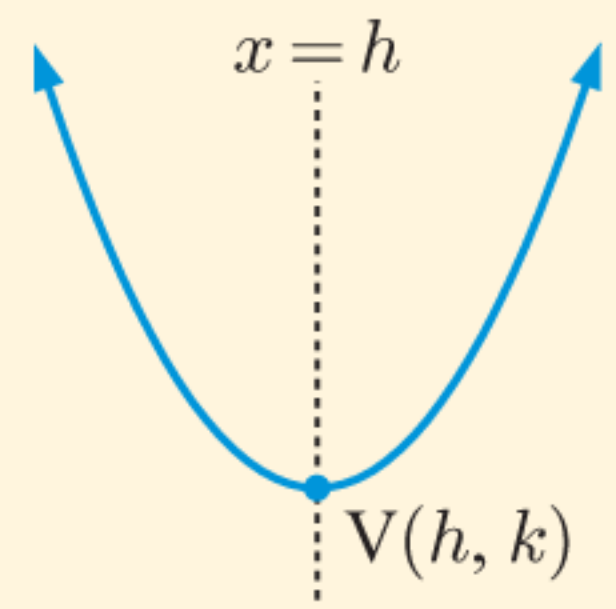
INVESTIGATION 2
GRAPHING $y = a(x - h)^2 + k$

In this Investigation we consider the properties of the graph of a quadratic stated in completed square form. It is best done using a **graphing package** or **graphics calculator**.

What to do:



- 1
 - a Use technology to help you to sketch:
 $y = (x - 3)^2 + 2$, $y = 2(x - 3)^2 + 2$, $y = -2(x - 3)^2 + 2$,
 $y = -(x - 3)^2 + 2$, and $y = -\frac{1}{3}(x - 3)^2 + 2$
 - b Find the coordinates of the vertex for each function in **a**.
 - c What is the geometrical significance of a in $y = a(x - 3)^2 + 2$?
- 2
 - a Use technology to help you to sketch:
 $y = 2(x - 1)^2 + 3$, $y = 2(x - 2)^2 + 4$, $y = 2(x - 3)^2 + 1$,
 $y = 2(x + 1)^2 + 4$, $y = 2(x + 2)^2 - 5$, and $y = 2(x + 3)^2 - 2$
 - b Find the coordinates of the vertex for each function in **a**.
 - c What is the geometrical significance of h and k in $y = 2(x - h)^2 + k$?
- 3 Copy and complete:
 If a quadratic has the form $y = a(x - h)^2 + k$ then its vertex has coordinates

 GRAPHING
PACKAGE


| Quadratic form, $a \neq 0$ | Graph | Facts |
|--|--|--|
| $y = a(x - p)(x - q)$ where $p, q \in \mathbb{R}$ |  | <ul style="list-style-type: none"> x-intercepts are p and q axis of symmetry is $x = \frac{p+q}{2}$ vertex has x-coordinate $\frac{p+q}{2}$ |
| $y = a(x - h)^2$ where $h \in \mathbb{R}$ |  | <ul style="list-style-type: none"> touches x-axis at h axis of symmetry is $x = h$ vertex is $(h, 0)$ |
| $y = a(x - h)^2 + k$ where $h, k \in \mathbb{R}$ |  | <ul style="list-style-type: none"> axis of symmetry is $x = h$ vertex is (h, k) |

You should have found that a , the coefficient of x^2 , controls the width of the graph and whether it opens upwards or downwards.

For a quadratic function $y = ax^2 + bx + c$, $a \neq 0$:

- $a > 0$ produces the shape  called concave up.
- $a < 0$ produces the shape  called concave down.
- If $-1 < a < 1$, $a \neq 0$ the graph is wider than $y = x^2$.
If $a < -1$ or $a > 1$ the graph is narrower than $y = x^2$.

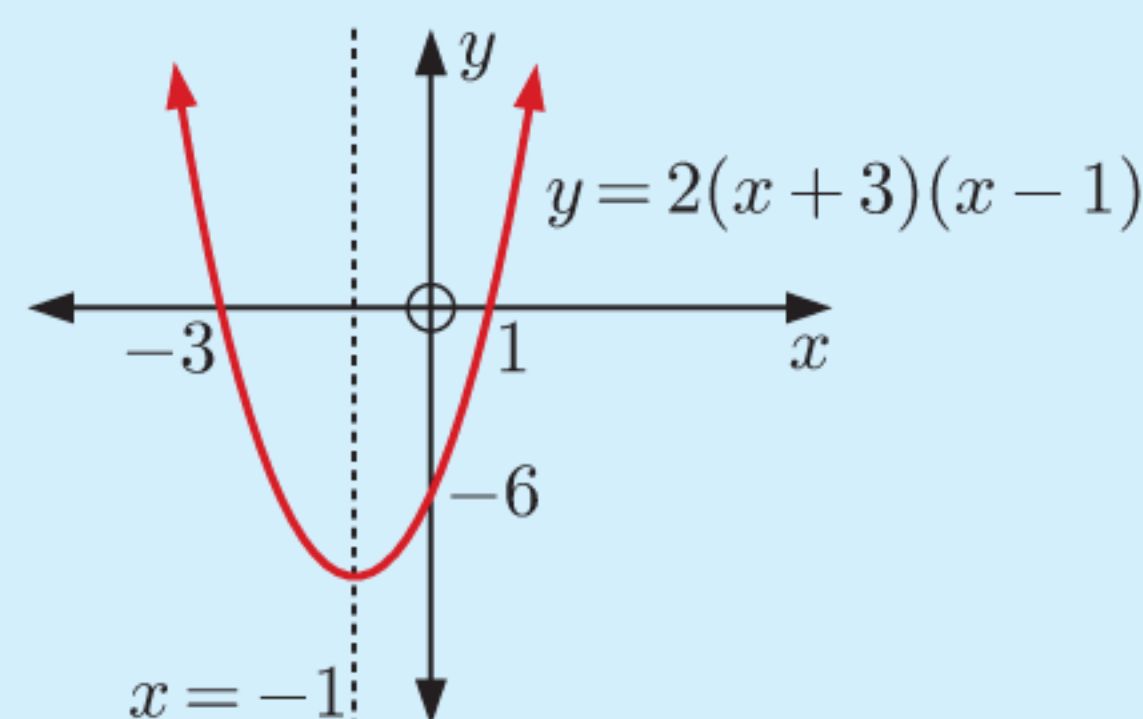
Example 3

Self Tutor

Sketch the graph using axes intercepts, and state the equation of the axis of symmetry:

a $y = 2(x + 3)(x - 1)$ **b** $y = -2(x - 1)(x - 2)$ **c** $y = \frac{1}{2}(x + 2)^2$

- a** $y = 2(x + 3)(x - 1)$
has x -intercepts $-3, 1$
When $x = 0$, $y = 2(3)(-1)$
 $= -6$
 \therefore y -intercept is -6

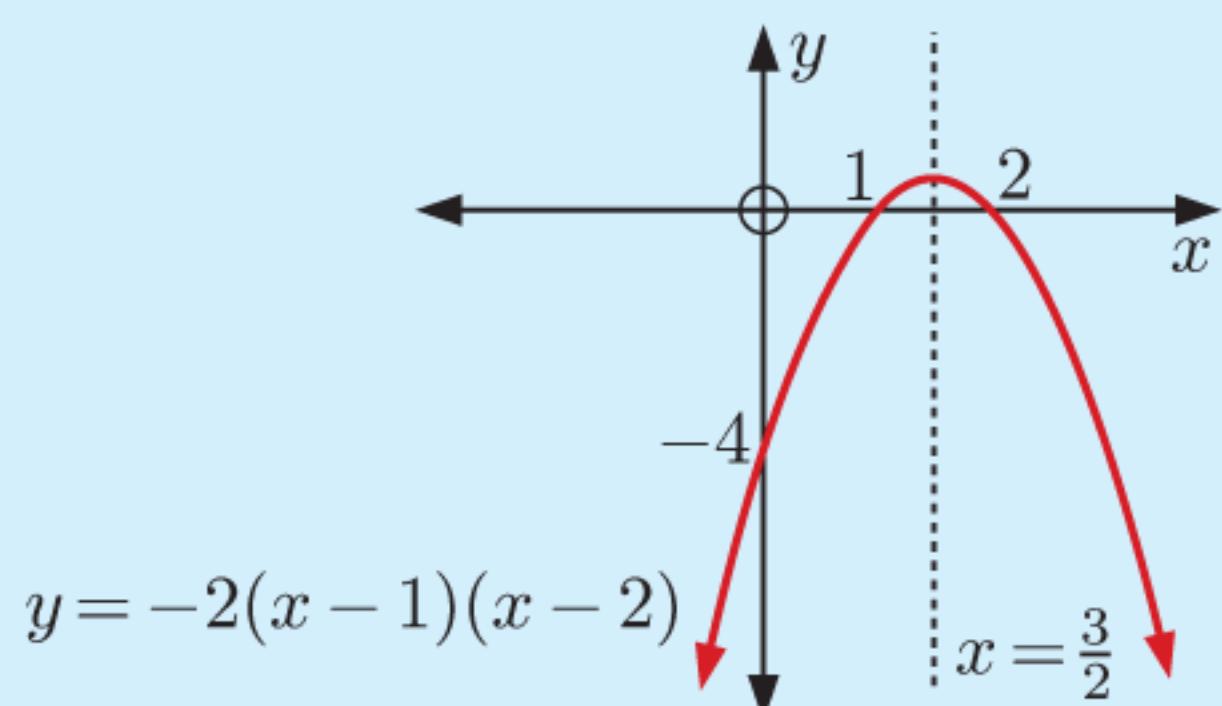


b $y = -2(x - 1)(x - 2)$

has x -intercepts 1, 2

When $x = 0$, $y = -2(-1)(-2)$
 $= -4$

\therefore y -intercept is -4

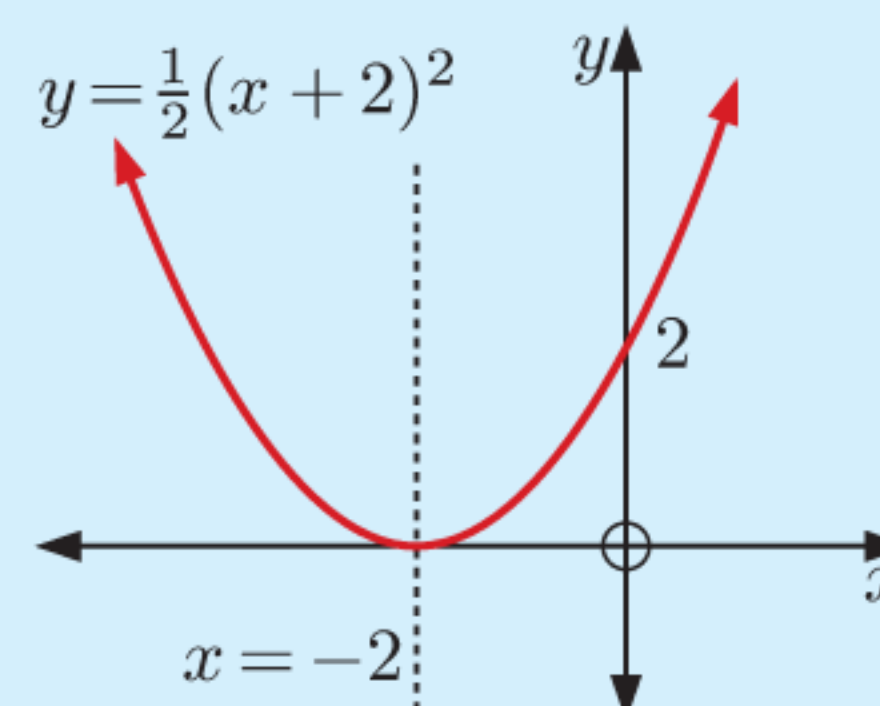


c $y = \frac{1}{2}(x + 2)^2$

touches x -axis at -2

When $x = 0$, $y = \frac{1}{2}(2)^2$
 $= 2$

\therefore y -intercept is 2



EXERCISE 14B.1

1 Sketch the graph using axes intercepts, and state the equation of the axis of symmetry:

a $y = (x - 4)(x + 2)$

b $y = -(x - 4)(x + 2)$

c $y = 2(x + 3)(x + 5)$

d $y = -3(x + 1)(x + 5)$

e $y = 2(x + 3)^2$

f $y = -\frac{1}{4}(x + 2)^2$

The axis of symmetry is midway between the x -intercepts.



2 Match each quadratic function with its corresponding graph.

a $y = 2(x - 1)(x - 4)$

b $y = -(x + 1)(x - 4)$

c $y = (x - 1)(x - 4)$

d $y = (x + 1)(x - 4)$

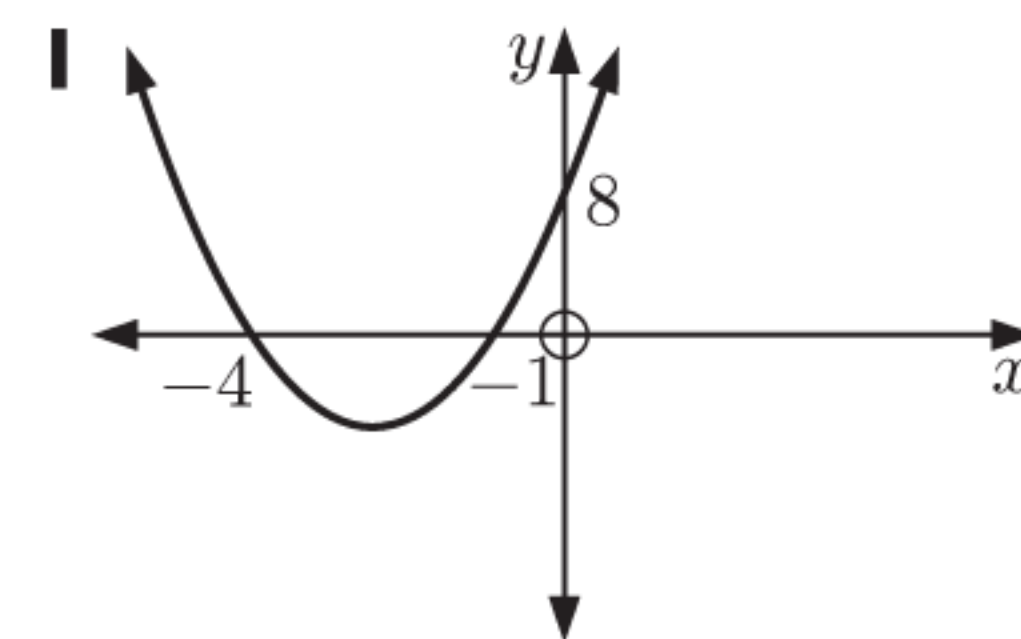
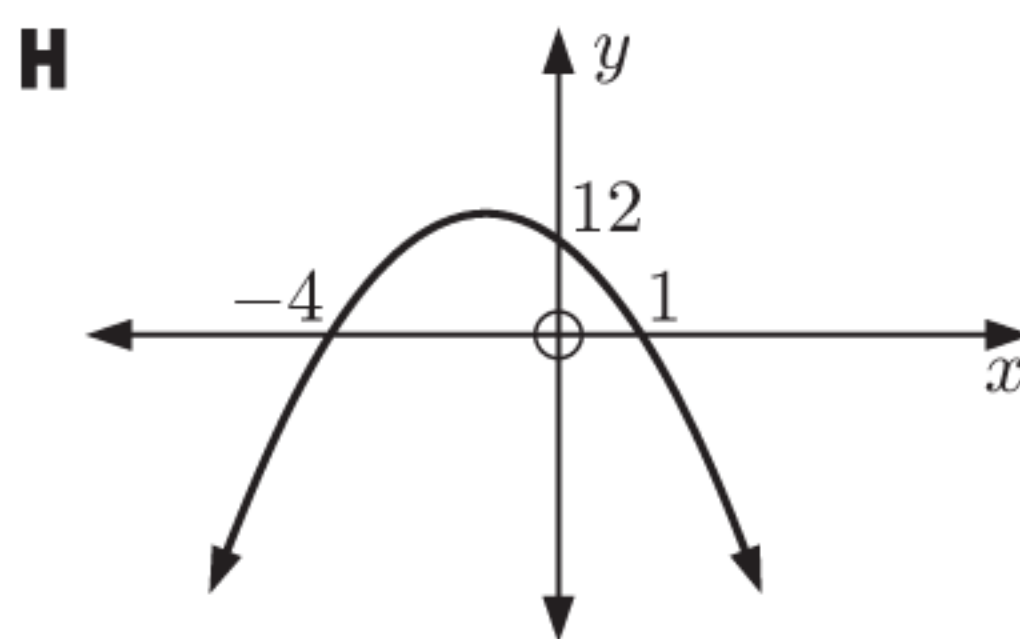
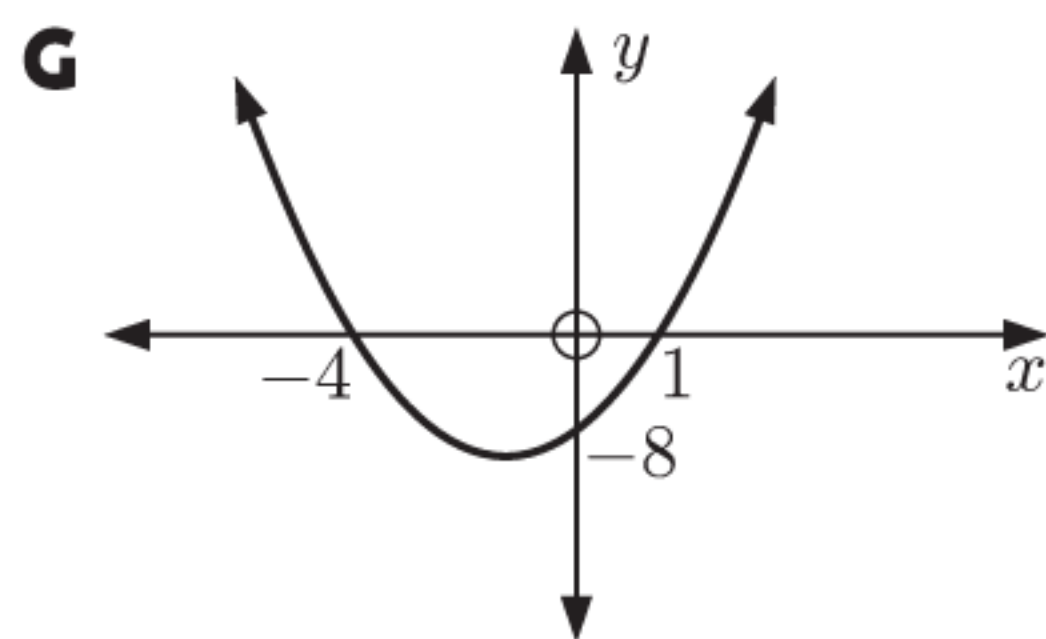
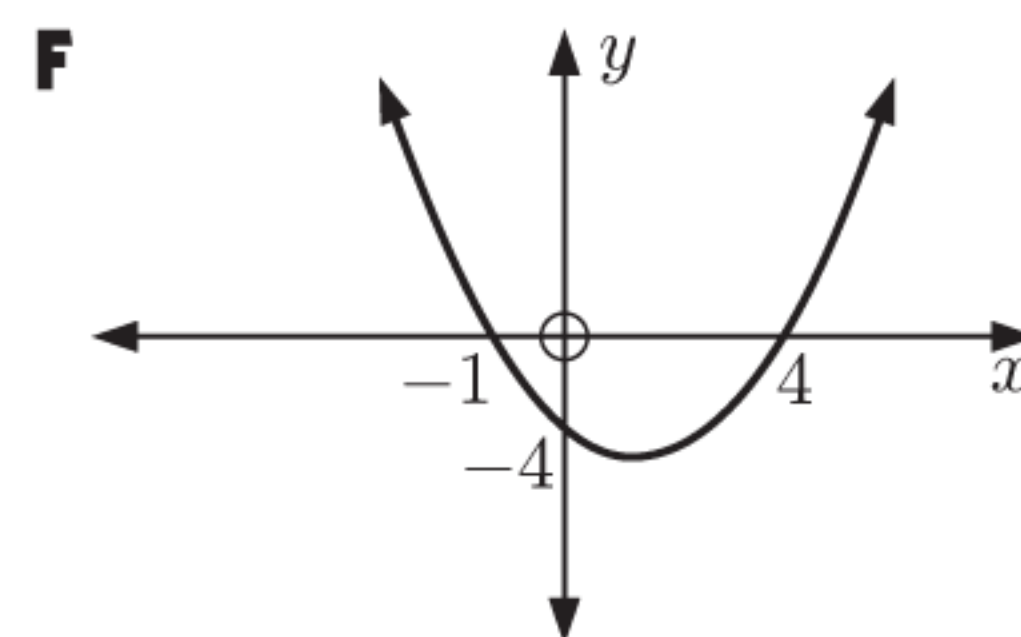
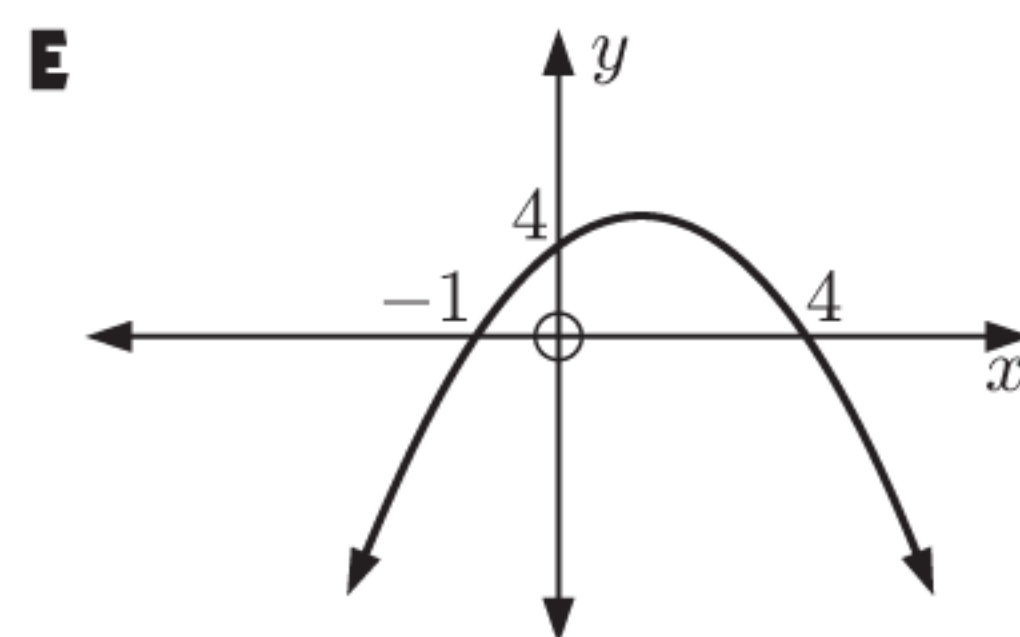
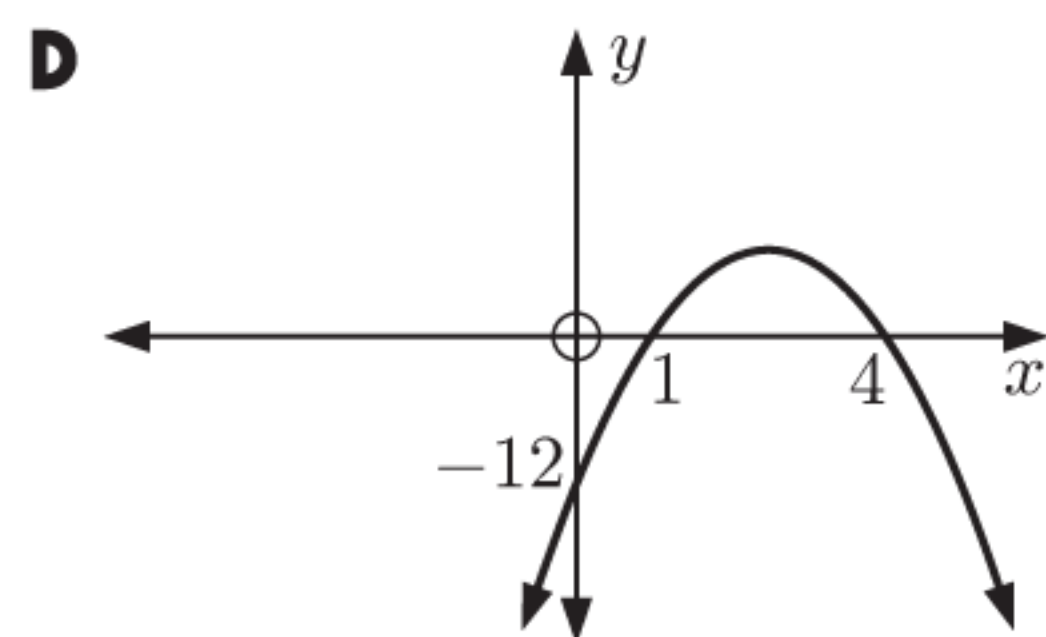
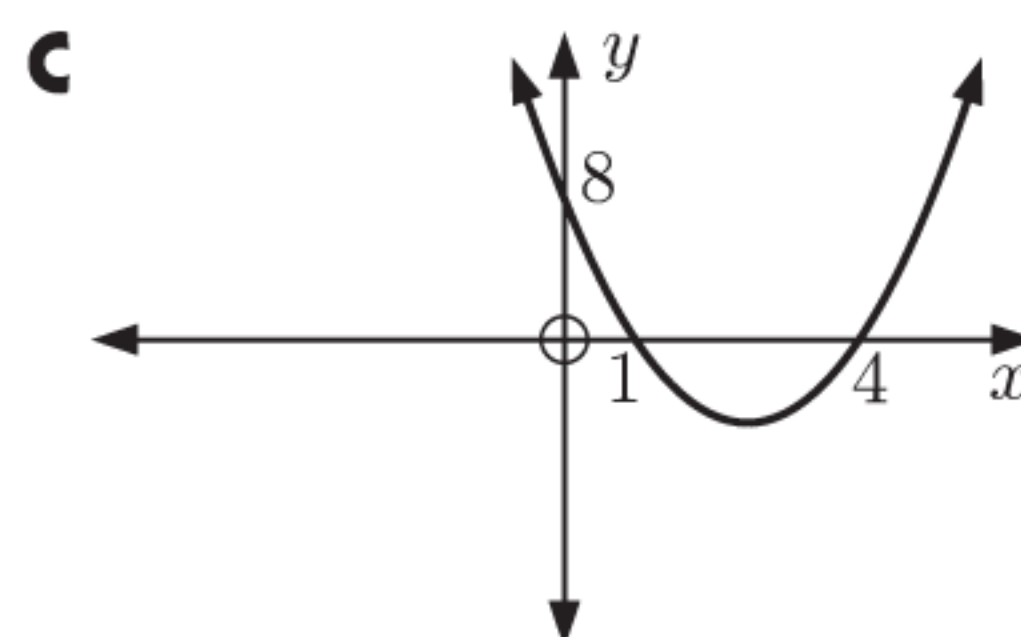
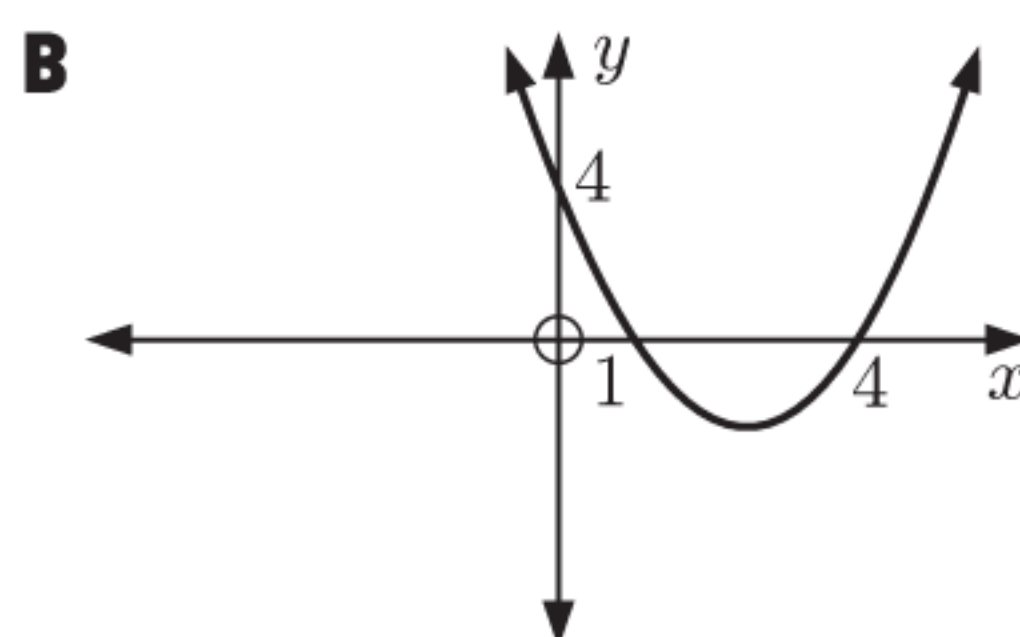
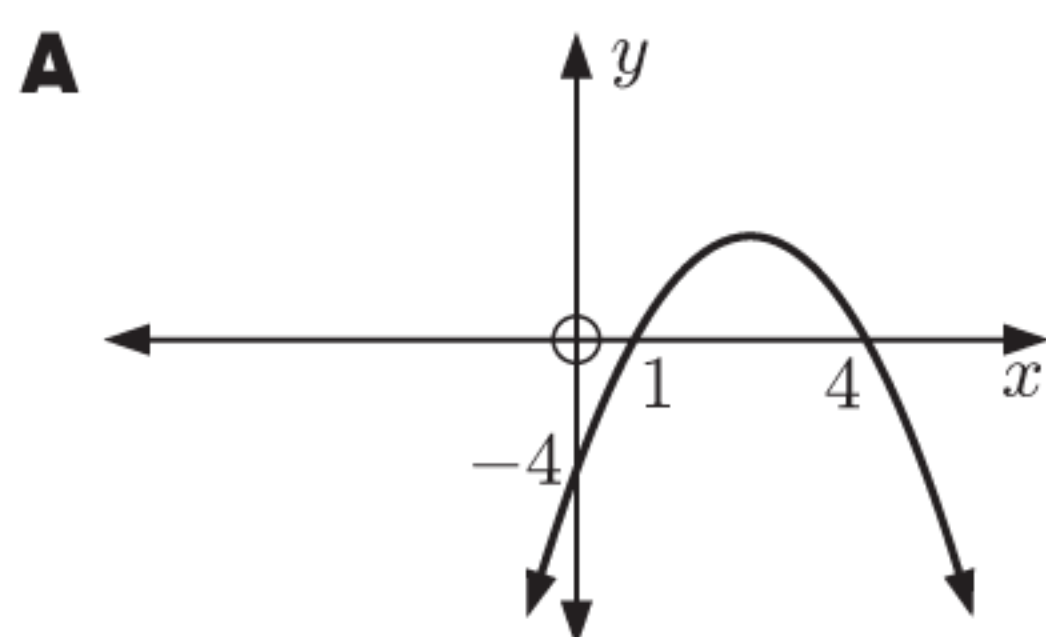
e $y = 2(x + 4)(x - 1)$

f $y = -3(x + 4)(x - 1)$

g $y = 2(x + 1)(x + 4)$

h $y = -(x - 1)(x - 4)$

i $y = -3(x - 1)(x - 4)$




Example 4**Self Tutor**

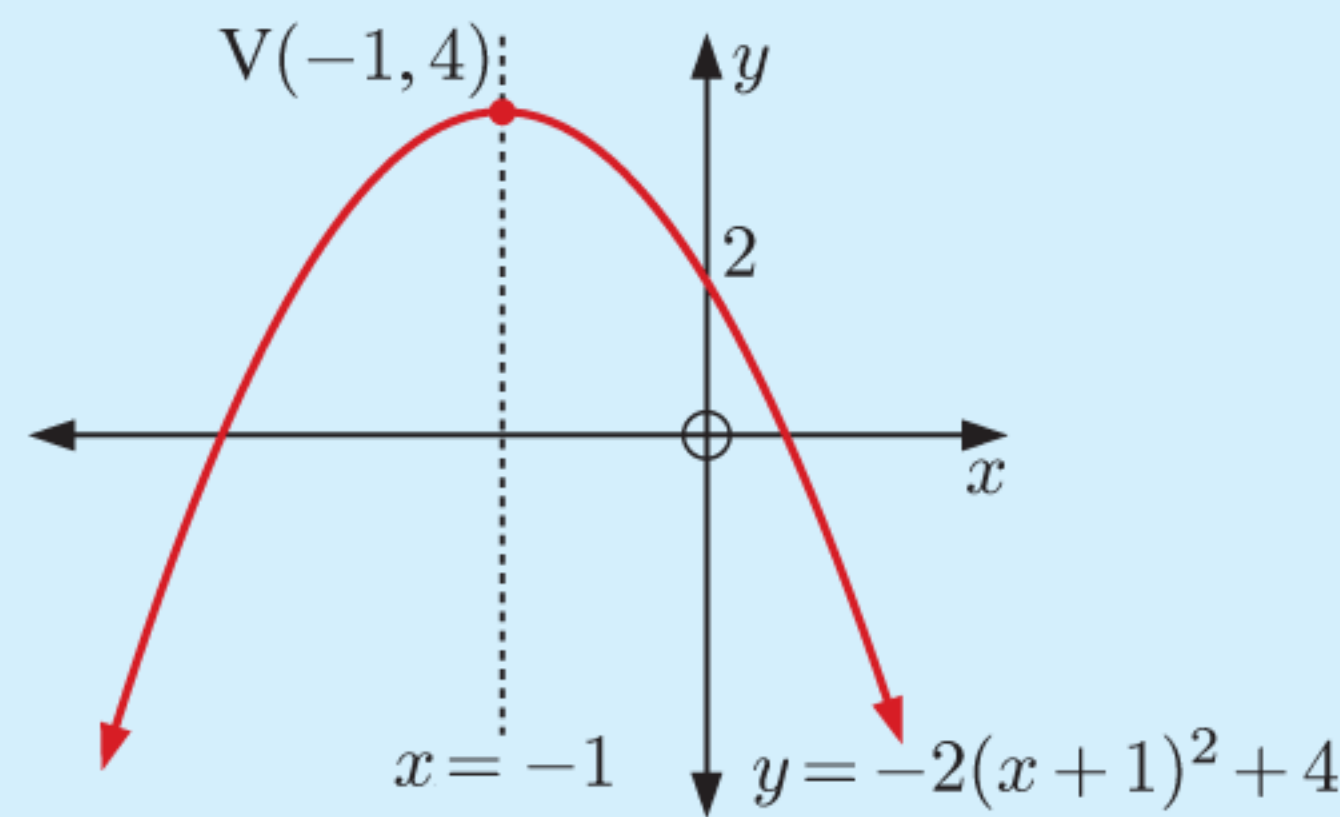
Use the vertex, axis of symmetry, and y -intercept to graph
 $y = -2(x + 1)^2 + 4$.

The axis of symmetry is $x = -1$.

The vertex is $(-1, 4)$.

When $x = 0$, $y = -2(1)^2 + 4$
 $= 2$

$a < 0$ so the shape is 



$y = a(x - h)^2 + k$
 is called **completed square form**.



3 Use the vertex, axis of symmetry, and y -intercept to graph:

a $y = (x - 1)^2 + 3$

b $y = (x + 3)^2 - 4$

c $y = -(x + 4)^2 + 2$

d $y = 2(x + 2)^2 + 1$

e $y = -2(x - 1)^2 - 3$

f $y = \frac{1}{3}(x + 6)^2 - 1$

g $y = \frac{1}{2}(x - 3)^2 + 2$

h $y = -\frac{1}{3}(x - 1)^2 + 4$

i $y = -\frac{1}{10}(x + 2)^2 - 3$

SKETCHING GRAPHS BY “COMPLETING THE SQUARE”

If we wish to graph a quadratic given in general form $y = ax^2 + bx + c$, one approach is to use “**completing the square**” to convert it to the completed square form $y = a(x - h)^2 + k$. We can then read off the coordinates of the vertex (h, k) .

Example 5**Self Tutor**

Write $y = x^2 + 4x + 3$ in the form $y = (x - h)^2 + k$ by “completing the square”.

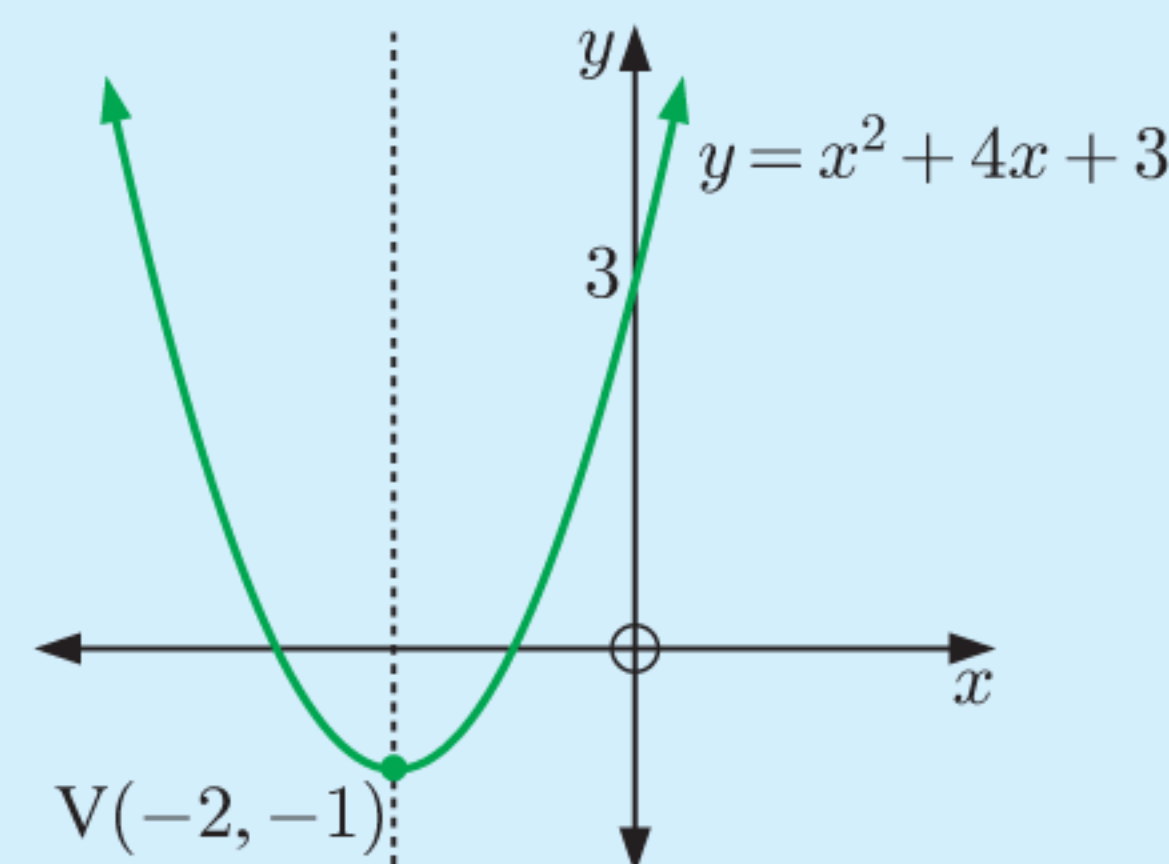
Hence sketch $y = x^2 + 4x + 3$, stating the coordinates of the vertex.

$$\begin{aligned} y &= x^2 + 4x + 3 \\ \therefore &= x^2 + 4x + 2^2 + 3 - 2^2 \\ \therefore y &= (x + 2)^2 - 1 \end{aligned}$$

So, the axis of symmetry is $x = -2$
 and the vertex is $(-2, -1)$.

When $x = 0$, $y = 3$

\therefore the y -intercept is 3.

**EXERCISE 14B.2**

1 Write the following quadratics in the form $y = (x - h)^2 + k$ by “completing the square”. Hence sketch each function, stating the coordinates of the vertex.

a $y = x^2 - 2x + 3$

b $y = x^2 + 4x - 2$

c $y = x^2 - 4x$

d $y = x^2 + 3x$

e $y = x^2 + 5x - 2$

f $y = x^2 - 3x + 2$

g $y = x^2 - 6x + 5$

h $y = x^2 + 8x - 2$

i $y = x^2 - 5x + 1$

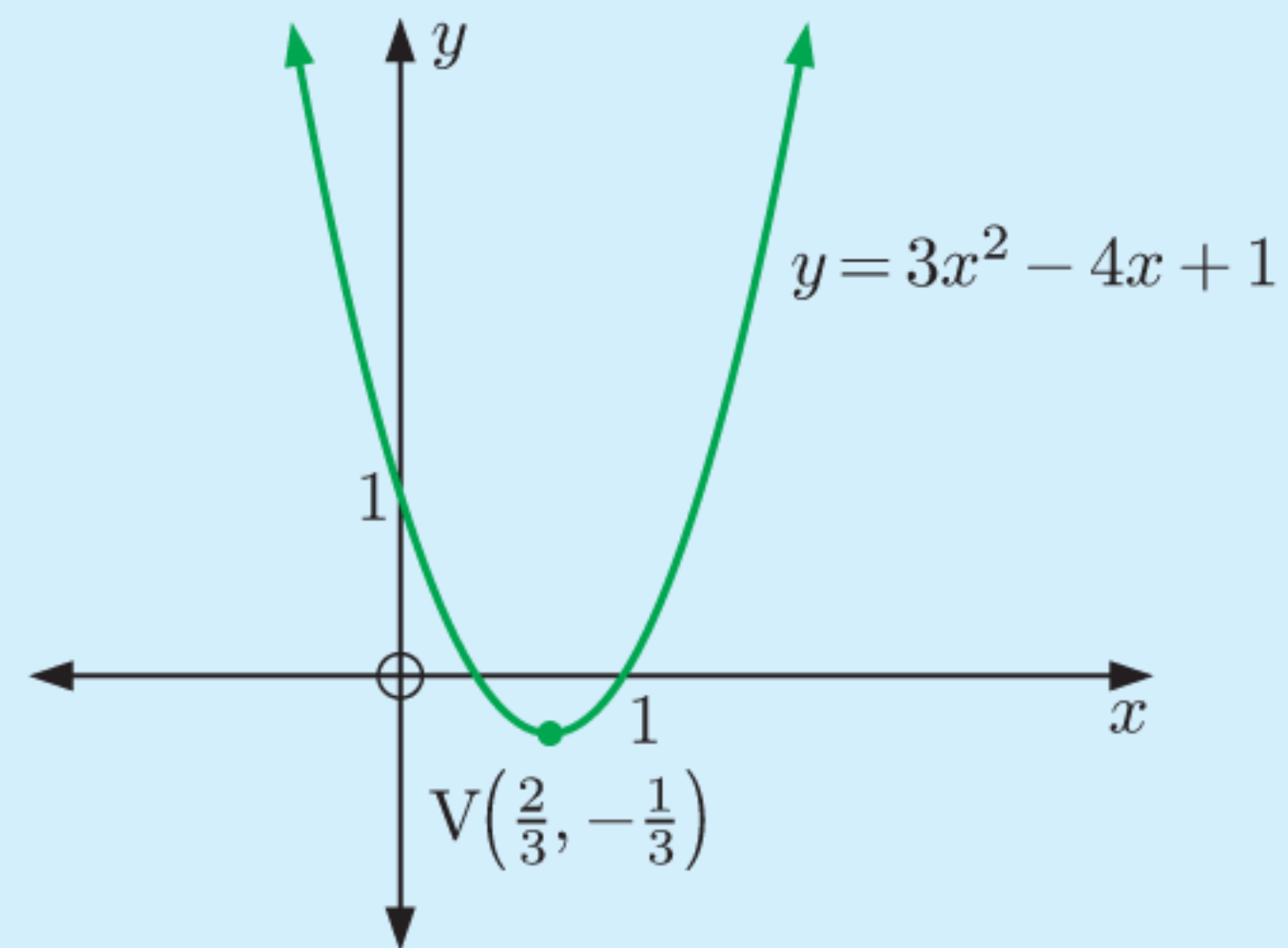
Example 6

Self Tutor

- a** Convert $y = 3x^2 - 4x + 1$ to the completed square form $y = a(x - h)^2 + k$.
- b** Hence write down the coordinates of the vertex, and sketch the quadratic.

a $y = 3x^2 - 4x + 1$
 $= 3[x^2 - \frac{4}{3}x + \frac{1}{3}]$
 $= 3[x^2 - 2(\frac{2}{3})x + (\frac{2}{3})^2 + \frac{1}{3} - (\frac{2}{3})^2]$
 $= 3[(x - \frac{2}{3})^2 + \frac{3}{9} - \frac{4}{9}]$
 $= 3[(x - \frac{2}{3})^2 - \frac{1}{9}]$
 $= 3(x - \frac{2}{3})^2 - \frac{1}{3}$

- b** The vertex is $(\frac{2}{3}, -\frac{1}{3})$
 and the y -intercept is 1.



2 For each of the following quadratics:

- i** Write the quadratic in the completed square form $y = a(x - h)^2 + k$.
- ii** State the coordinates of the vertex.
- iii** Find the y -intercept.
- iv** Sketch the graph of the quadratic.

- | | |
|---|--|
| a $y = 2x^2 + 4x + 5$ | b $y = 2x^2 - 8x + 3$ |
| c $y = 2x^2 - 6x + 1$ | d $y = 3x^2 - 6x + 5$ |
| e $y = -x^2 + 4x + 2$ | f $y = -2x^2 - 5x + 3$ |
| g $y = -\frac{1}{3}x^2 + 2x - 3$ | h $y = \frac{1}{2}x^2 + 3x - 4$ |

Take out the factor a , then complete the square.



SKETCHING QUADRATICS IN THE GENERAL FORM $y = ax^2 + bx + c$

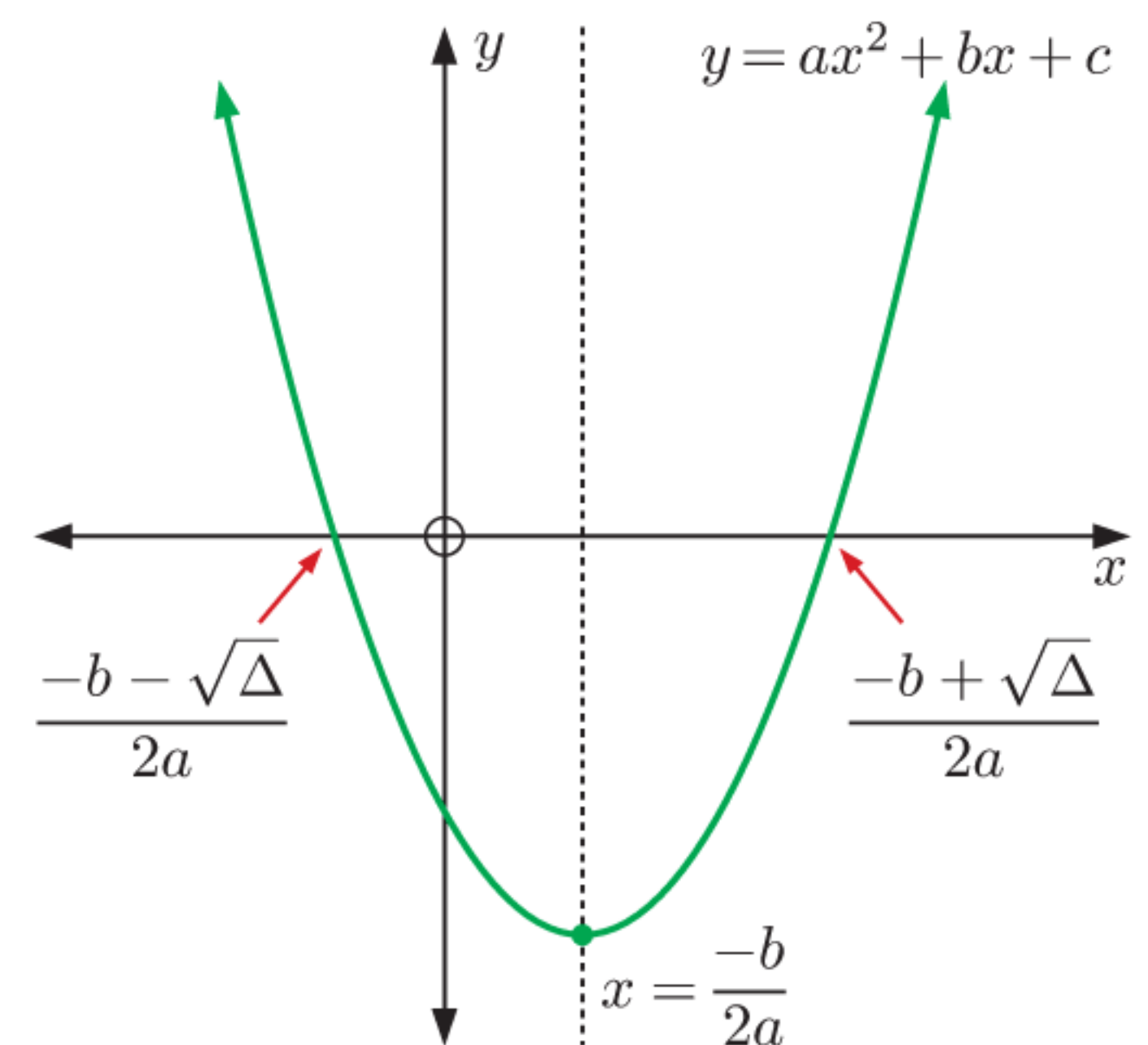
We now consider a method of graphing quadratics of the form $y = ax^2 + bx + c$ directly, without having to first convert them to a different form.

We know that the quadratic equation $ax^2 + bx + c = 0$ has solutions $\frac{-b \pm \sqrt{\Delta}}{2a}$ where $\Delta = b^2 - 4ac$.

If $\Delta \geq 0$, these are the x -intercepts of the graph of the quadratic function $y = ax^2 + bx + c$.

The average of the values is $\frac{-b}{2a}$, so we conclude that:

- the axis of symmetry is $x = \frac{-b}{2a}$
- the vertex of the quadratic has x -coordinate $\frac{-b}{2a}$.




To graph a quadratic of the form $y = ax^2 + bx + c$, we:

- Find the axis of symmetry $x = \frac{-b}{2a}$.
- Substitute this value to find the y -coordinate of the vertex.
- State the y -intercept c .
- Find the x -intercepts by solving $ax^2 + bx + c = 0$, either by factorisation or using the quadratic formula.
- Graph the quadratic using the information you have found.

Example 7**Self Tutor**

Consider the quadratic $y = 2x^2 + 8x - 10$.

- a** Find the axis of symmetry. **b** Find the coordinates of the vertex.
c Find the axes intercepts. **d** Hence sketch the quadratic.

$y = 2x^2 + 8x - 10$ has $a = 2$, $b = 8$, and $c = -10$. Since $a > 0$, the shape is 

a $\frac{-b}{2a} = \frac{-8}{2(2)} = -2$

The axis of symmetry is $x = -2$.

- c** The y -intercept is -10 .

When $y = 0$, $2x^2 + 8x - 10 = 0$

$$\therefore 2(x^2 + 4x - 5) = 0$$

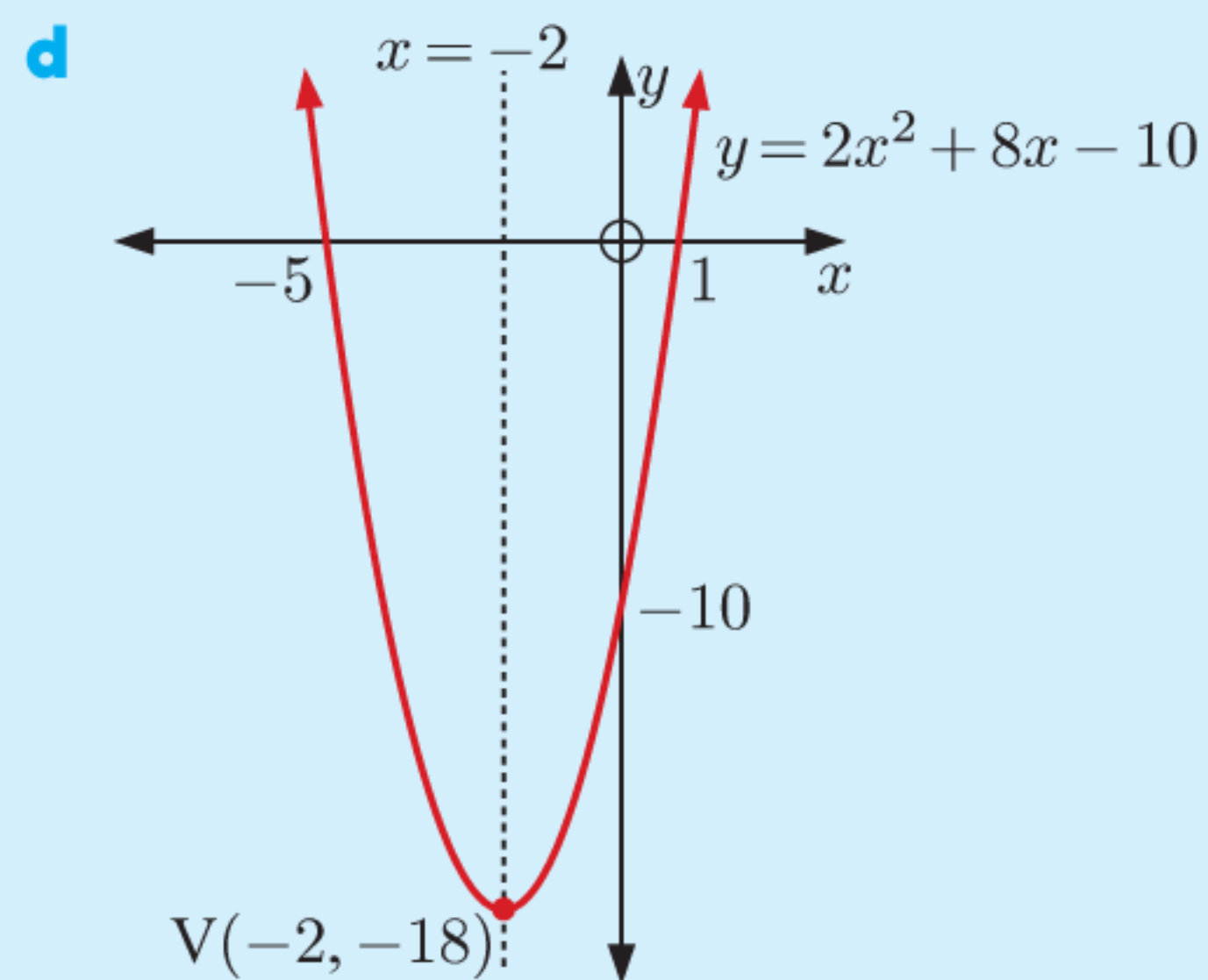
$$\therefore 2(x + 5)(x - 1) = 0$$

$$\therefore x = -5 \text{ or } 1$$

\therefore the x -intercepts are -5 and 1 .

- b** When $x = -2$,
- $$y = 2(-2)^2 + 8(-2) - 10 = -18$$

The vertex is $(-2, -18)$.

**EXERCISE 14B.3**

- 1** For each of the following quadratics:
- i** Locate the turning point or vertex.
 - ii** State whether the vertex is a minimum turning point or a maximum turning point.
- a** $y = x^2 - 4x + 2$ **b** $y = x^2 + 2x - 3$
c $y = 2x^2 + 4$ **d** $y = -3x^2 + 1$
e $y = 2x^2 + 8x - 7$ **f** $y = -x^2 - 4x - 9$
g $y = 2x^2 + 6x - 1$ **h** $y = 2x^2 - 10x + 3$
i $y = -\frac{1}{2}x^2 + x - 5$ **j** $y = \frac{1}{4}x^2 - 7x + 6$

The vertex lies on the axis of symmetry.



2 For each of the following quadratics:

i State the axis of symmetry.

ii Find the coordinates of the vertex.

iii Find the axes intercepts.

iv Hence sketch the quadratic.

a $y = x^2 - 8x + 7$

b $y = -x^2 - 6x - 8$

c $y = 6x - x^2$

d $y = -x^2 + 3x - 2$

e $y = 2x^2 + 4x - 24$

f $y = -3x^2 + 4x - 1$

g $y = 2x^2 - 5x + 2$

h $y = 4x^2 - 8x - 5$

i $y = -\frac{1}{4}x^2 + 2x - 3$

3 For the quadratic function $y = ax^2 + bx + c$, suppose a and c remain constant, but b is allowed to vary. As b varies, the vertex of the quadratic changes.

Show that the path formed by the vertex is itself a quadratic function, and that the vertex of this quadratic function always lies on the y -axis.

DYNAMIC GEOMETRY PACKAGE



ACTIVITY 2

Click on the icon to run a card game for quadratic functions.

CARD GAME



C

USING THE DISCRIMINANT

The discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $\Delta = b^2 - 4ac$.

We have used Δ to determine the number of real roots of the equation. If they exist, these roots correspond to zeros of the quadratic $y = ax^2 + bx + c$. Δ therefore tells us about the relationship between the graph of a quadratic function and the x -axis.

The graphs of $y = x^2 - 2x - 3$, $y = x^2 - 2x + 1$, and $y = x^2 - 2x + 3$ all have the same axis of symmetry, $x = 1$.

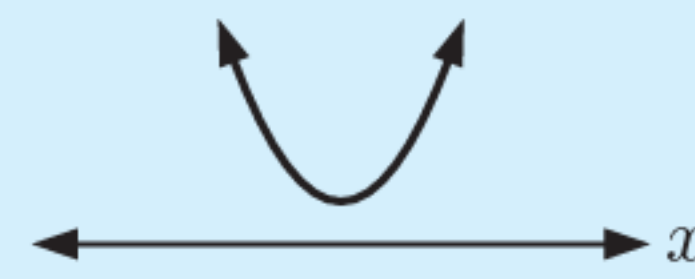
| $y = x^2 - 2x - 3$ | $y = x^2 - 2x + 1$ | $y = x^2 - 2x + 3$ |
|---|--|--|
| | | |
| $\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(-3)$ $= 16$ <p>cuts the x-axis twice</p> | $\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(1)$ $= 0$ <p>touches the x-axis</p> | $\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(3)$ $= -8$ <p>does not cut the x-axis</p> |

For a quadratic function $y = ax^2 + bx + c$, we consider the discriminant $\Delta = b^2 - 4ac$.

- If $\Delta > 0$, the graph cuts the x -axis twice.
- If $\Delta = 0$, the graph *touches* the x -axis.
- If $\Delta < 0$, the graph does not cut the x -axis.

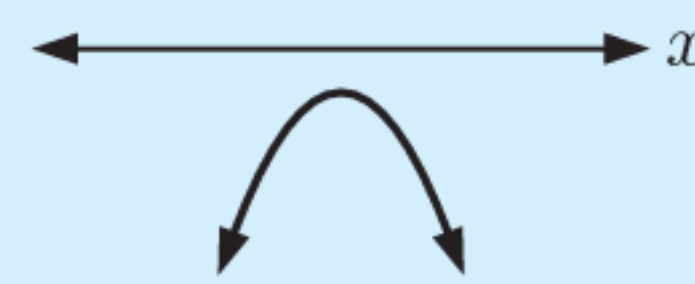
POSITIVE DEFINITE AND NEGATIVE DEFINITE QUADRATICS

Positive definite quadratics are quadratics which are positive for all values of x . So, $ax^2 + bx + c > 0$ for all $x \in \mathbb{R}$.



A quadratic is **positive definite** if and only if $a > 0$ and $\Delta < 0$.

Negative definite quadratics are quadratics which are negative for all values of x . So, $ax^2 + bx + c < 0$ for all $x \in \mathbb{R}$.



A quadratic is **negative definite** if and only if $a < 0$ and $\Delta < 0$.

Example 8

Self Tutor

Use the discriminant to determine the relationship between the graph of each function and the x -axis:

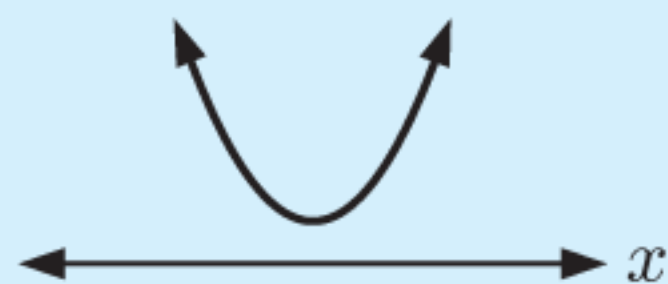
a $y = x^2 + 3x + 4$

b $y = -2x^2 + 5x + 1$

a $a = 1, b = 3, c = 4$
 $\therefore \Delta = b^2 - 4ac$
 $= 9 - 4(1)(4)$
 $= -7$

Since $\Delta < 0$, the graph does not cut the x -axis.

Since $a > 0$, the graph is concave up. The graph is positive definite.



b $a = -2, b = 5, c = 1$
 $\therefore \Delta = b^2 - 4ac$
 $= 25 - 4(-2)(1)$
 $= 33$

Since $\Delta > 0$, the graph cuts the x -axis twice.

Since $a < 0$, the graph is concave down. The quadratic is neither positive definite nor negative definite.



EXERCISE 14C

1 Use the discriminant to determine the relationship between the graph of each function and the x -axis:

a $y = x^2 + x - 2$

b $y = x^2 - 4x + 1$

c $y = -x^2 - 3$

d $y = x^2 + 7x - 2$

e $y = x^2 + 8x + 16$

f $y = -2x^2 + 3x + 1$

g $y = 6x^2 + 5x - 4$

h $y = -x^2 + x + 6$

i $y = 9x^2 + 6x + 1$

2 Consider the graph of $y = 2x^2 - 5x + 1$.

a Describe the shape of the graph.

b Use the discriminant to show that the graph cuts the x -axis twice.

c Find the x -intercepts, rounding your answers to 2 decimal places.

d State the y -intercept.

e Hence sketch the function.

- 3** Consider the graph of $y = -x^2 + 4x - 7$.
- Use the discriminant to show that the graph does not cut the x -axis.
 - Is the graph positive definite or negative definite? Explain your answer.
 - Find the vertex and y -intercept.
 - Hence sketch the function.

4 Show that:

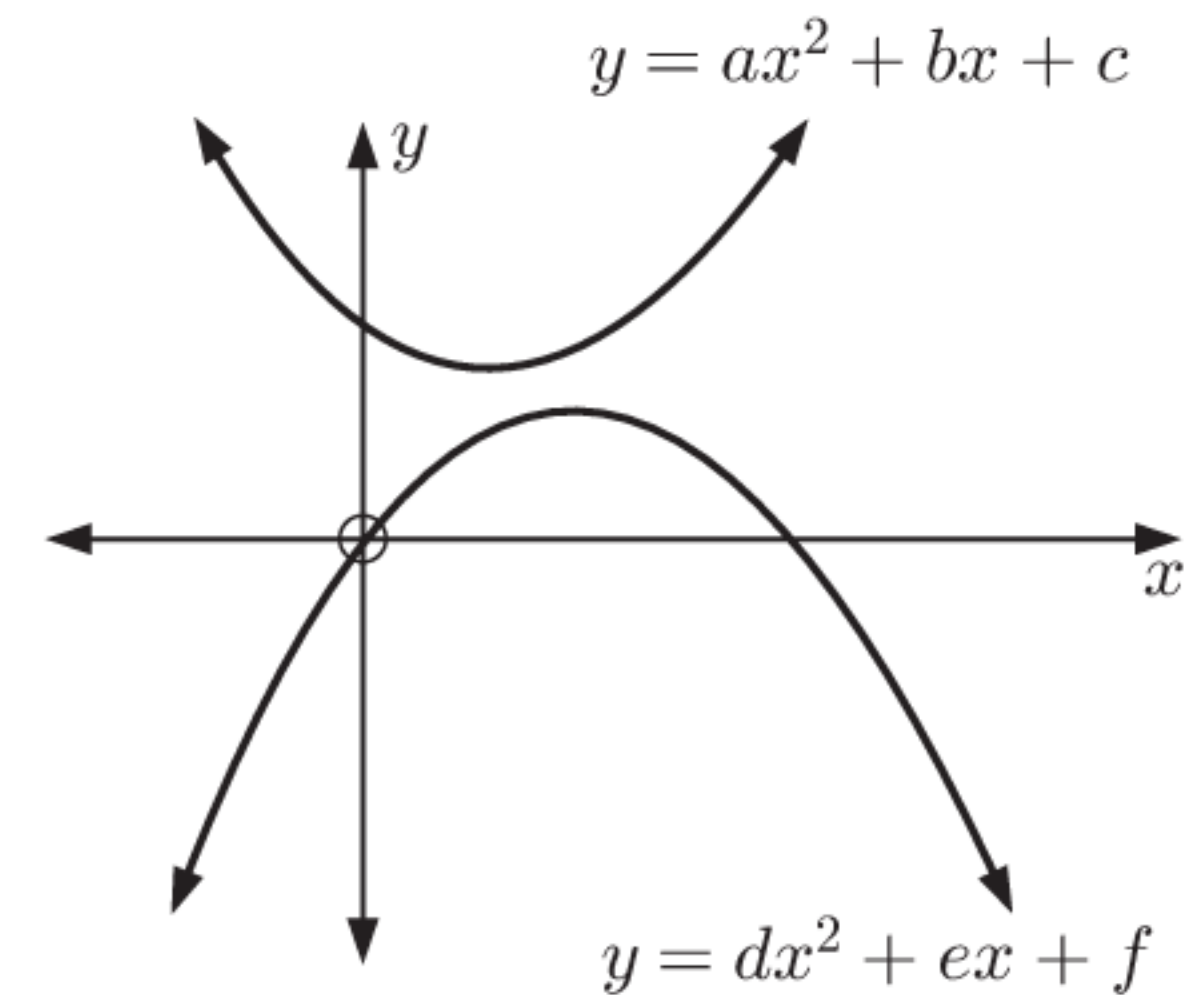
- $2x^2 - 4x + 7$ is positive definite
- $-2x^2 + 3x - 4$ is negative definite
- $x^2 - 3x + 6 > 0$ for all x
- $4x - x^2 - 6 < 0$ for all x .

5 Consider the graphs illustrated.

Let $y = ax^2 + bx + c$ have discriminant Δ_1 , and $y = dx^2 + ex + f$ have discriminant Δ_2 .

Copy and complete the following table by indicating whether each constant is positive, negative, or zero:

| Constant | a | b | c | d | e | f | Δ_1 | Δ_2 |
|----------|-----|-----|-----|-----|-----|-----|------------|------------|
| Sign | | | | | | | | |



Example 9

Self Tutor

Find the value(s) of k for which the function $y = x^2 - 6x + k$:

- cuts the x -axis twice
- touches the x -axis
- misses the x -axis.

$$a = 1, \quad b = -6, \quad c = k$$

$$\begin{aligned} \therefore \Delta &= b^2 - 4ac \\ &= (-6)^2 - 4(1)(k) \\ &= 36 - 4k \end{aligned}$$

- a** The graph cuts the x -axis twice if $\Delta > 0$.

$$\begin{aligned} \therefore 36 - 4k &> 0 \\ \therefore 4k &< 36 \\ \therefore k &< 9 \end{aligned}$$

- b** The graph touches the x -axis twice if $\Delta = 0$.

$$\begin{aligned} \therefore 36 - 4k &= 0 \\ \therefore k &= 9 \end{aligned}$$

- c** The graph does not cut the x -axis if $\Delta < 0$.

$$\begin{aligned} \therefore 36 - 4k &< 0 \\ \therefore 4k &> 36 \\ \therefore k &> 9 \end{aligned}$$

6 For each quadratic function, find the value(s) of k for which the function:

- cuts the x -axis twice
- touches the x -axis
- misses the x -axis.

$$\mathbf{a} \quad y = x^2 + 3x + k \quad \mathbf{b} \quad y = kx^2 - 4x + 1 \quad \mathbf{c} \quad y = (k + 1)x^2 - 2kx + (k - 4)$$

7 Explain why $3x^2 + kx - 1$ is never positive definite for any value of k .

8 Find the value of k such that $y = \frac{1}{2}x^2 + (k - 2)x + k^2 + 4$ is *not* positive definite. What relationship does the graph have with the x -axis in this case?

9 $b_1, c_1, b_2,$ and c_2 are real, non-zero numbers such that $b_1b_2 = 2(c_1 + c_2)$. Show that at least one of the quadratics $y = x^2 + b_1x + c_1$, and $y = x^2 + b_2x + c_2$ cuts the x -axis twice.

D

FINDING A QUADRATIC FROM ITS GRAPH

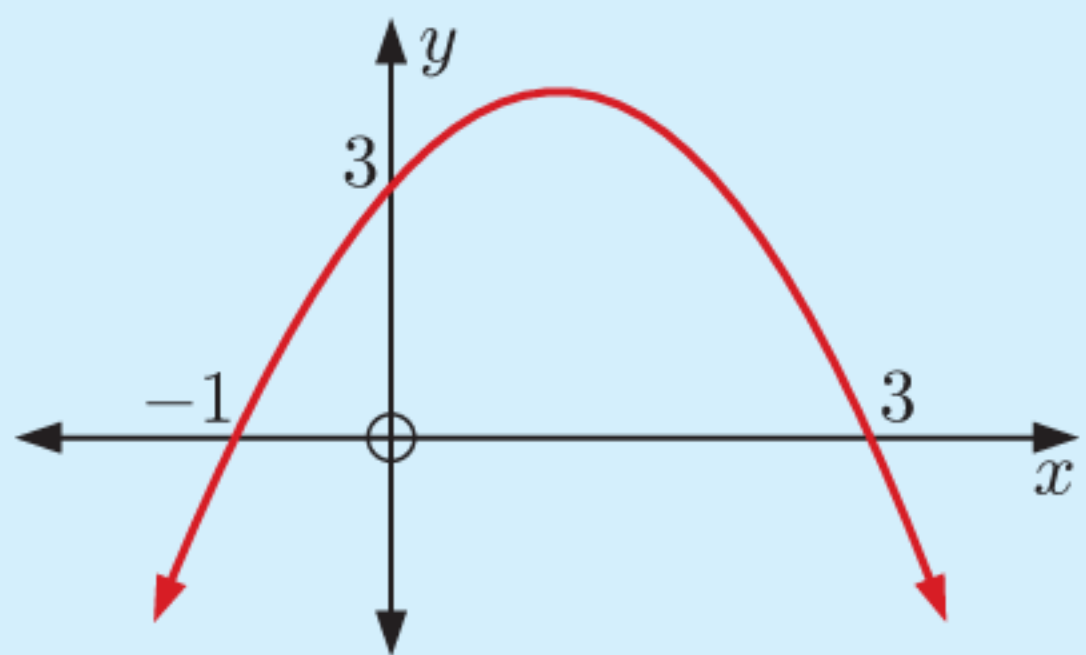
If we are given sufficient information on or about a graph, we can determine the quadratic in whatever form is required.

Example 10

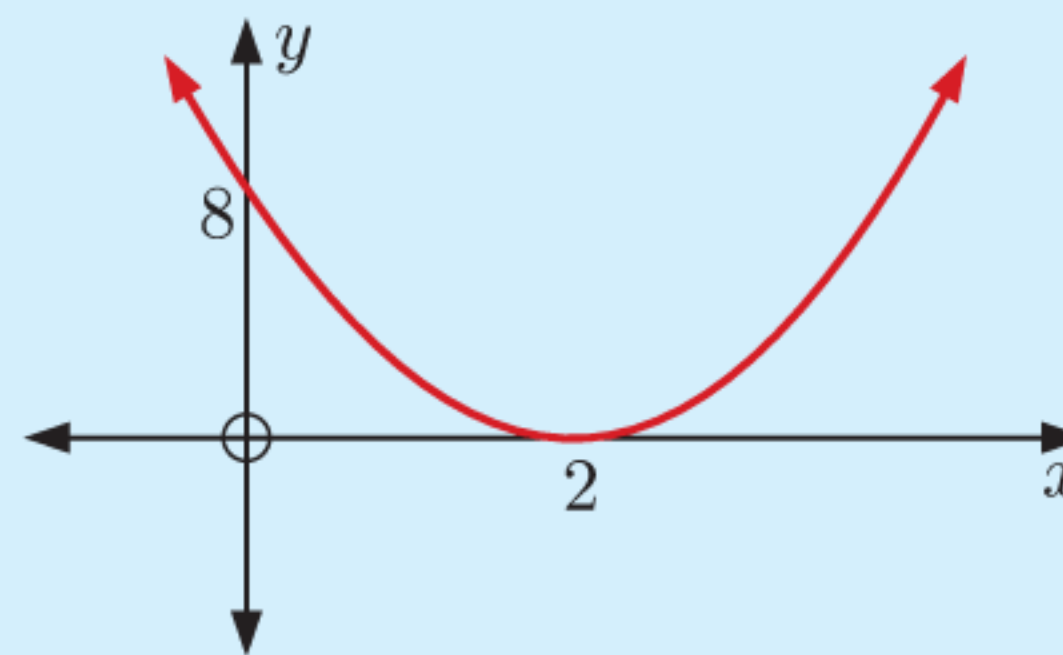
Self Tutor

Find the equation of the quadratic with graph:

a



b



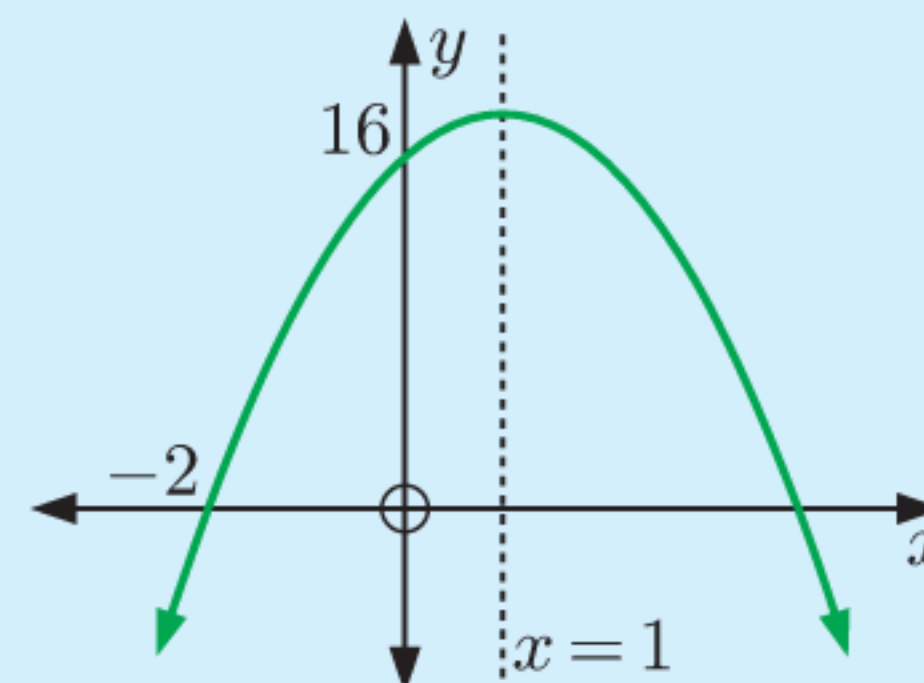
- a Since the x -intercepts are -1 and 3 ,
 $y = a(x + 1)(x - 3)$.
 The graph is concave down, so $a < 0$.
 When $x = 0$, $y = 3$
 $\therefore 3 = a(1)(-3)$
 $\therefore a = -1$
 The quadratic is $y = -(x + 1)(x - 3)$.

- b The graph touches the x -axis at $x = 2$,
 so $y = a(x - 2)^2$.
 The graph is concave up, so $a > 0$.
 When $x = 0$, $y = 8$
 $\therefore 8 = a(-2)^2$
 $\therefore a = 2$
 The quadratic is $y = 2(x - 2)^2$.

Example 11

Self Tutor

Find the equation of the quadratic with graph:



The axis of symmetry $x = 1$ lies midway between the x -intercepts.

\therefore the other x -intercept is 4.

\therefore the quadratic has the form

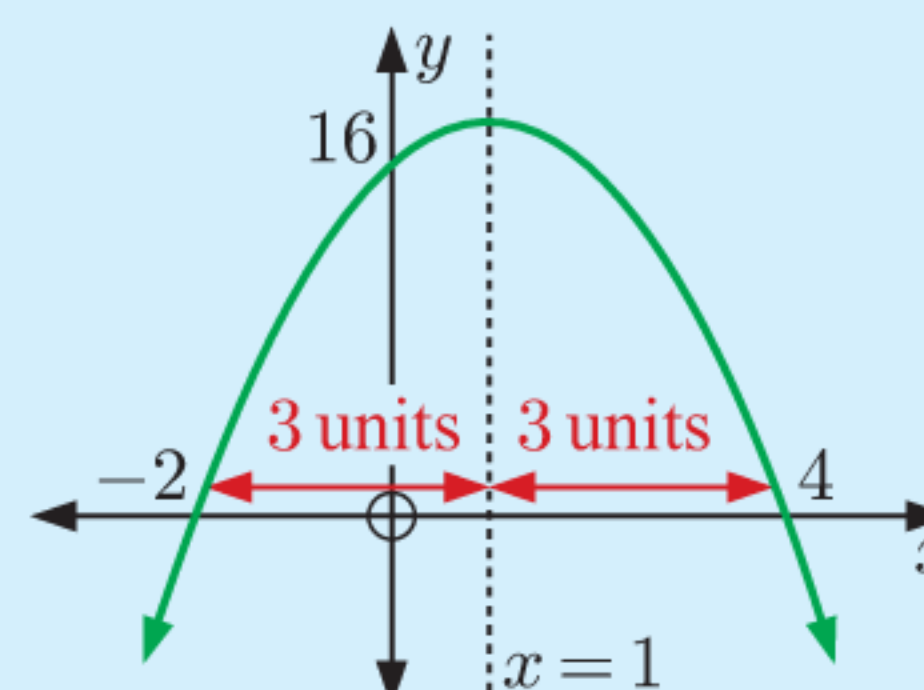
$$y = a(x + 2)(x - 4) \quad \text{where } a < 0$$

But when $x = 0$, $y = 16$

$$\therefore 16 = a(2)(-4)$$

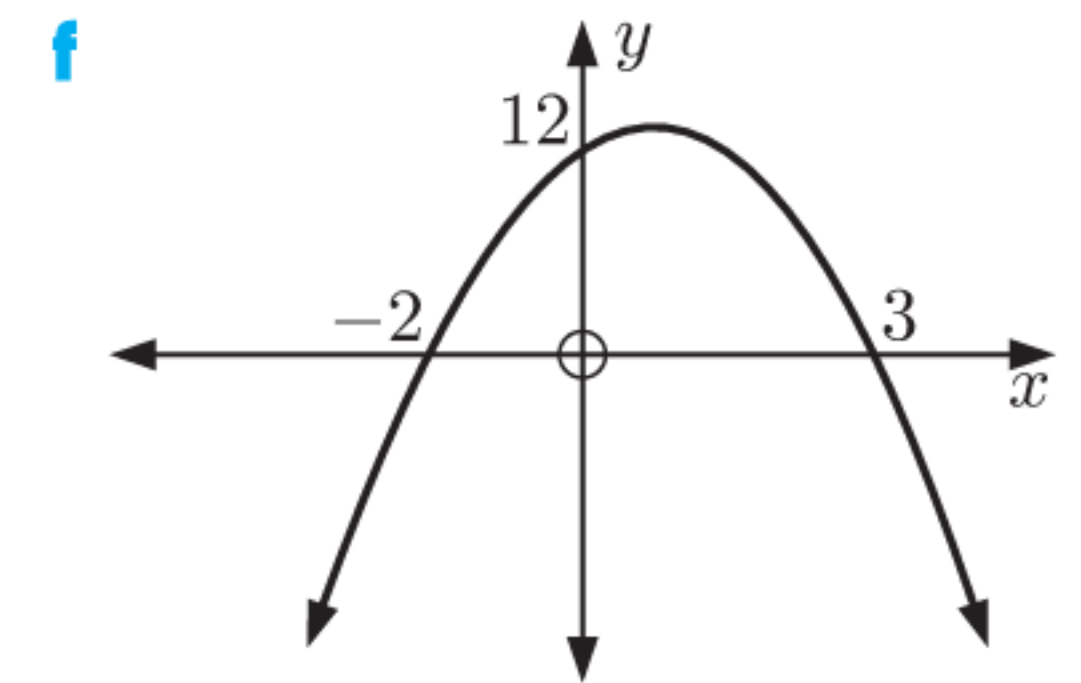
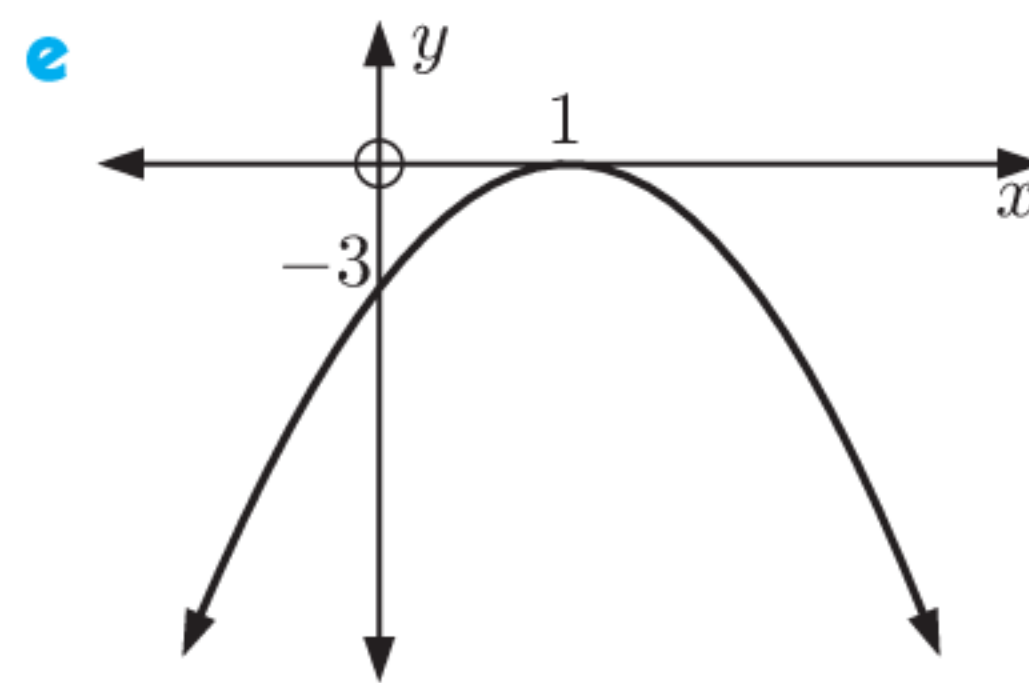
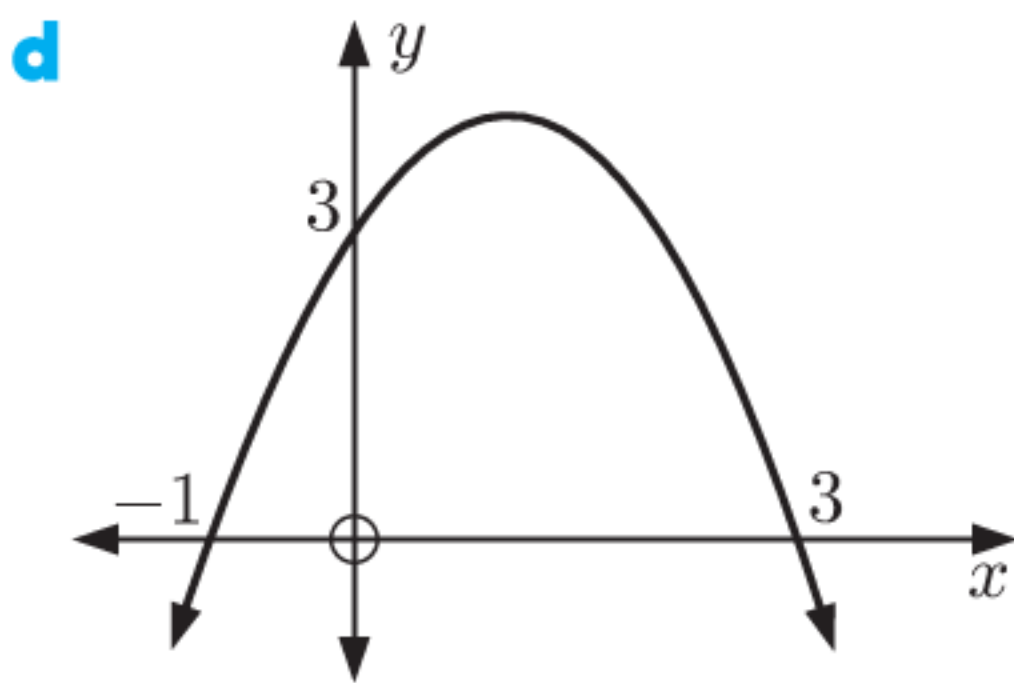
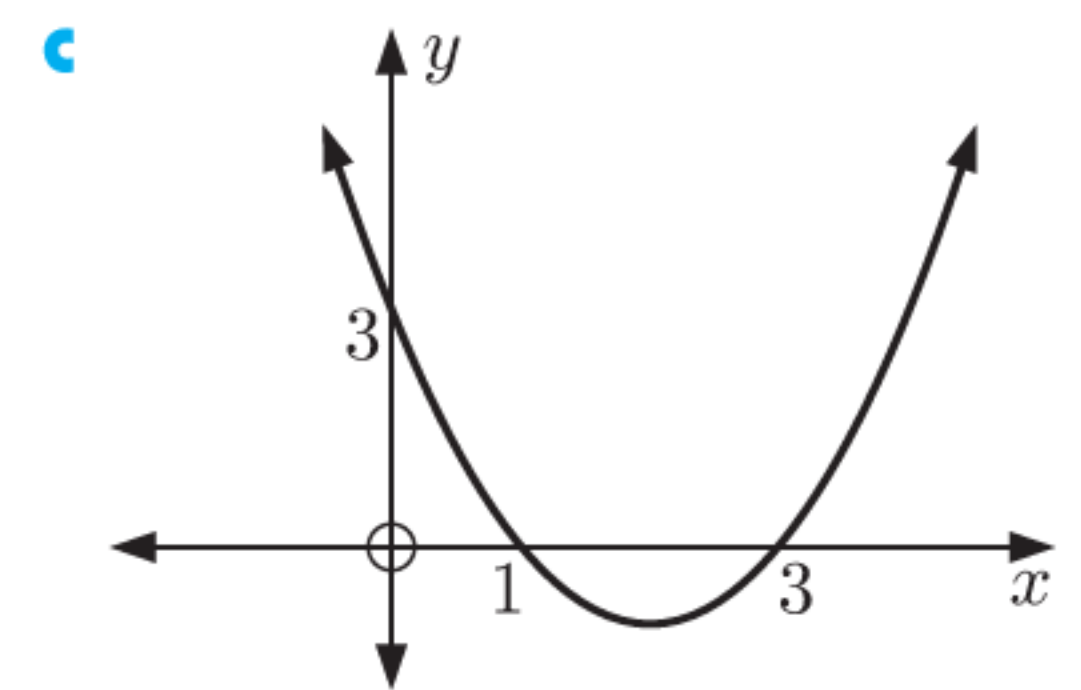
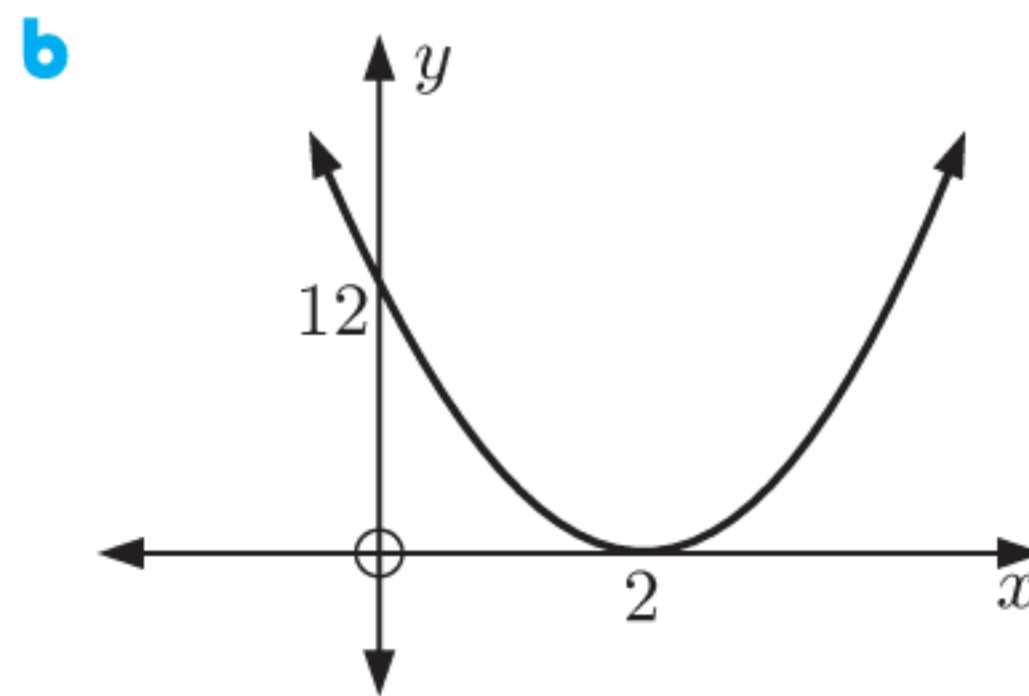
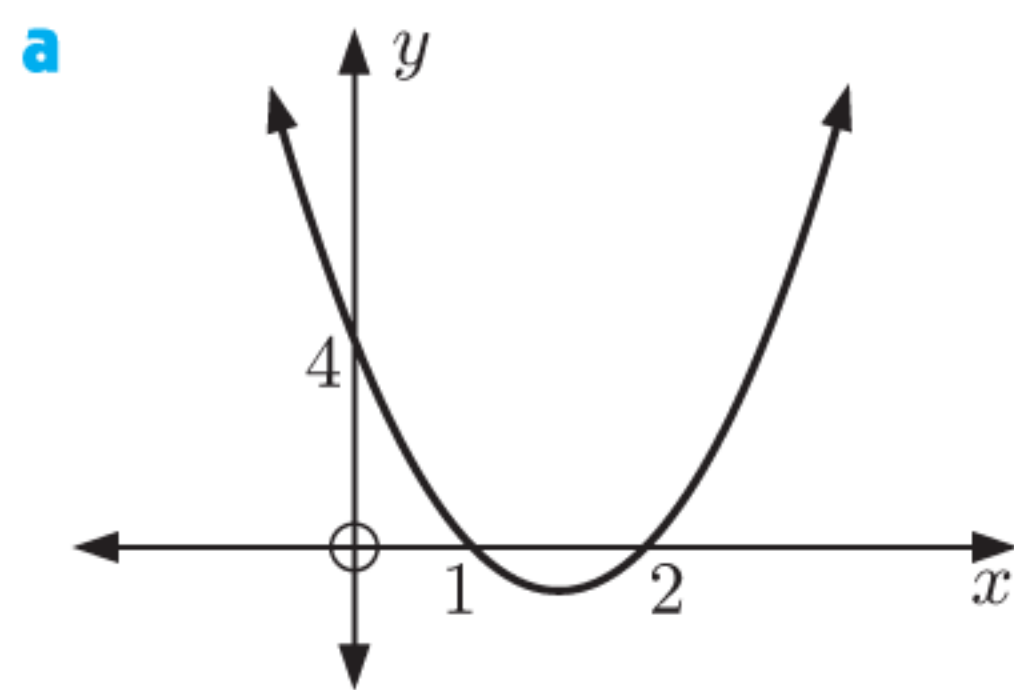
$$\therefore a = -2$$

The quadratic is $y = -2(x + 2)(x - 4)$.

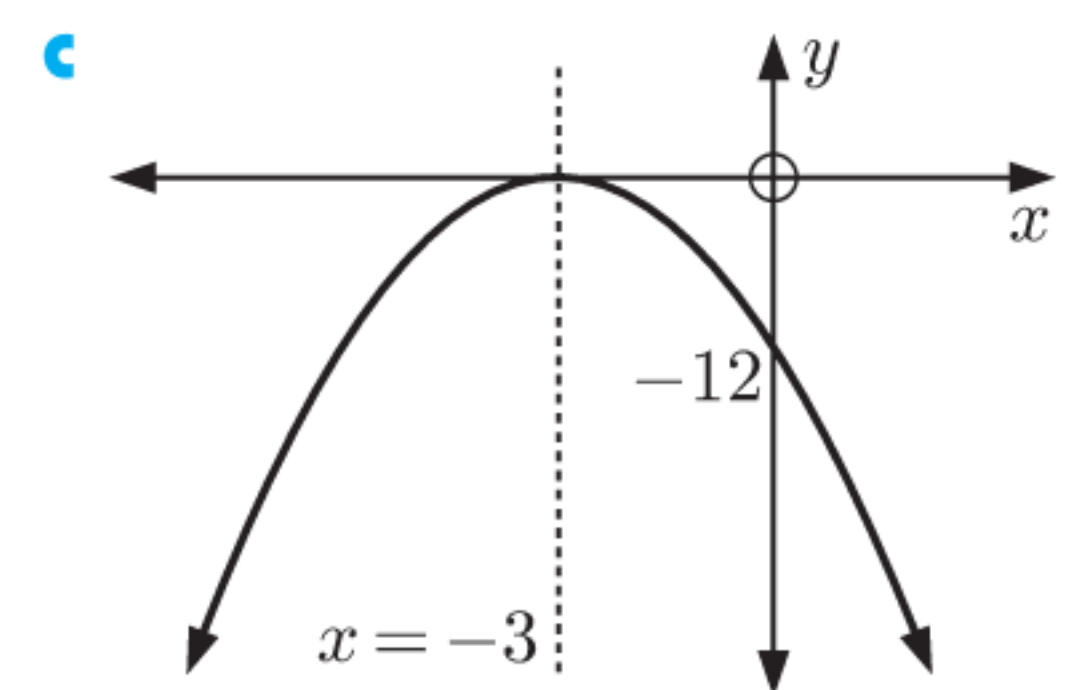
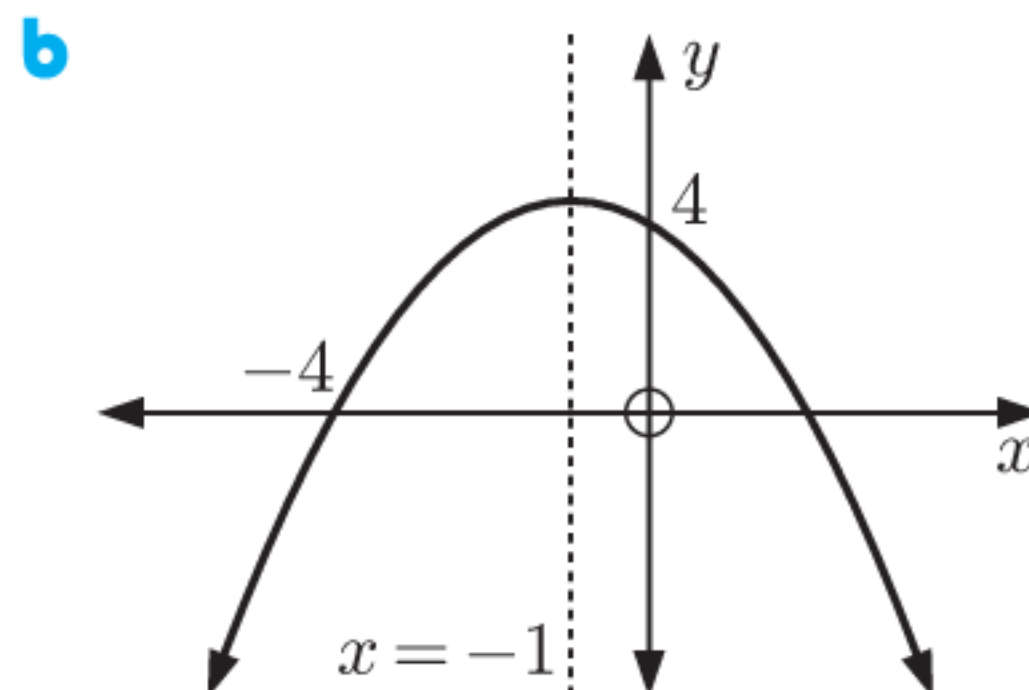
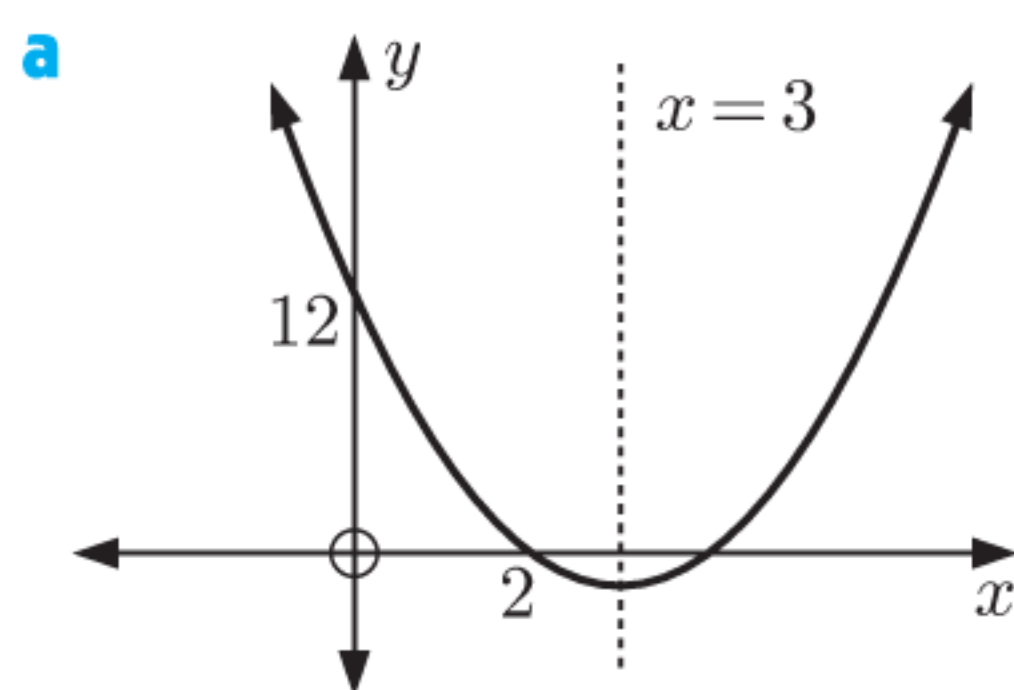


EXERCISE 14D

1 Find the equation of the quadratic with graph:



2 Find the equation of the quadratic with graph:


Example 12
Self Tutor

Find the equation of the quadratic whose graph cuts the x -axis at 4 and -3 , and which passes through the point $(2, -20)$. Give your answer in the form $y = ax^2 + bx + c$.

Since the x -intercepts are 4 and -3 , the quadratic has the form $y = a(x - 4)(x + 3)$, $a \neq 0$.

When $x = 2$, $y = -20$

$$\therefore -20 = a(2 - 4)(2 + 3)$$

$$\therefore -20 = a(-2)(5)$$

$$\therefore a = 2$$

The quadratic is $y = 2(x - 4)(x + 3)$

$$= 2(x^2 - x - 12)$$

$$= 2x^2 - 2x - 24$$

3 Find, in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph:

a cuts the x -axis at 5 and 1, and passes through $(2, -9)$

b cuts the x -axis at 2 and $-\frac{1}{2}$, and passes through $(3, -14)$

c touches the x -axis at 3 and passes through $(-2, -25)$

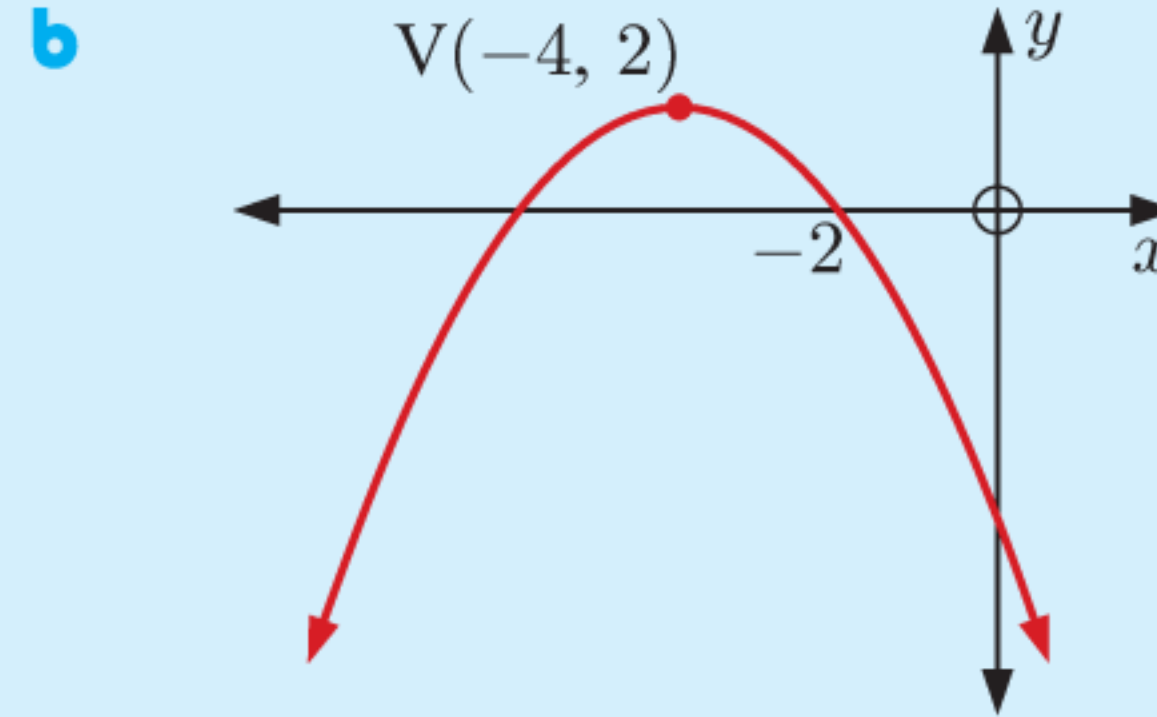
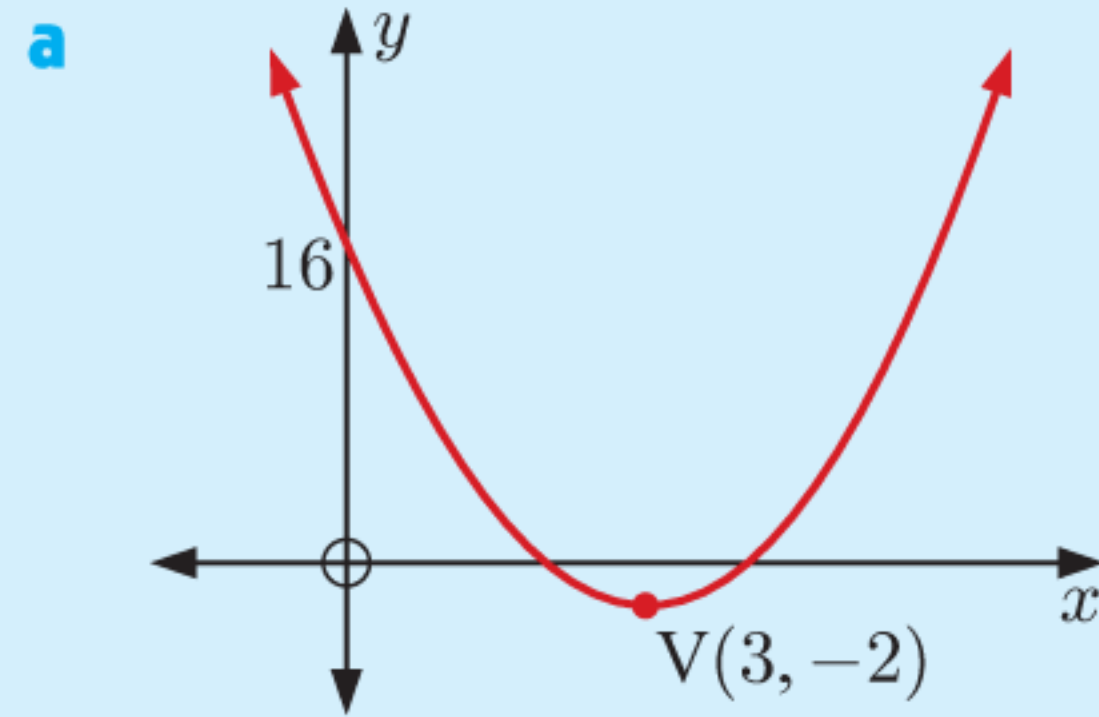
d touches the x -axis at -2 and passes through $(-1, 4)$

e cuts the x -axis at 3, passes through $(5, 12)$, and has axis of symmetry $x = 2$

f cuts the x -axis at 5, passes through $(2, 5)$, and has axis of symmetry $x = 1$.

Example 13**Self Tutor**

Find the equation of the quadratic with graph:



a Since the vertex is $(3, -2)$, the quadratic has the form $y = a(x - 3)^2 - 2$ where $a > 0$.

When $x = 0$, $y = 16$

$$\therefore 16 = a(-3)^2 - 2$$

$$\therefore 16 = 9a - 2$$

$$\therefore 18 = 9a$$

$$\therefore a = 2$$

The quadratic is $y = 2(x - 3)^2 - 2$.

b Since the vertex is $(-4, 2)$, the quadratic has the form $y = a(x + 4)^2 + 2$ where $a < 0$.

When $x = -2$, $y = 0$

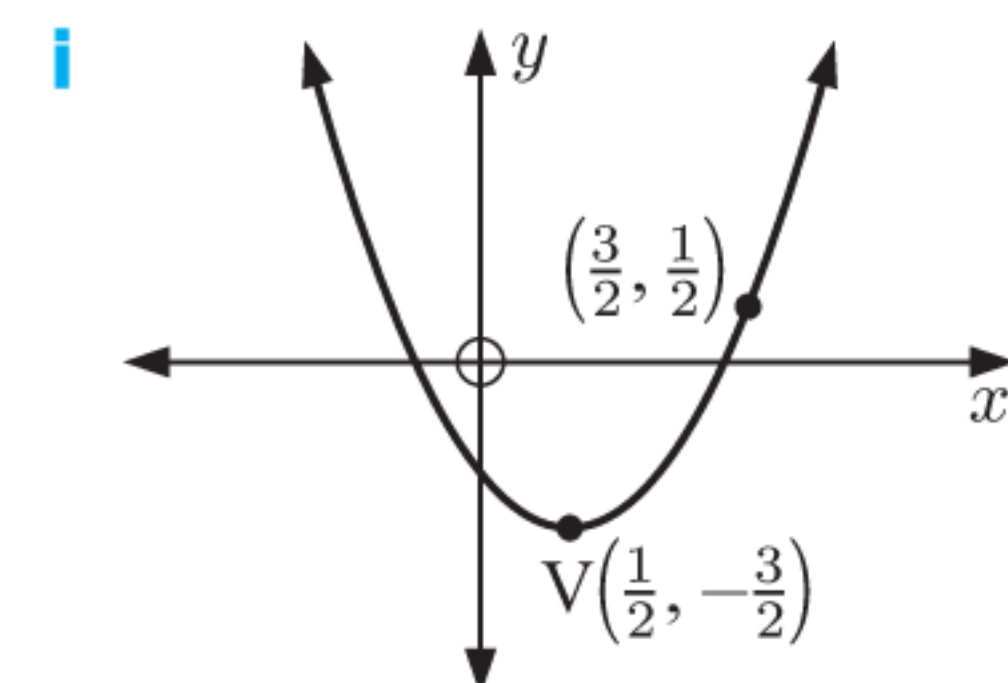
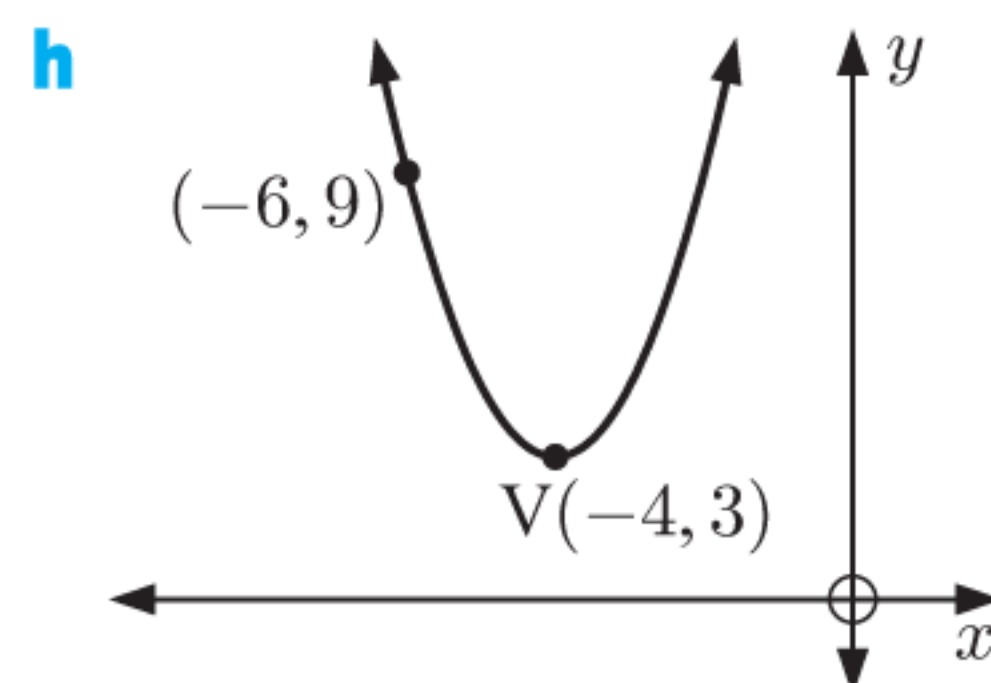
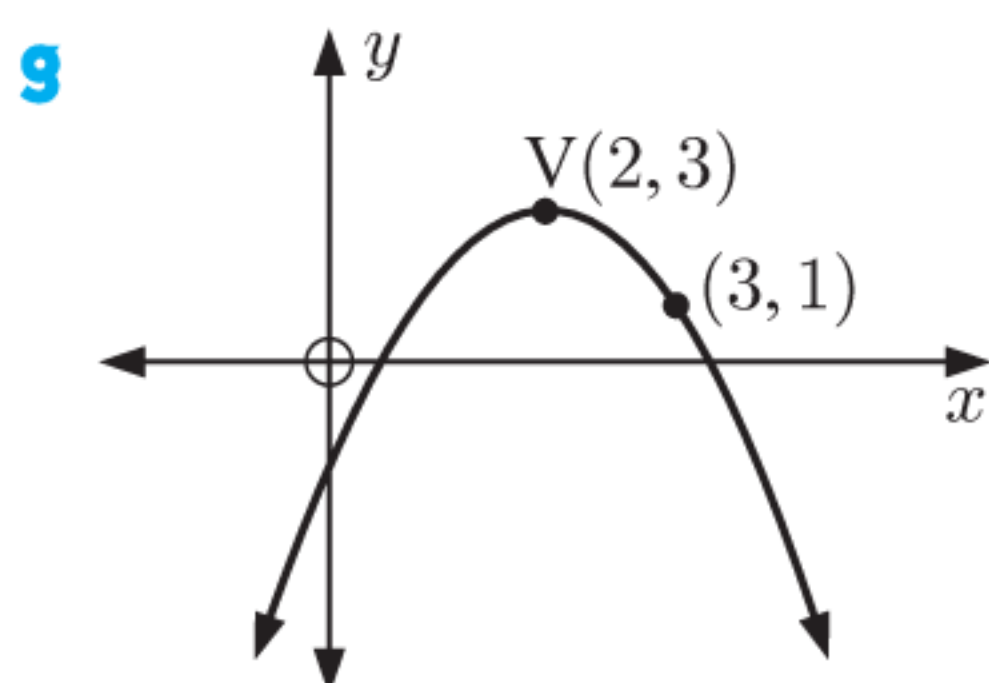
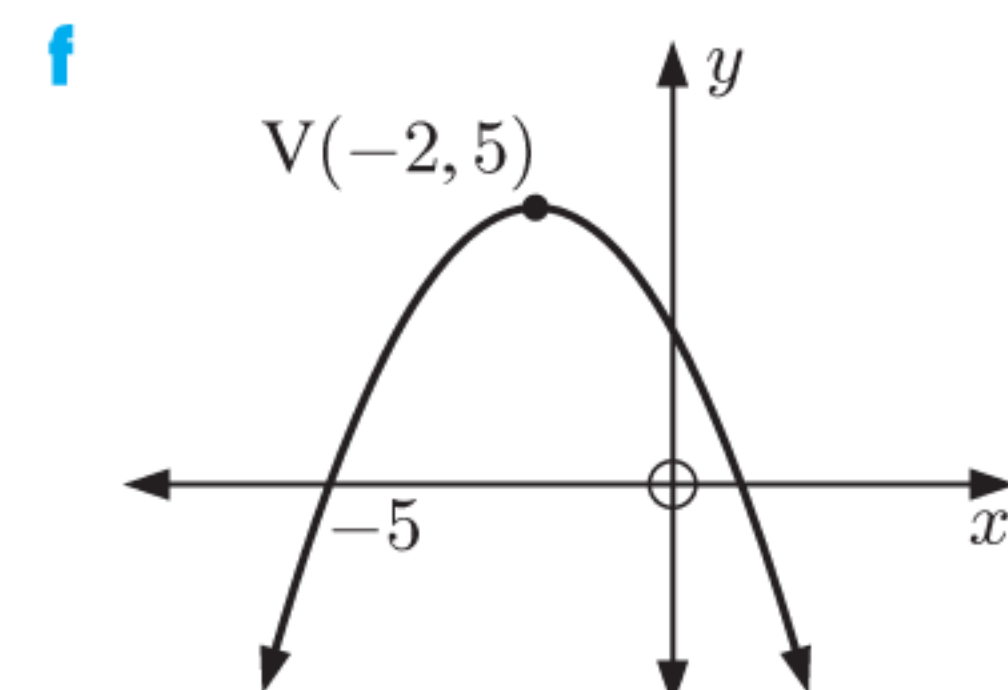
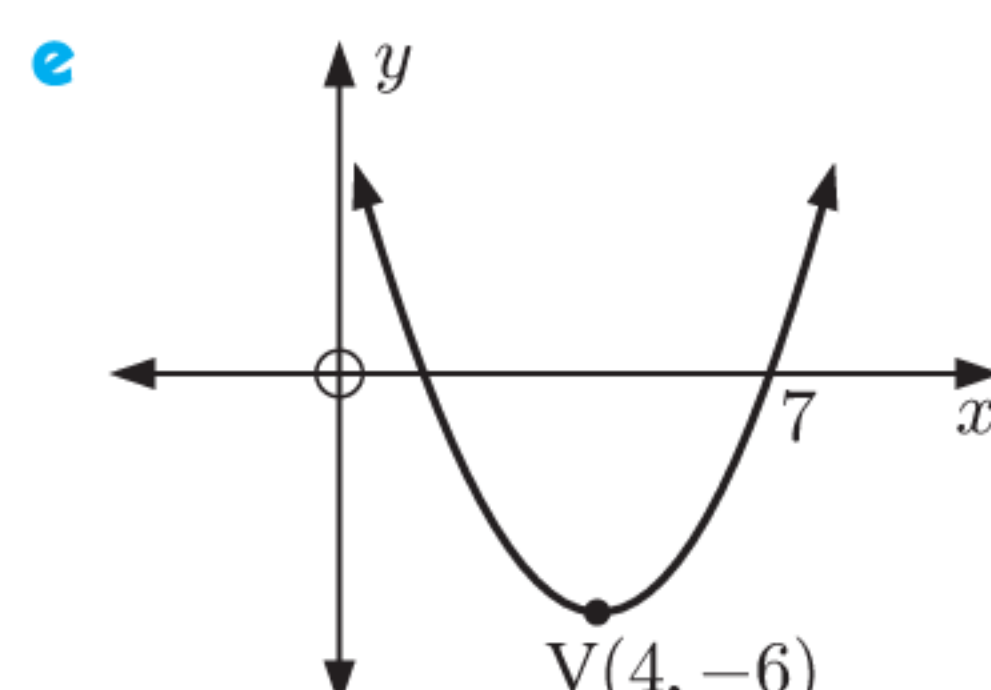
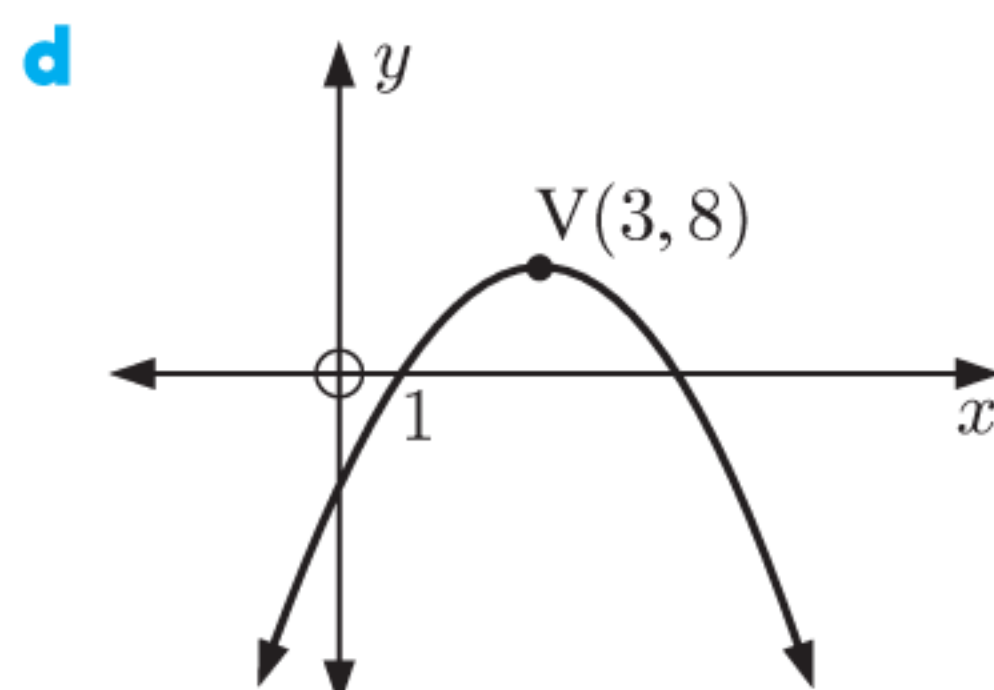
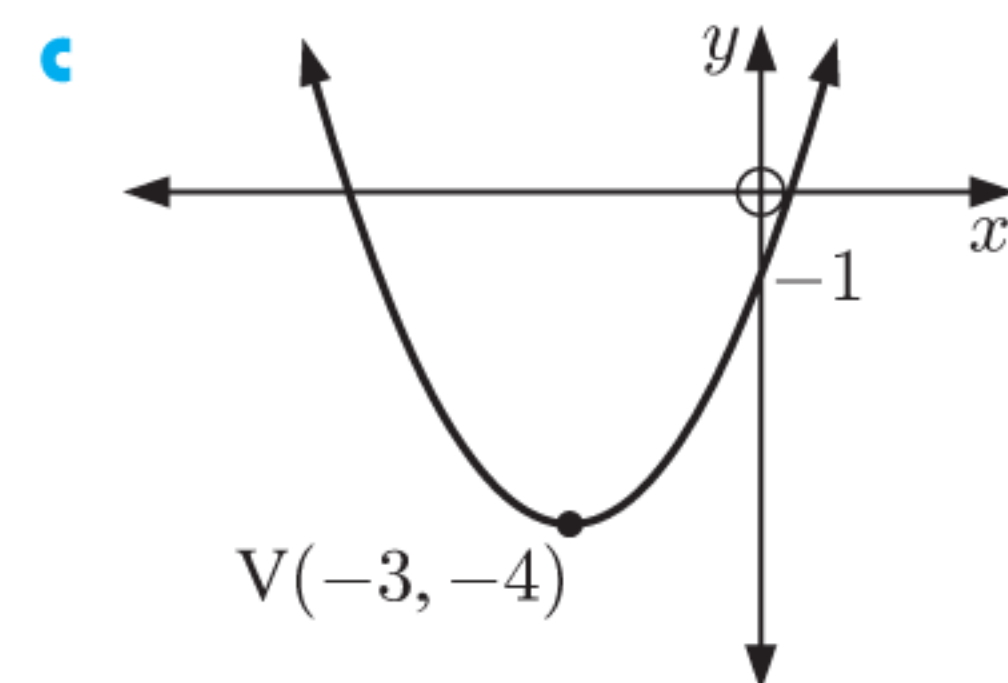
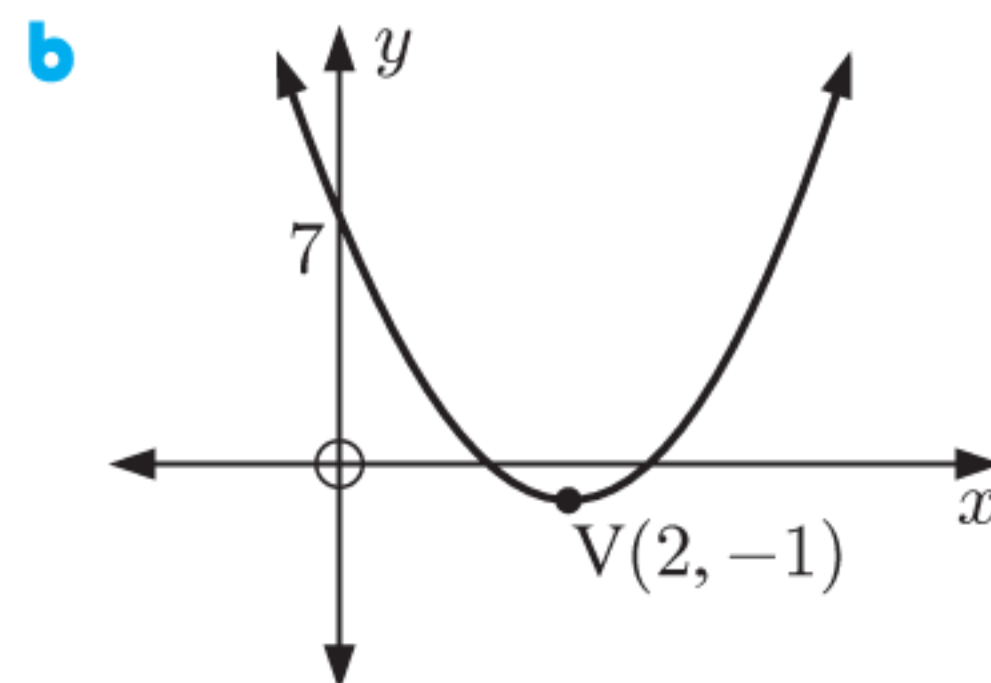
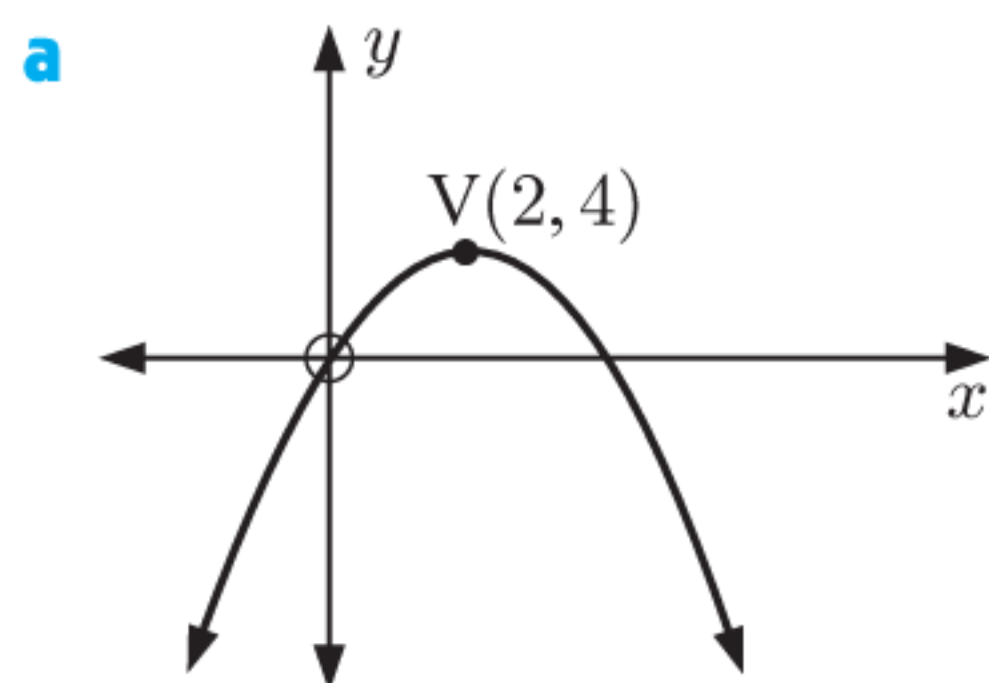
$$\therefore 0 = a(2)^2 + 2$$

$$\therefore 4a = -2$$

$$\therefore a = -\frac{1}{2}$$

The quadratic is $y = -\frac{1}{2}(x + 4)^2 + 2$.

4 If V is the vertex, find the equation of the quadratic with graph:



5 A quadratic has vertex $(2, -5)$, and passes through the point $(-1, 13)$. Find the value of the quadratic when $x = 4$.

INVESTIGATION 3

For the quadratic $y = 2x^2 + 3x + 7$ we can construct a table of values for $x = 0, 1, 2, 3, 4, 5$.

We turn this table into a **difference table** by adding two further rows:

- the row Δ_1 gives the differences between successive y -values
- the row Δ_2 gives the differences between successive Δ_1 -values.

FINDING QUADRATICS

| | | | | | | |
|-----|---|----|----|----|----|----|
| x | 0 | 1 | 2 | 3 | 4 | 5 |
| y | 7 | 12 | 21 | 34 | 51 | 72 |

| | | | | | | |
|------------|---|----|----|----|----|----|
| x | 0 | 1 | 2 | 3 | 4 | 5 |
| y | 7 | 12 | 21 | 34 | 51 | 72 |
| Δ_1 | | 5 | 9 | 13 | 17 | 21 |
| Δ_2 | | | 4 | 4 | 4 | 4 |

\uparrow \uparrow \uparrow
 $9 - 5$ $34 - 21$ $72 - 51$

What to do:

- Construct difference tables for $x = 0, 1, 2, 3, 4, 5$ for each of the following quadratics:
 - $y = x^2 + 4x + 3$
 - $y = 3x^2 - 4x$
 - $y = 5x - x^2$
 - $y = 4x^2 - 5x + 2$
- What do you notice about the Δ_2 row for each quadratic in 1?
- Consider the general quadratic $y = ax^2 + bx + c$, $a \neq 0$.
 - Copy and complete the following difference table:

| | | | | | | |
|------------|---|-------------|---------------|-------|-------|-------|
| x | 0 | 1 | 2 | 3 | 4 | 5 |
| y | ⓐ | $a + b + c$ | $4a + 2b + c$ | | | |
| Δ_1 | ○ | | | | | |
| Δ_2 | | ○ | | | | |

- Comment on the Δ_2 row.
 - What can the circled numbers be used for?
- Use your observations in 3 to determine, if possible, the quadratics with the following tables of values:

a

| | | | | | |
|-----|---|---|---|----|----|
| x | 0 | 1 | 2 | 3 | 4 |
| y | 6 | 5 | 8 | 15 | 26 |

b

| | | | | | |
|-----|---|----|----|----|----|
| x | 0 | 1 | 2 | 3 | 4 |
| y | 8 | 10 | 18 | 32 | 52 |

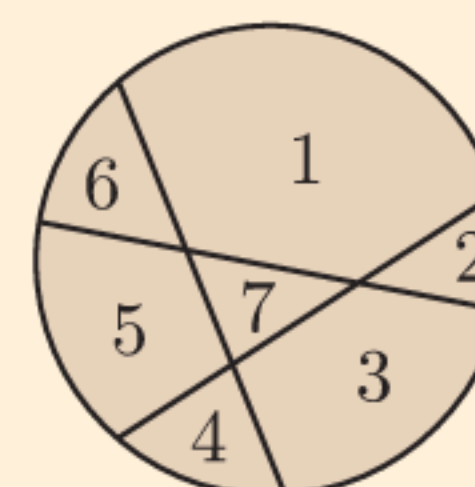
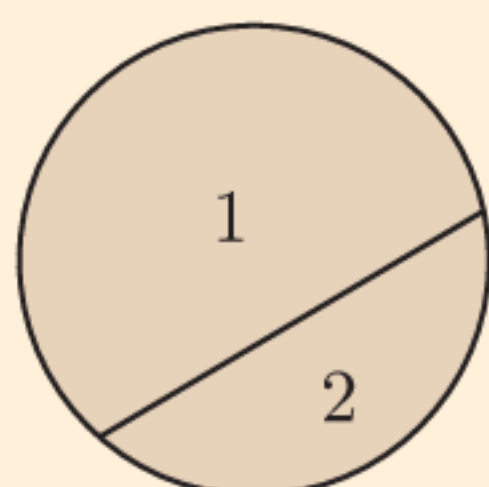
c

| | | | | | |
|-----|---|---|----|----|-----|
| x | 0 | 1 | 2 | 3 | 4 |
| y | 1 | 2 | -1 | -8 | -19 |

d

| | | | | | |
|-----|---|---|----|----|-----|
| x | 0 | 1 | 2 | 3 | 4 |
| y | 5 | 3 | -1 | -7 | -15 |

- We wish to determine the **maximum** number of pieces into which a pizza can be cut using n straight cuts across it. For example:
 - for $n = 1$ we can make 2 pieces
 - for $n = 3$ we can make 7 pieces.



a Copy and complete:

| | | | | | | |
|---------------------------------|---|---|---|---|---|---|
| Number of cuts, n | 0 | 1 | 2 | 3 | 4 | 5 |
| Maximum number of pieces, P_n | | | | | | |

b Complete the Δ_1 and Δ_2 rows. Hence determine a quadratic formula for P_n .

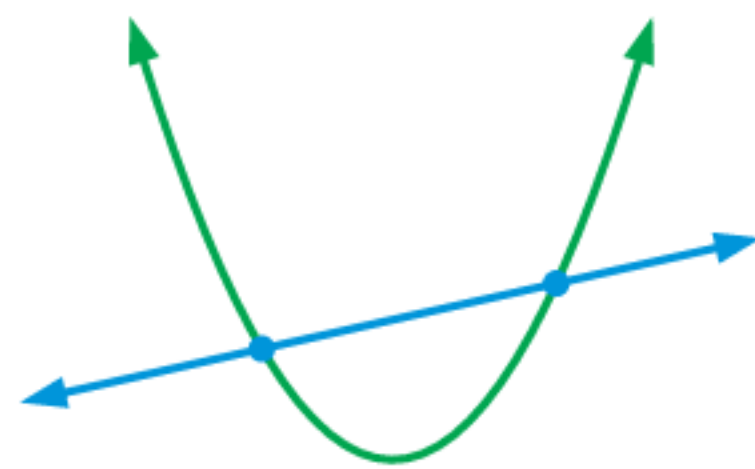
c For a huge pizza with 12 cuts across it, find the maximum number of pieces which can result.

E

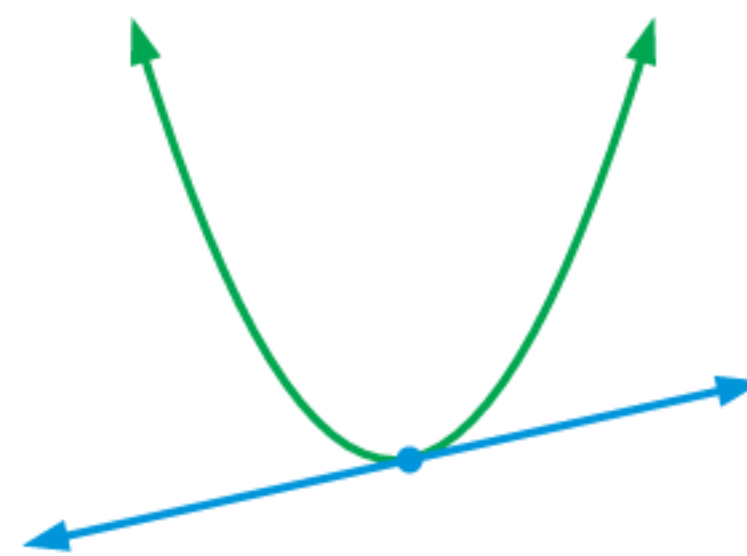
THE INTERSECTION OF GRAPHS

Consider the graphs of a quadratic function and a linear function on the same set of axes.

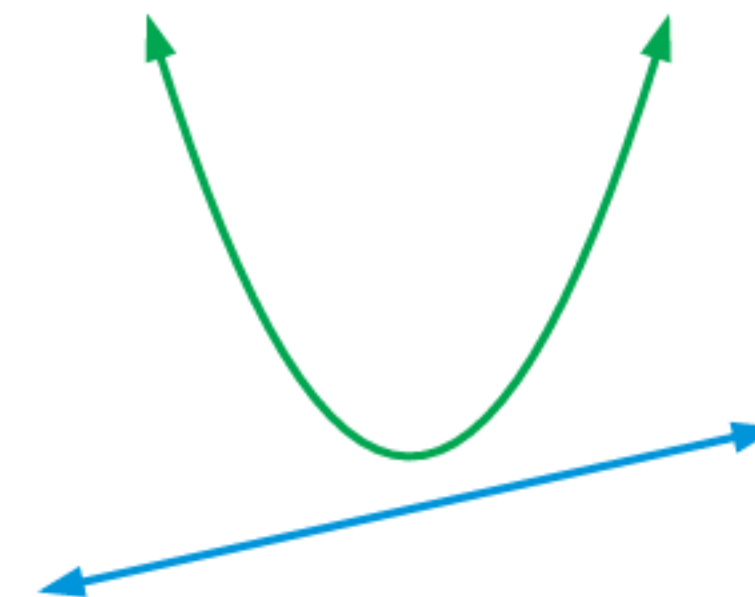
There are three possible scenarios for intersection:



cutting
(2 points of intersection)



touching
(1 point of intersection)



missing
(no points of intersection)

If the line *touches* the curve, we say that the line is a **tangent** to the curve.

The x -coordinates of any intersection points of the graphs can be found by solving the two equations **simultaneously**.

Example 14

Self Tutor

Find the coordinates of the point(s) of intersection of the graphs with equations $y = x^2 - x - 18$ and $y = x - 3$.

$y = x^2 - x - 18$ meets $y = x - 3$ where

$$x^2 - x - 18 = x - 3$$

$$\therefore x^2 - 2x - 15 = 0 \quad \{\text{RHS} = 0\}$$

$$\therefore (x - 5)(x + 3) = 0 \quad \{\text{factorising}\}$$

$$\therefore x = 5 \text{ or } -3$$

Substituting into $y = x - 3$, when $x = 5$, $y = 2$ and when $x = -3$, $y = -6$.

\therefore the graphs meet at $(5, 2)$ and $(-3, -6)$.

Graphing each side of an inequality helps us to illustrate its solutions. Any points where the graphs intersect will lie at the endpoints of the interval(s) in the solution.

Example 15
 **Self Tutor**

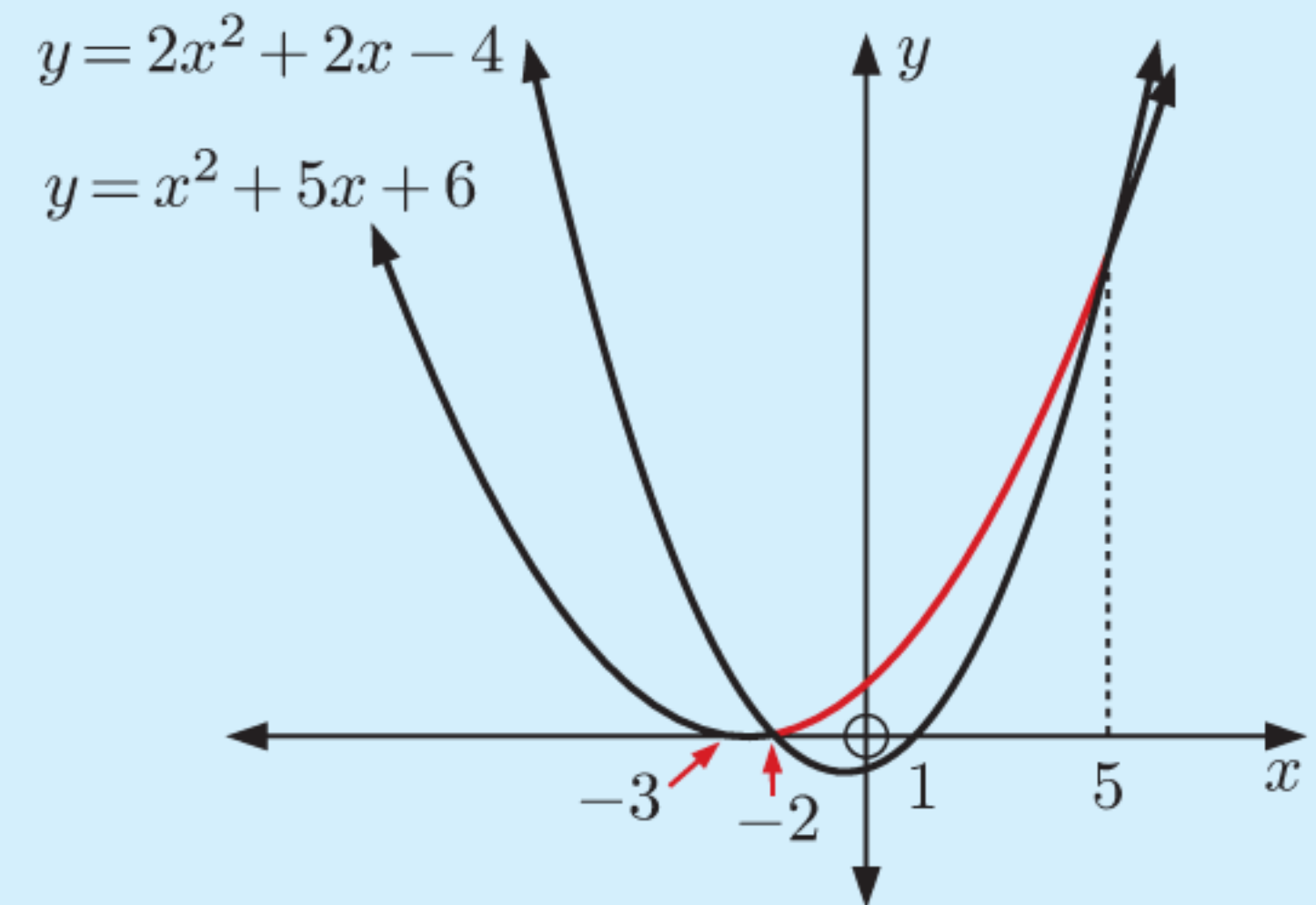
Consider the curves $y = x^2 + 5x + 6$ and $y = 2x^2 + 2x - 4$.

- Solve for x : $x^2 + 5x + 6 = 2x^2 + 2x - 4$.
- Graph the two curves on the same set of axes.
- Hence solve for x : $x^2 + 5x + 6 > 2x^2 + 2x - 4$.

$$\begin{aligned} \text{a} \quad x^2 + 5x + 6 &= 2x^2 + 2x - 4 \\ \therefore x^2 - 3x - 10 &= 0 \\ \therefore (x + 2)(x - 5) &= 0 \\ \therefore x &= -2 \text{ or } 5 \end{aligned}$$

$$\begin{aligned} \text{b} \quad y &= x^2 + 5x + 6 \\ &= (x + 2)(x + 3) \quad \text{has zeros } -2 \text{ and } -3. \end{aligned}$$

$$\begin{aligned} y &= 2x^2 + 2x - 4 \\ &= 2(x^2 + x - 2) \\ &= 2(x + 2)(x - 1) \quad \text{has zeros } -2 \text{ and } 1. \end{aligned}$$



- If $x^2 + 5x + 6 > 2x^2 + 2x - 4$, the graph of $y = x^2 + 5x + 6$ is above the graph of $y = 2x^2 + 2x - 4$.
This occurs when $-2 < x < 5$.

EXERCISE 14E

- Find the coordinates of the point(s) of intersection of:
 - $y = x^2 - 2x + 8$ and $y = x + 6$
 - $y = -x^2 + 3x + 9$ and $y = 2x - 3$
 - $y = x^2 - 4x + 3$ and $y = 2x - 6$
 - $y = -x^2 + 4x - 7$ and $y = 5x - 4$
- Use technology to find the coordinates of the points of intersection of the graphs with equations:
 - $y = x^2 - 3x + 7$ and $y = x + 5$
 - $y = x^2 - 5x + 2$ and $y = x - 7$
 - $y = -x^2 - 2x + 4$ and $y = x + 8$
 - $y = -x^2 + 4x - 2$ and $y = 5x - 6$.
- Consider the graphs with equations $y = x^2$ and $y = x + 2$.
 - Find the points where the graphs intersect.
 - Plot the graphs on the same set of axes.
 - Hence solve for x : $x^2 > x + 2$.
- Consider the graphs with equations $y = x^2 + 2x - 3$ and $y = x - 1$.
 - Find the points where the graphs intersect.
 - Plot the graphs on the same set of axes.
 - Hence solve for x : $x^2 + 2x - 3 > x - 1$.

GRAPHING PACKAGE


The solutions to $x^2 > x + 2$ are the values of x for which $y = x^2$ is above $y = x + 2$.



- 5 Consider the curves $y = 2x^2 - x + 3$ and $y = 2 + x + x^2$.
- Find the points where the curves intersect.
 - Plot the curves on the same set of axes.
 - Hence solve for x : $2x^2 - x + 3 > 2 + x + x^2$.
- 6 Consider the graphs with equations $y = \frac{4}{x}$ and $y = x + 3$.
- Solve $\frac{4}{x} = x + 3$ using algebra.
 - Use technology to plot the graphs on the same set of axes.
 - Hence solve for x : $\frac{4}{x} > x + 3$.

Example 16**Self Tutor**

$y = 2x + k$ is a tangent to $y = 2x^2 - 3x + 4$. Find k .

$$y = 2x + k \text{ meets } y = 2x^2 - 3x + 4 \text{ where}$$

$$2x^2 - 3x + 4 = 2x + k$$

$$\therefore 2x^2 - 5x + (4 - k) = 0$$

Since the graphs touch, this quadratic has $\Delta = 0$

$$\therefore (-5)^2 - 4(2)(4 - k) = 0$$

$$\therefore 25 - 8(4 - k) = 0$$

$$\therefore 25 - 32 + 8k = 0$$

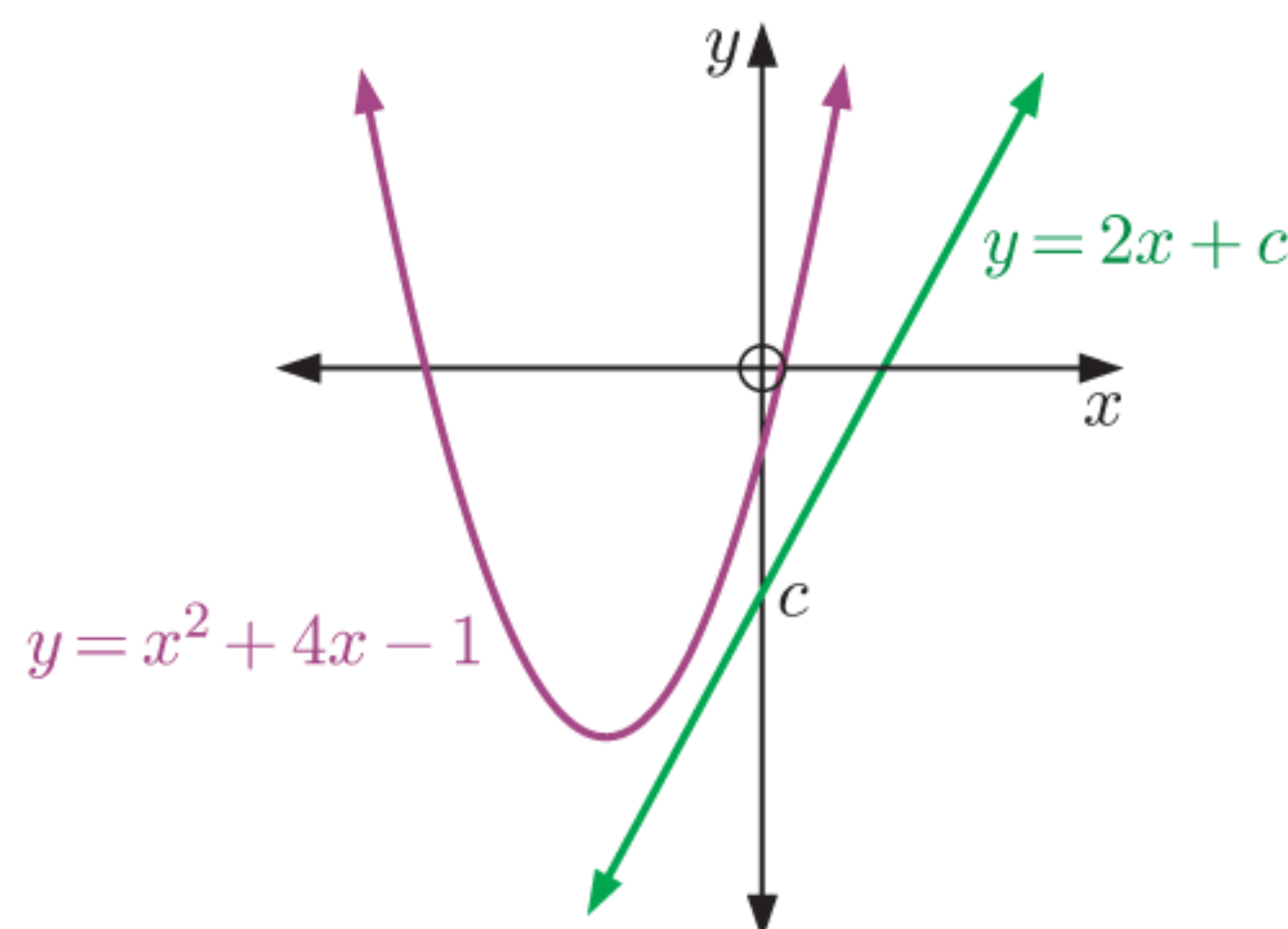
$$\therefore 8k = 7$$

$$\therefore k = \frac{7}{8}$$

A *tangent* is a line which touches the curve.



- 7 For what value of c is the line $y = 3x + c$ a tangent to the parabola with equation $y = x^2 - 5x + 7$?
- 8 Find the values of m for which the lines $y = mx - 2$ are tangents to the curve with equation $y = x^2 - 4x + 2$.
- 9 Find the gradients of the lines with y -intercept 1 that are tangents to the curve $y = 3x^2 + 5x + 4$.
- 10 **a** For what values of c do the lines $y = x + c$ never meet the parabola with equation $y = 2x^2 - 3x - 7$?
- b** Choose one of the values of c found in part **a**. Illustrate with a sketch that these graphs never meet.
- 11 Prove that two quadratic functions can intersect at most twice.
- 12 Consider the curve $y = x^2 + 4x - 1$ and the line $y = 2x + c$. Find the values of c for which the line:
- meets the curve twice
 - is a tangent to the curve
 - does not meet the curve.



- 13** Show that any linear function passing through $(0, 3)$ will meet the curve $y = 2x^2 - x - 2$ twice.
- 14** The graphs of $y = (x - 2)^2$ and $y = -x^2 + bx + c$ touch when $x = 3$. Find the values of b and c .
- 15** Consider the quadratic function $y = x^2$. The point $P(a, a^2)$ lies on the function.
- The line $y = m(x - a) + c$ also passes through P . What, if anything, can you say about the values of m and c ?
 - Suppose $y = m(x - a) + a^2$ touches $y = x^2$ at P . Find the value of m .

F
PROBLEM SOLVING WITH QUADRATICS

Some real-world problems can be solved using a quadratic equation.

Any answer we obtain must be checked to see if it is reasonable. For example:

- if we are finding a length then it must be positive and we reject any negative solutions
- if we are finding “how many people are present” then the answer must be a positive integer.

We employ the following general problem solving method:

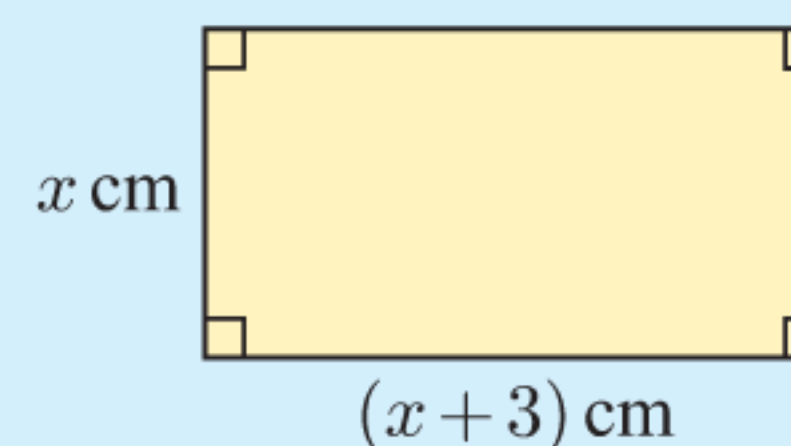
- Step 1:* If the information is given in words, translate it into algebra using a variable such as x . Be sure to define what x represents, and include units if appropriate. Write down the resulting equation.
- Step 2:* Solve the equation by a suitable method.
- Step 3:* Examine the solutions carefully to see if they are acceptable.
- Step 4:* Give your answer in a sentence, making sure you answer the question.

Example 17
Self Tutor

A rectangle has length 3 cm longer than its width, and its area is 42 cm^2 . Find the width of the rectangle.

If the width is x cm then the length is $(x + 3)$ cm.

$$\begin{aligned} \therefore x(x + 3) &= 42 \quad \{\text{equating areas}\} \\ \therefore x^2 + 3x - 42 &= 0 \\ \therefore x &= \frac{-3 \pm \sqrt{3^2 - 4(1)(-42)}}{2} \\ \therefore x &= \frac{-3 \pm \sqrt{177}}{2} \\ \therefore x &\approx -8.15 \text{ or } 5.15 \end{aligned}$$



We reject the negative solution as lengths are positive.

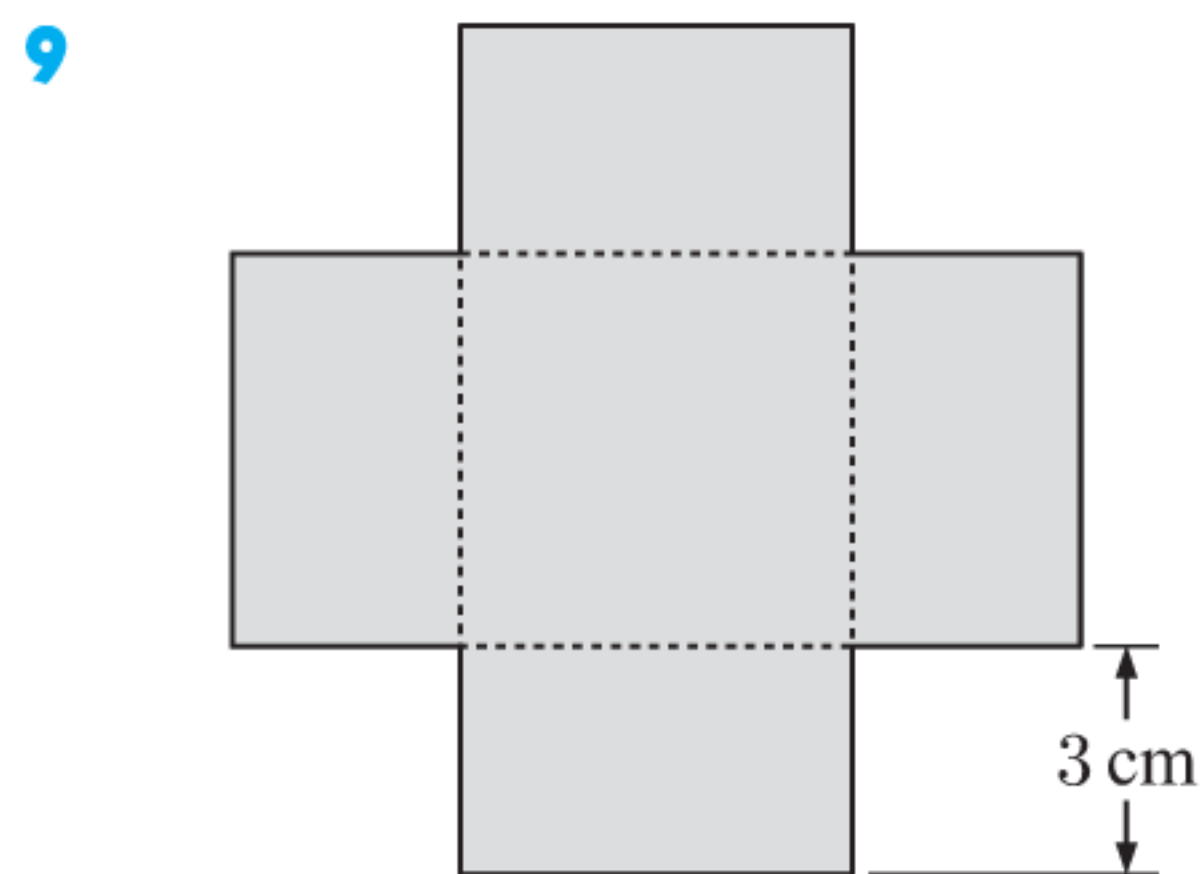
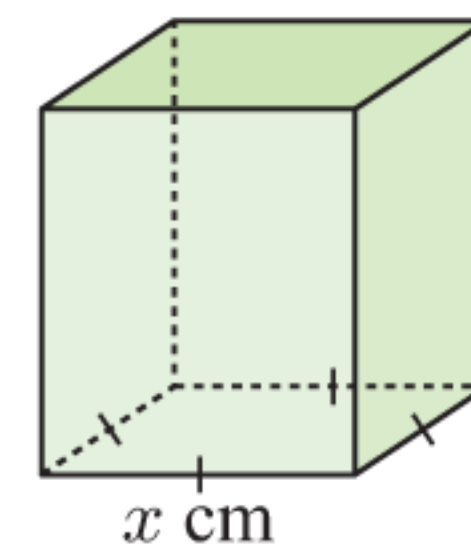
The width is about 5.15 cm.

EXERCISE 14F

- Two integers differ by 12, and the sum of their squares is 74. Find the integers.
- The sum of a number and its reciprocal is $\frac{26}{5}$. Find the number.
- The sum of a natural number and its square is 210. Find the number.
- The product of two consecutive even numbers is 360. Find the numbers.
- The product of two consecutive odd numbers is 255. Find the numbers.
- The number of diagonals of an n -sided polygon is given by the formula $D = \frac{n}{2}(n - 3)$.
A polygon has 90 diagonals. How many sides does it have?
- The length of a rectangle is 4 cm longer than its width. The rectangle has area 26 cm^2 . Find its width.

- A rectangular box has a square base. Its height is 1 cm longer than its base side length. The total surface area of the box is 240 cm^2 .
Suppose the sides of the base are x cm long.

- Show that the total surface area is given by $A = 6x^2 + 4x \text{ cm}^2$.
- Find the dimensions of the box.



An open box can hold 80 cm^3 . It is made from a square piece of tinfoil with 3 cm squares cut from each of its 4 corners. Find the dimensions of the original piece of tinfoil.

Example 18**Self Tutor**

Is it possible to bend a 12 cm length of wire to form the perpendicular sides of a right angled triangle with area 20 cm^2 ?

Suppose the wire is bent $x \text{ cm}$ from one end.

The area $A = \frac{1}{2}x(12 - x)$

$$\therefore \frac{1}{2}x(12 - x) = 20$$

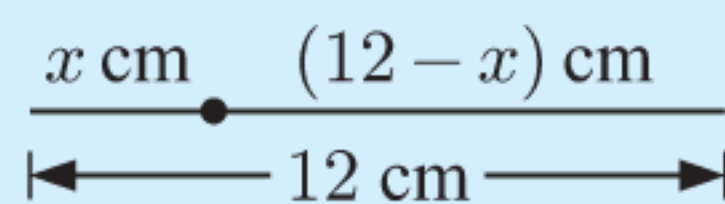
$$\therefore x(12 - x) = 40$$

$$\therefore 12x - x^2 - 40 = 0$$

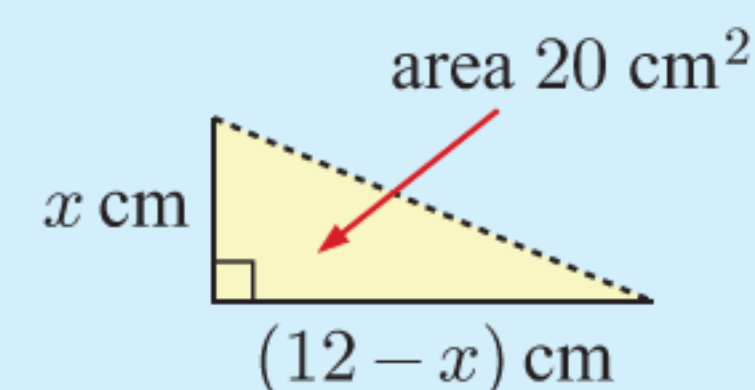
$$\therefore x^2 - 12x + 40 = 0$$

$$\begin{aligned} \text{Now } \Delta &= (-12)^2 - 4(1)(40) \\ &= -16 \text{ which is } < 0 \end{aligned}$$

There are no real solutions, indicating that this situation is **impossible**.



becomes



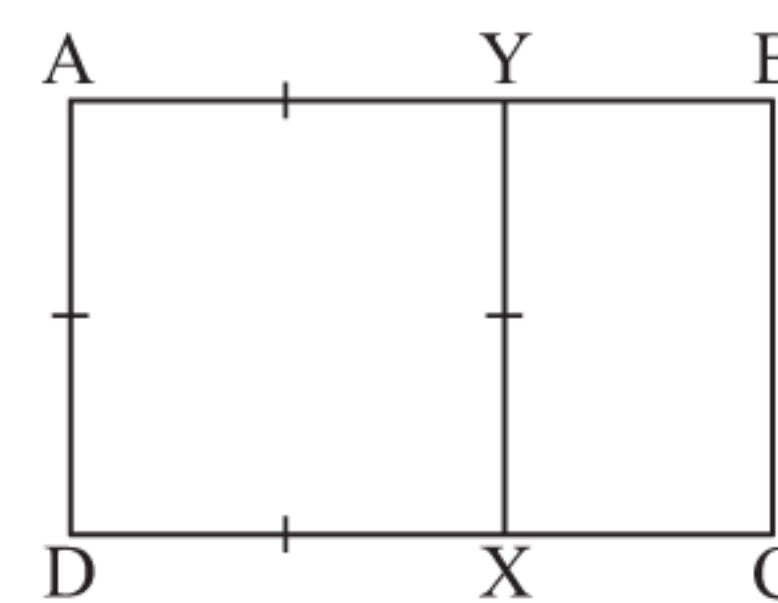
- Is it possible to bend a 20 cm length of wire into a rectangle with area 30 cm^2 ?

- 11** The rectangle ABCD is divided into a square and a smaller rectangle by [XY] which is parallel to its shorter sides. The smaller rectangle BCXY is *similar* to the original rectangle, so rectangle ABCD is a **golden rectangle**.

The ratio $\frac{AB}{AD}$ is called the **golden ratio**.

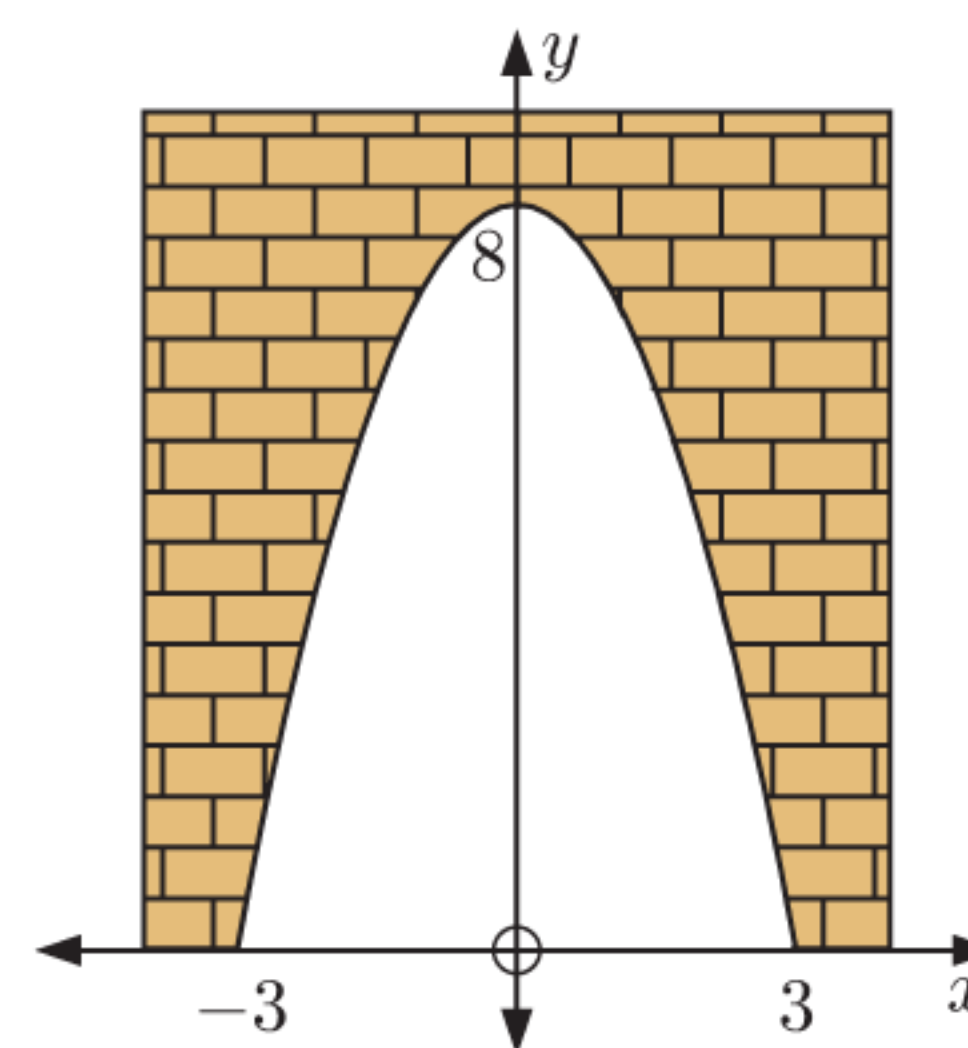
Show that the golden ratio is $\frac{1 + \sqrt{5}}{2}$.

Hint: Let $AB = x$ units and $AD = 1$ unit.



- 12** Two trains travel along a 160 km track each day. The express travels 10 km h^{-1} faster and takes 30 minutes less time than the normal train. Find the speed of the express.
- 13** A group of elderly citizens chartered a bus for \$160. Unfortunately, 8 of them fell ill and had to miss the trip. As a consequence, the other citizens had to pay an extra \$1 each. How many elderly citizens went on the trip?
- 14** A truck carrying a wide load needs to pass through the parabolic tunnel shown. The units are metres. The truck is 5 m high and 4 m wide.

- Find the quadratic function which describes the shape of the tunnel.
- Determine whether the truck will fit.



- 15** A stone is thrown into the air from the top of a cliff 60 m above sea level. The stone reaches a maximum height of 80 m above sea level after 2 seconds.
- Find the quadratic function which describes the stone's height above sea level.
 - Find the stone's height above sea level after 3 seconds.
 - How long will it take for the stone to hit the water?

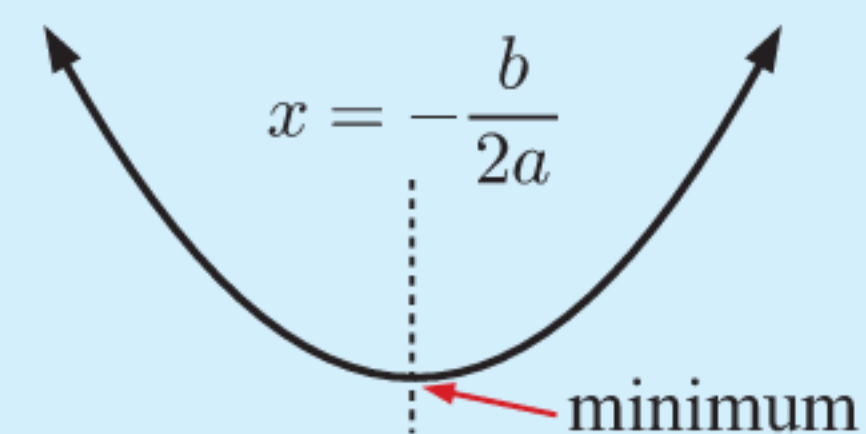
G

OPTIMISATION WITH QUADRATICS

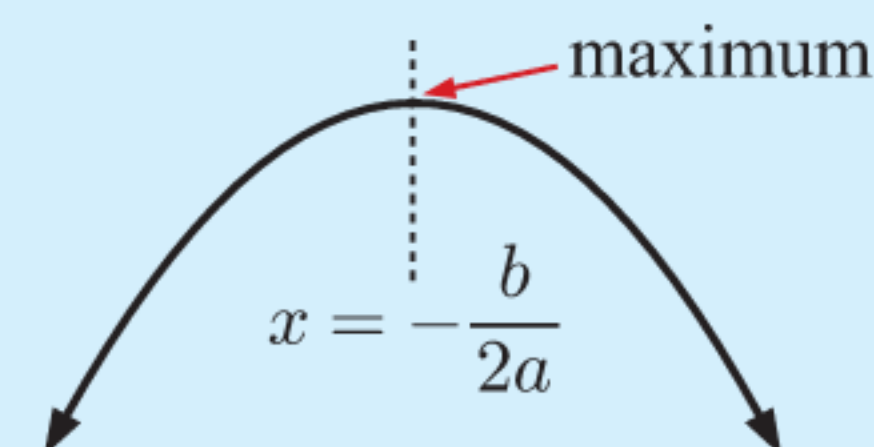
The process of finding a maximum or minimum value is called **optimisation**.

For the quadratic $y = ax^2 + bx + c$, we have seen that the vertex has x -coordinate $-\frac{b}{2a}$.

- If $a > 0$, the **minimum** value of y occurs at $x = -\frac{b}{2a}$.



- If $a < 0$, the **maximum** value of y occurs at $x = -\frac{b}{2a}$.



Example 19**Self Tutor**

Find the maximum or minimum value of the quadratic, and the corresponding value of x :

a $y = x^2 + x - 3$

b $y = 3 + 3x - 2x^2$

a $y = x^2 + x - 3$ has
 $a = 1$, $b = 1$, and $c = -3$.

Since $a > 0$, the shape is



The minimum value occurs

when $x = \frac{-b}{2a} = -\frac{1}{2}$

and $y = (-\frac{1}{2})^2 + (-\frac{1}{2}) - 3 = -3\frac{1}{4}$

So, the minimum value of y is $-3\frac{1}{4}$,
occurring when $x = -\frac{1}{2}$.

b $y = -2x^2 + 3x + 3$ has
 $a = -2$, $b = 3$, and $c = 3$.

Since $a < 0$, the shape is



The maximum value occurs

when $x = \frac{-b}{2a} = \frac{-3}{-4} = \frac{3}{4}$

and $y = -2(\frac{3}{4})^2 + 3(\frac{3}{4}) + 3 = 4\frac{1}{8}$

So, the maximum value of y is $4\frac{1}{8}$,
occurring when $x = \frac{3}{4}$.

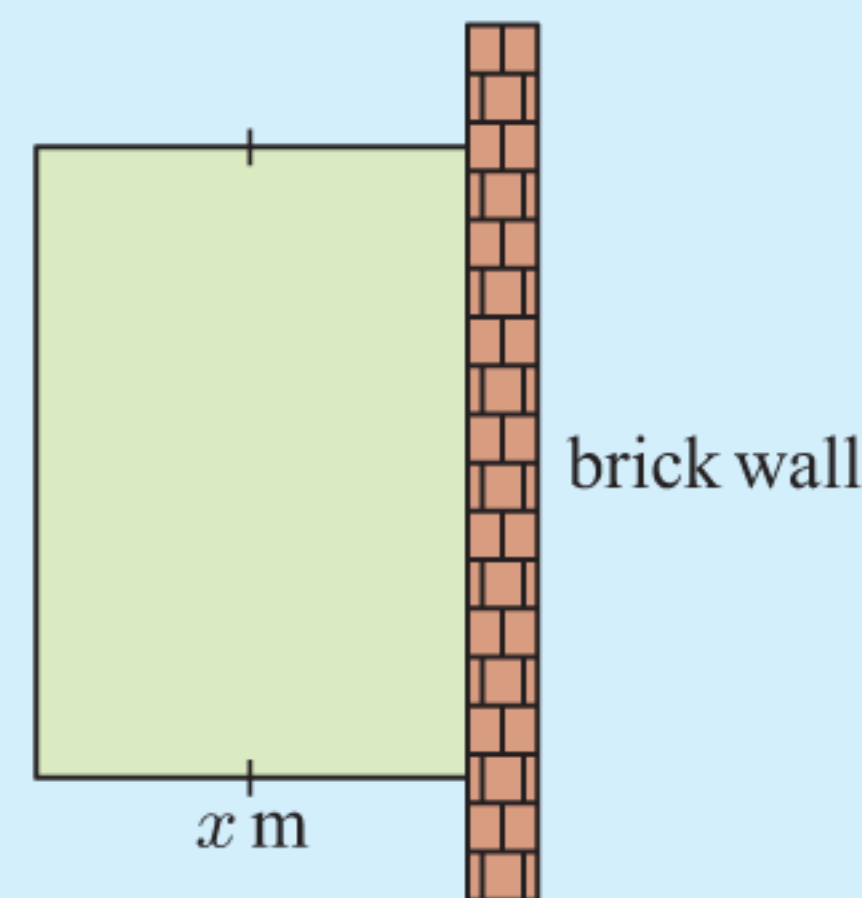
EXERCISE 14G

- Find the maximum or minimum value for each quadratic, and the corresponding value of x :
 - $y = x^2 - 2x$
 - $y = 7 - 2x - x^2$
 - $y = 8 + 2x - 3x^2$
 - $y = 2x^2 + x - 1$
 - $y = 4x^2 - x + 5$
 - $y = 7x - 2x^2$
- The profit in manufacturing x refrigerators per day, is given by $P = -3x^2 + 240x - 800$ euros.
 - How many refrigerators should be made each day to maximise the total profit?
 - What is the maximum profit?

Example 20**Self Tutor**

A gardener has 40 m of fencing to enclose a rectangular garden plot, where one side is an existing brick wall. Suppose the two new equal sides are x m long.

- Show that the area enclosed is given by $A = x(40 - 2x)$ m².
- Find the dimensions of the garden of maximum area.

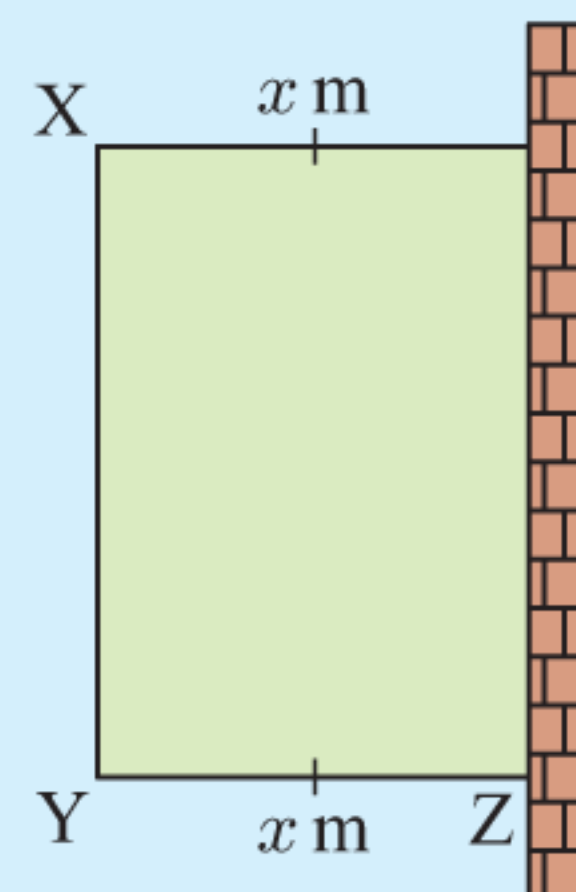


- Side [XY] has length $(40 - 2x)$ m.
Now, area = length \times width
 $\therefore A = x(40 - 2x)$ m²
- $A = 0$ when $x = 0$ or 20.
The vertex of the function lies midway between these values, so $x = 10$.

Since $a < 0$, the shape is

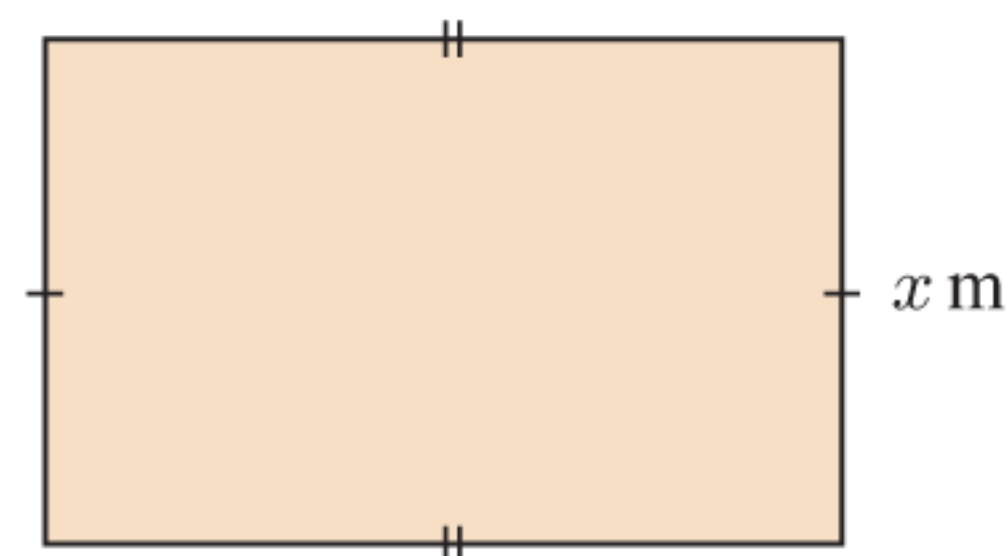


\therefore the area is maximised when $YZ = 10$ m and $XY = 20$ m.



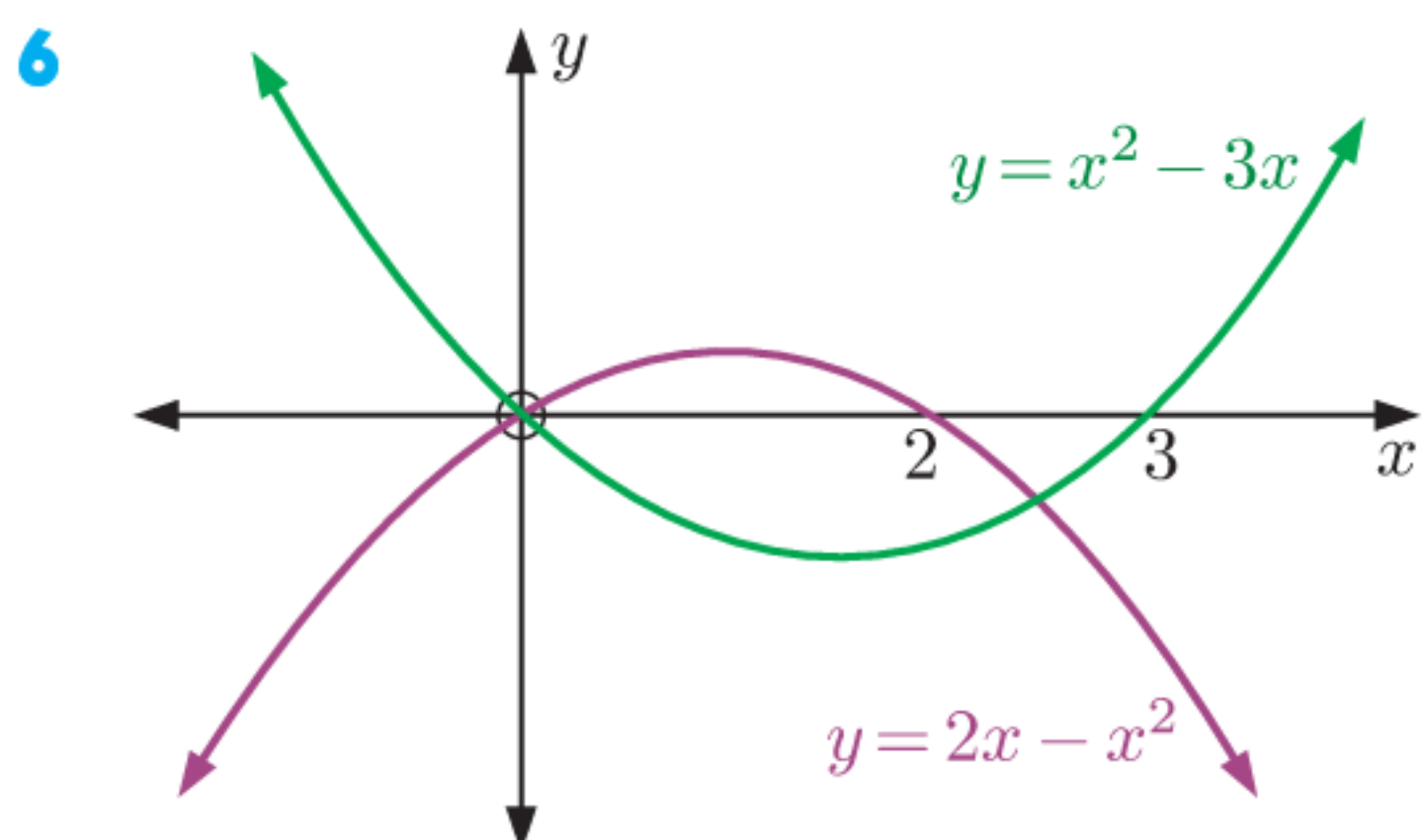
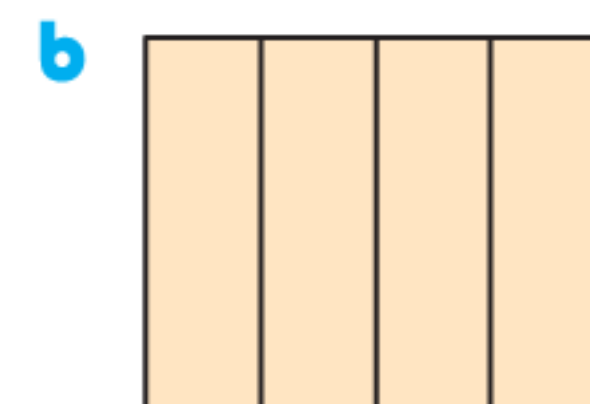
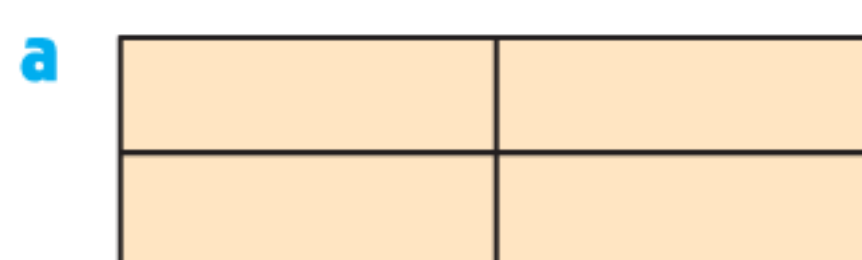
3 A rectangular plot is enclosed by 200 m of fencing and has an area of A square metres. Show that:

- a $A = 100x - x^2$ where x m is the length of one of its sides
- b the area is maximised if the rectangle is a square.



4 Three sides of a rectangular paddock are to be fenced, the fourth side being an existing straight water drain. If 1000 m of fencing is available, what dimensions should be used for the paddock to maximise its area?

5 500 m of fencing is available to make 4 rectangular pens of identical shape. Find the dimensions that maximise the area of each pen, if the plan is:

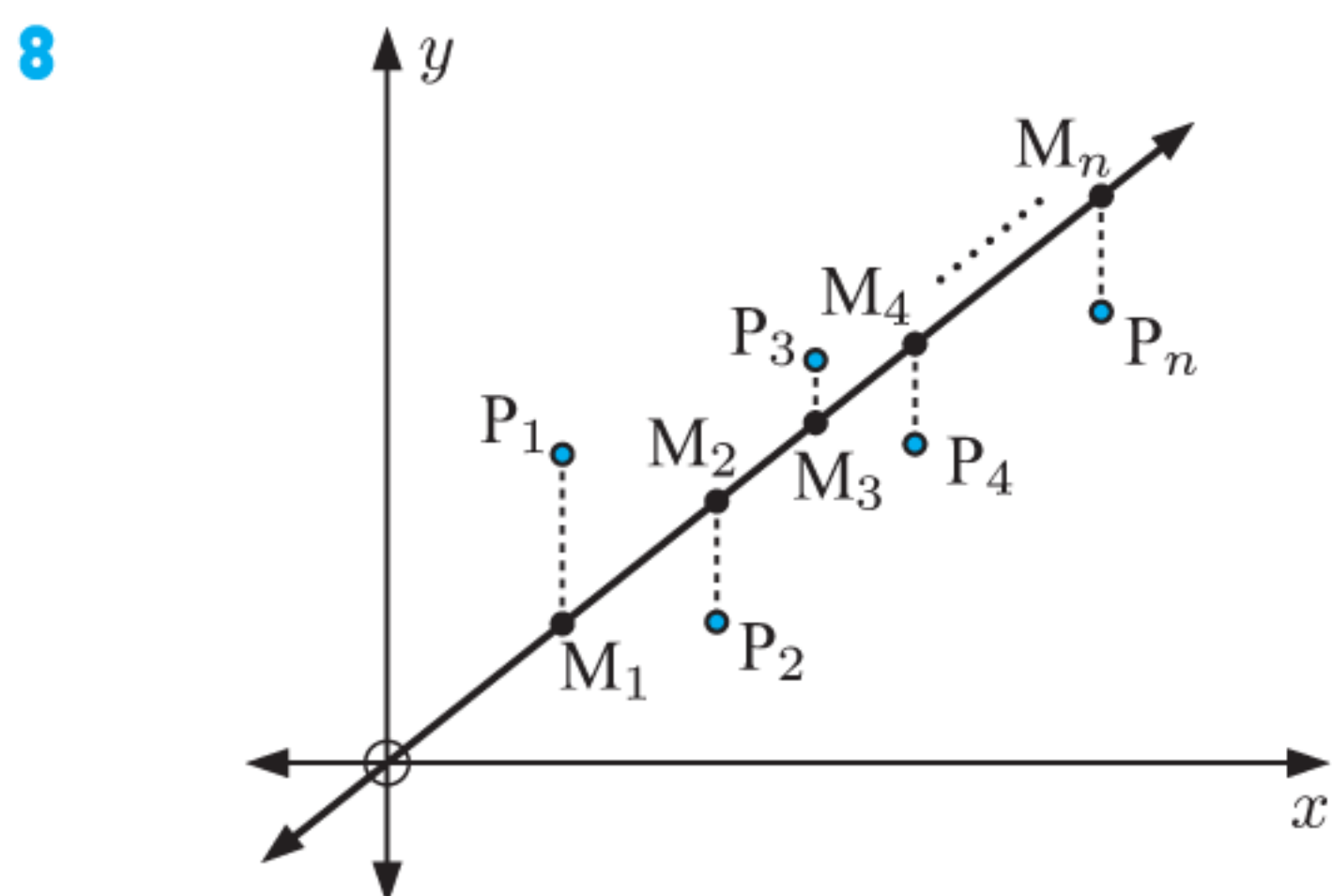
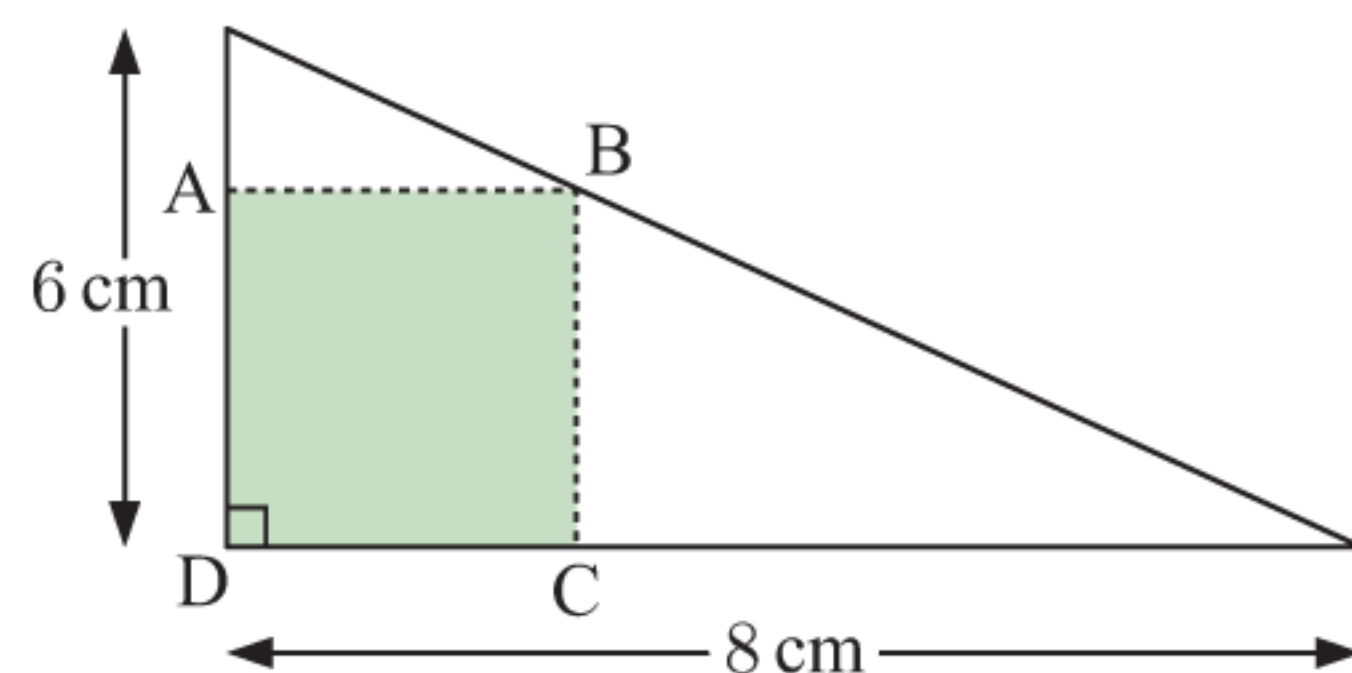


The graphs of $y = x^2 - 3x$ and $y = 2x - x^2$ are illustrated.

- a Show that the graphs meet where $x = 0$ and $x = 2\frac{1}{2}$.
- b Find the maximum vertical separation between the curves for $0 \leq x \leq 2\frac{1}{2}$.

7 Infinitely many rectangles may be inscribed within the right angled triangle shown alongside. One of them is illustrated.

- a Let $AB = x$ cm and $BC = y$ cm. Use similar triangles to find y in terms of x .
- b Find the dimensions of rectangle ABCD of maximum area.



$P_1(a_1, b_1), P_2(a_2, b_2), P_3(a_3, b_3), \dots, P_n(a_n, b_n)$ are a set of data points which are approximately linear through the origin $O(0, 0)$.

To find the equation of the “line of best fit” through the origin, we minimise

$$(P_1M_1)^2 + (P_2M_2)^2 + (P_3M_3)^2 + \dots + (P_nM_n)^2$$

where M_i lies on the line and has x -coordinate a_i .

Find the gradient m of the “line of best fit” in terms of a_i and $b_i, i = 1, 2, 3, 4, \dots, n$.

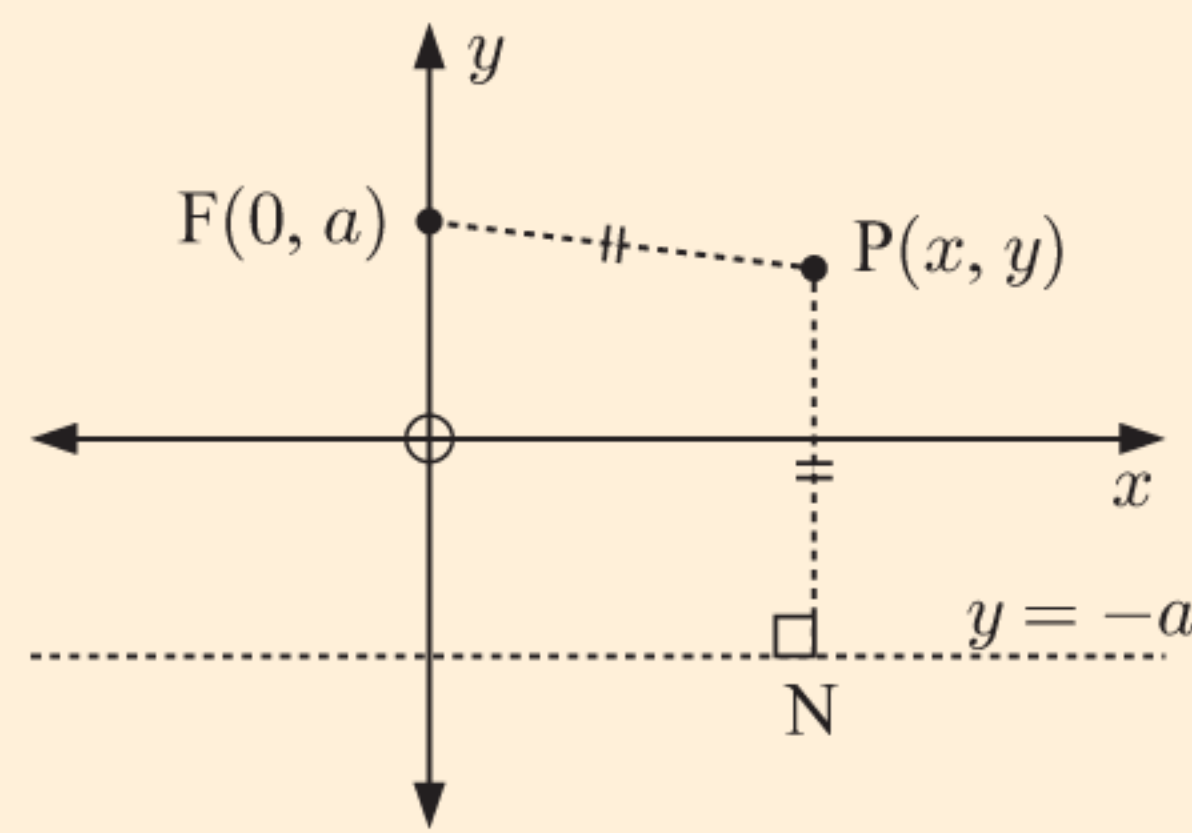
9 Assuming a and b are real constants, expand $y = (x - a - b)(x - a + b)(x + a - b)(x + a + b)$ and hence determine the least value of y .

10 By considering the function $y = (a_1x - b_1)^2 + (a_2x - b_2)^2$, use quadratic theory to prove the **Cauchy-Schwarz inequality** $|a_1b_1 + a_2b_2| \leq \sqrt{a_1^2 + a_2^2} \sqrt{b_1^2 + b_2^2}$.

INVESTIGATION 4 THE GEOMETRIC DEFINITION OF A PARABOLA

A **parabola** is defined as the locus of all points which are equidistant from a fixed point called the **focus** and a fixed line called the **directrix**.

Suppose the focus is $F(0, a)$ and the directrix is the horizontal line $y = -a$. The parabola is the set of all points P such that $FP = NP$ where N is the closest point on the directrix to P .



What to do:

- 1 Suggest why it is convenient to let the focus be at $(0, a)$ and the directrix be the line $y = -a$.
- 2 Use the circular-linear graph paper provided to graph the parabola which has focus $F(0, 2)$ and directrix $y = -2$.
- 3 Using the definition above:
 - a Write down the coordinates of N .
 - b Write expressions for FP and NP .
 - c Show that the parabola has the equation $y = \frac{x^2}{4a}$.
- 4 Consider a point $P\left(X, \frac{X^2}{4a}\right)$ on the parabola $y = \frac{x^2}{4a}$ with focus $F(0, a)$ and directrix $y = -a$.

PRINTABLE
GRAPH PAPER

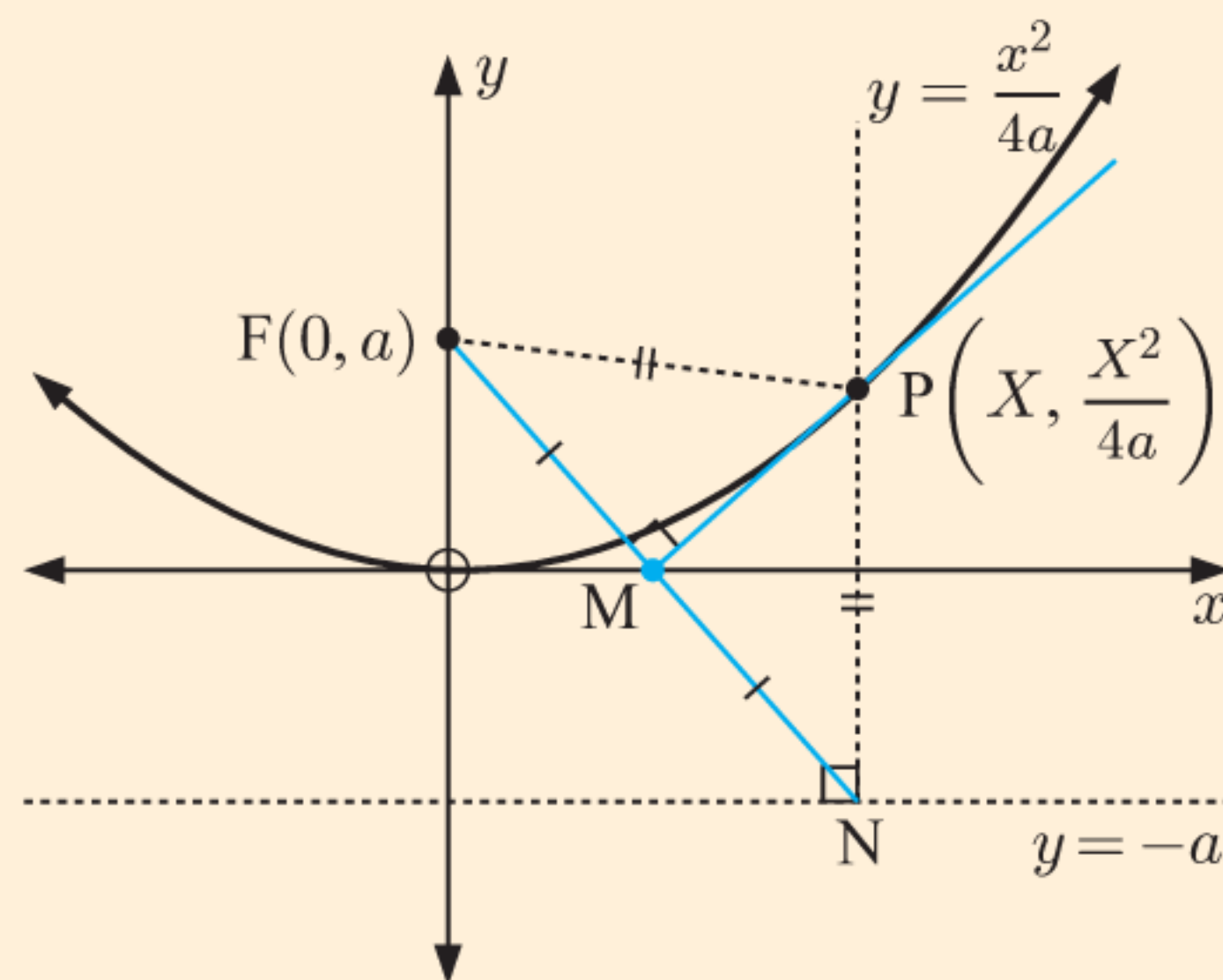


- a Let N be the closest point on the directrix to P , and M be the midpoint of $[FN]$.

- i Show that $[MP]$ has equation

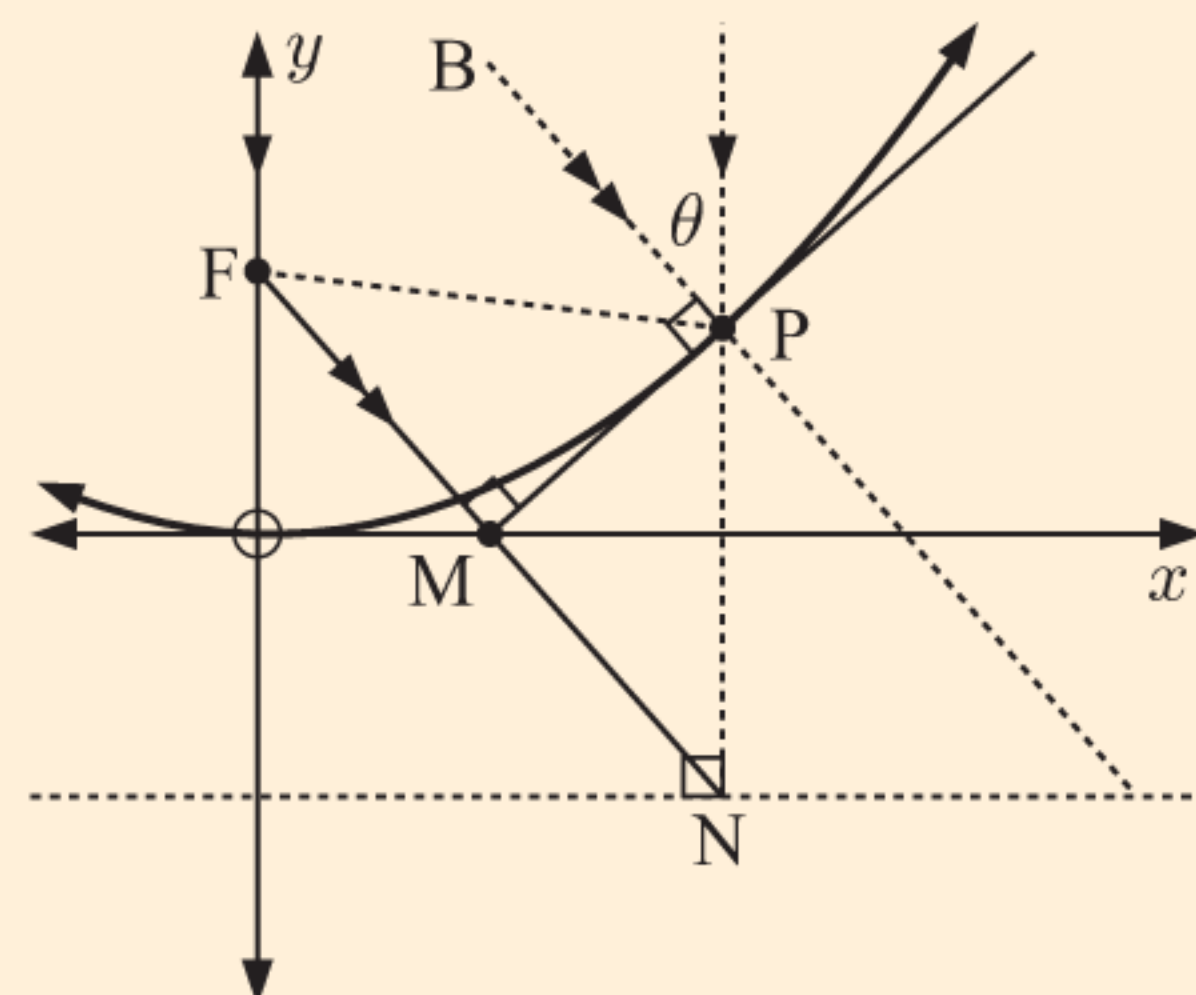
$$y = \frac{X}{2a} \left(x - \frac{X}{2} \right).$$

- ii Hence prove that (MP) is a tangent to the parabola.



- b Let B lie on the normal to the parabola. Suppose a ray of light shines vertically down onto the parabola with angle of incidence θ as shown.

- i Explain why \widehat{MNP} must equal θ .
- ii Hence explain why \widehat{MFP} must equal θ .
- iii Hence explain why \widehat{FPB} must equal θ .
- iv Hence explain why any vertical ray of light shining down onto a parabolic mirror will be reflected to the focus of the parabola F .



- v Explain what shape Misha needs in the **Opening Problem** and where he needs to place his cup.

This experiment was performed by Dr Jonathon Hare and Dr Ellen McCallie for the television series "Rough Science".



H QUADRATIC INEQUALITIES

A **quadratic inequality** can be written in either the form $ax^2 + bx + c \geq 0$ or $ax^2 + bx + c > 0$ where $a \neq 0$.

We have seen that the solutions to a quadratic equation are the x -intercepts of the corresponding quadratic function.

In a similar way, the solutions to a quadratic *inequality* are the values of x for which the corresponding function has a particular *sign*.

SIGN DIAGRAMS

A **sign diagram** is a number line which indicates the values of x for which a function is negative, zero, positive, or undefined.

A sign diagram consists of:

- a **horizontal line** which represents the x -axis
- **positive (+)** and **negative (-)** signs indicating where the graph is **above** and **below** the x -axis respectively
- the **zeros** of the function, which are the x -intercepts of its graph.

Consider the three functions below:

| Function | $y = (x + 2)(x - 1)$ | $y = (x + 3)^2 + 2$ | $y = -2(x - 1)^2$ |
|--------------|----------------------|---------------------|-------------------|
| Graph | | | |
| Sign diagram | | | |

You should notice that:

- A sign change occurs about a zero of the function for single linear factors such as $(x + 2)$ and $(x - 1)$. This indicates **cutting** of the x -axis.
- No sign change occurs about a zero of the function for squared linear factors such as $(x - 1)^2$. This indicates **touching** of the x -axis.



In general:

- when a linear factor has an **odd power** there is a change of sign about that zero
- when a linear factor has an **even power** there is no sign change about that zero.

Example 21

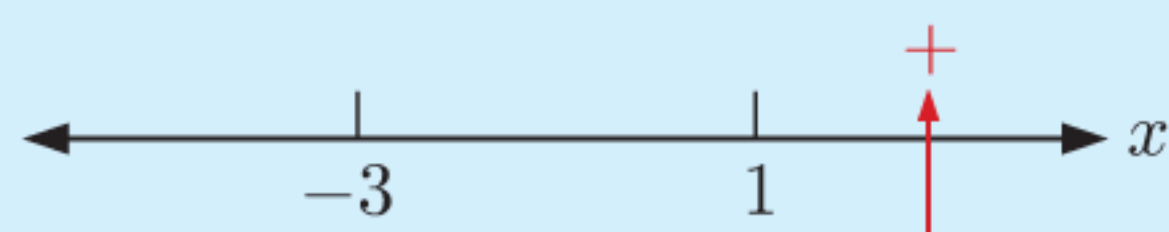
Self Tutor

Draw a sign diagram for:

a $x^2 + 2x - 3$

b $-4(x - 3)^2$

a $x^2 + 2x - 3 = (x + 3)(x - 1)$
which has zeros -3 and 1 .

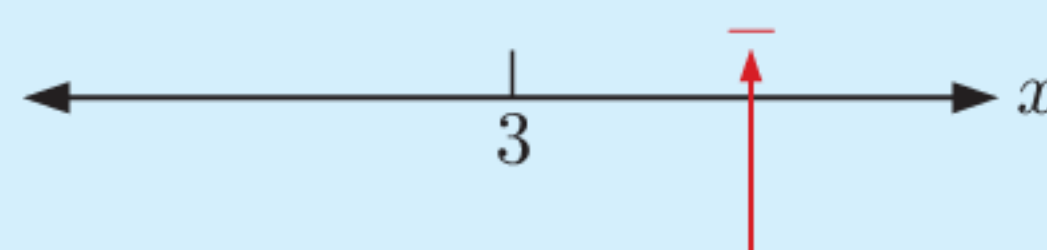


When $x = 2$ we have $(5)(1) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs alternate.

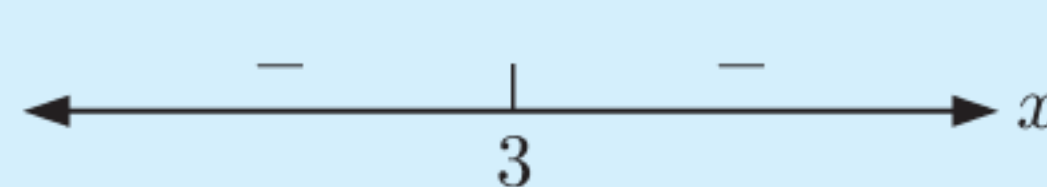


b $-4(x - 3)^2$ has zero 3 .



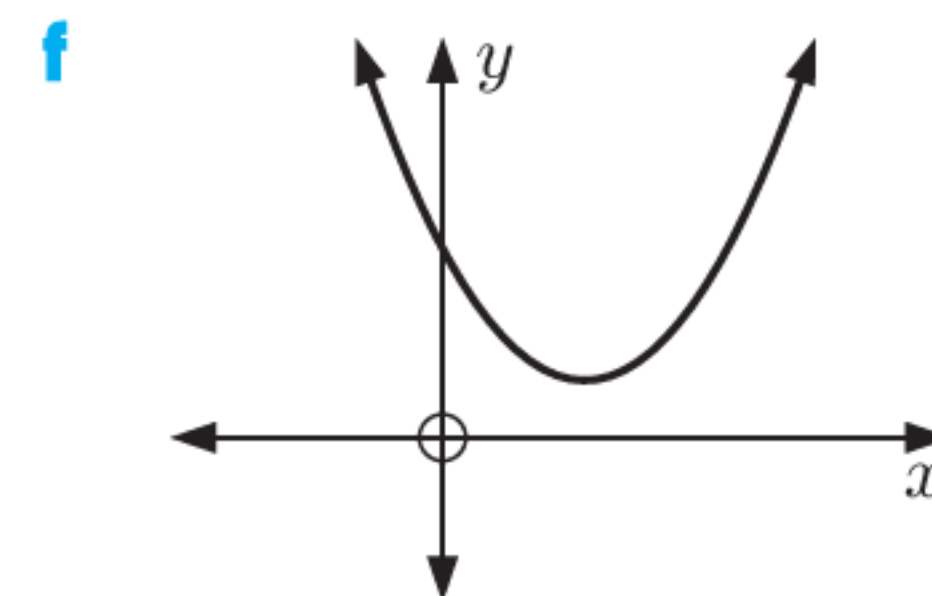
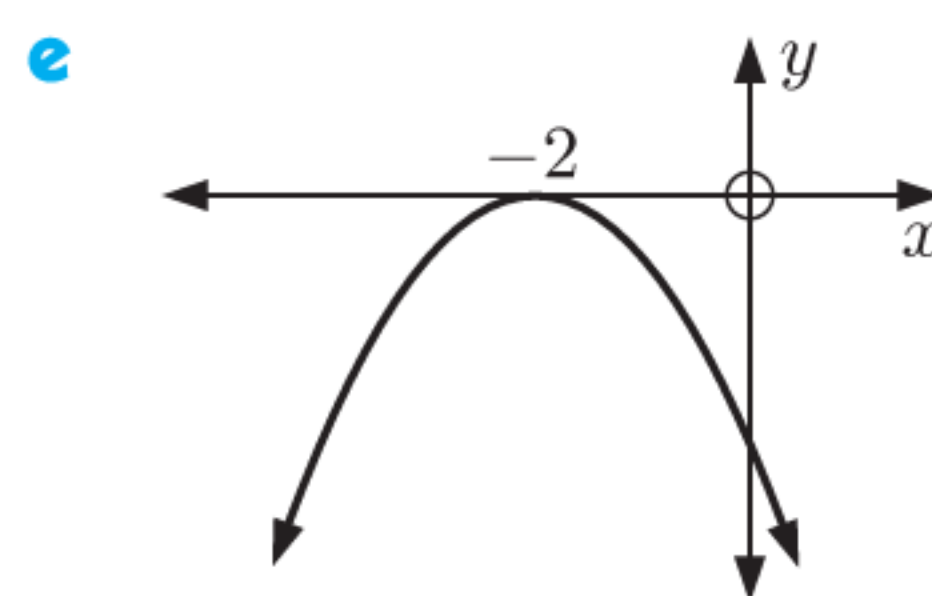
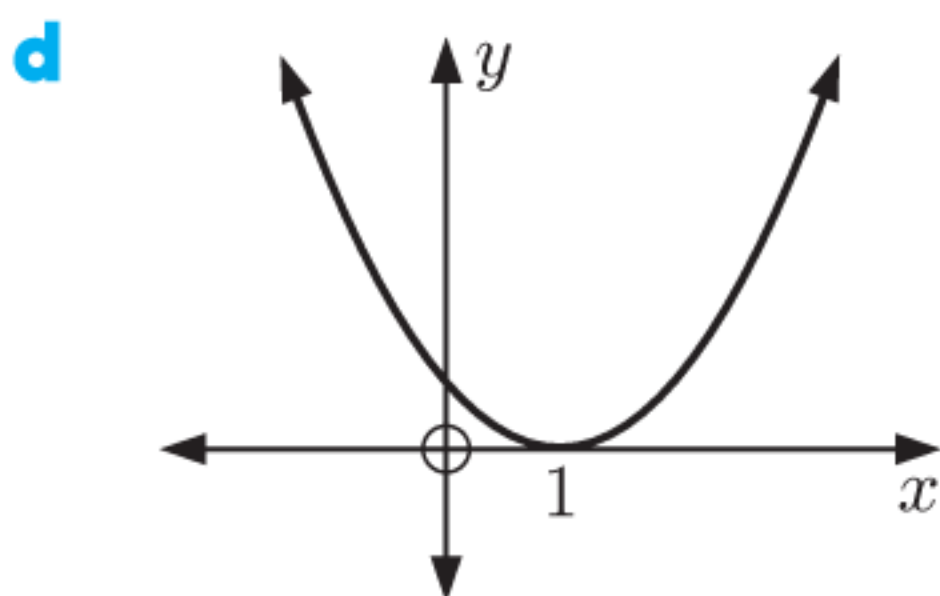
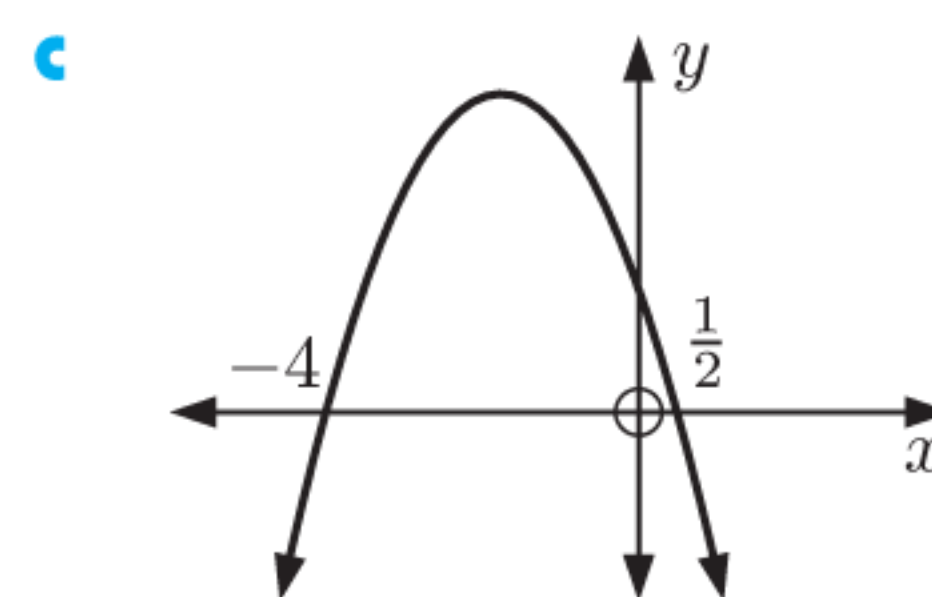
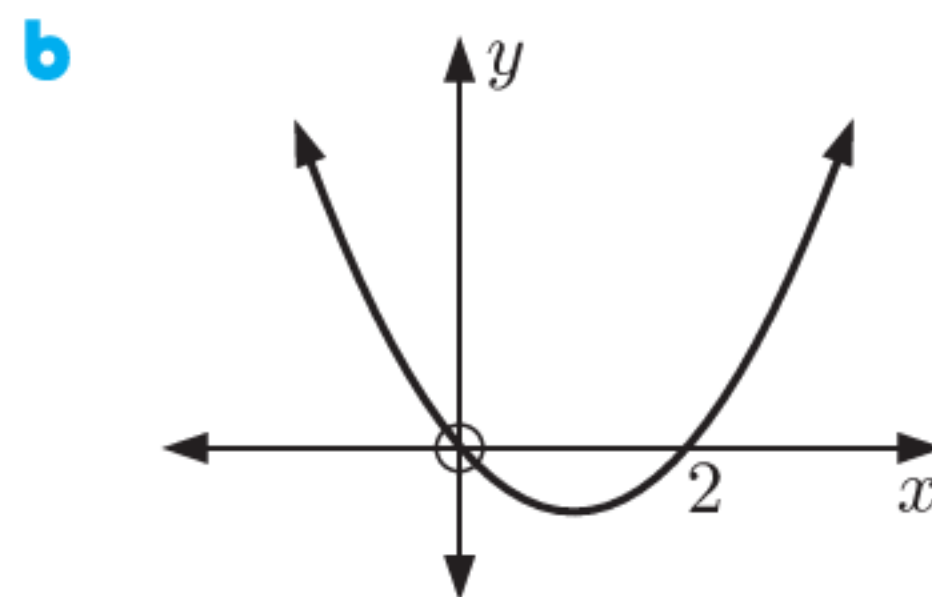
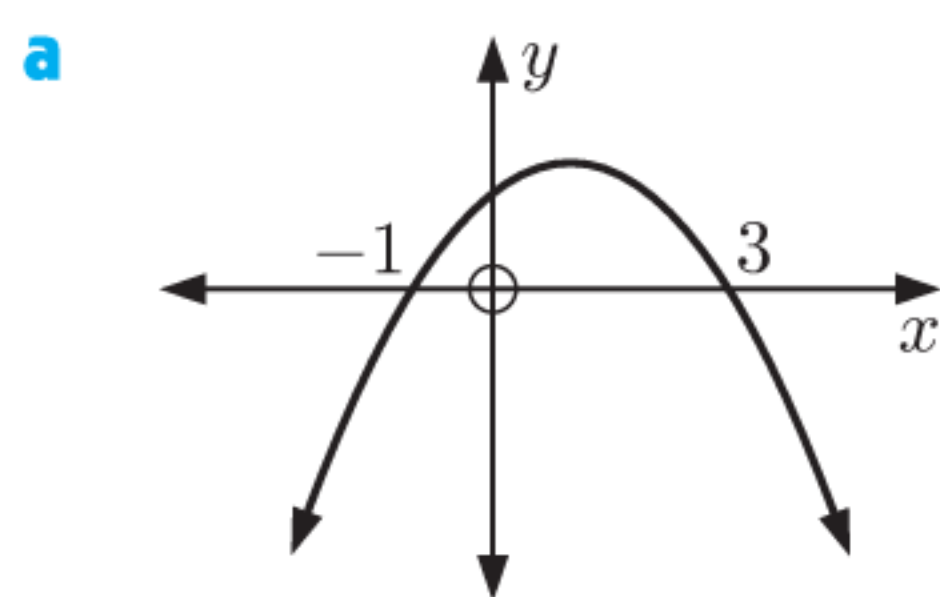
When $x = 4$ we have $-4(1)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.



EXERCISE 14H.1

1 Draw a sign diagram for each graph:



2 Draw a sign diagram for:

a $(x + 4)(x - 2)$

b $(x + 1)(x - 5)$

c $x(x - 3)$

d $x(x + 2)$

e $(2x + 1)(x - 4)$

f $-(x + 1)(x - 3)$

g $-(3x - 2)(x + 1)$

h $(2x - 1)(3 - x)$

i $(5 - x)(1 - 2x)$

3 Draw a sign diagram for:

a $(x + 2)^2$

b $(x - 3)^2$

c $-(x - 4)^2$

d $2(x + 1)^2$

e $-3(x + 4)^2$

f $-\frac{1}{2}(2x + 5)^2$

4 Draw a sign diagram for:

a $x^2 - 9$

b $4 - x^2$

c $5x - x^2$

d $x^2 - 3x + 2$

e $2 - 8x^2$

f $6x^2 + x - 2$

g $6 - 16x - 6x^2$

h $-2x^2 + 9x + 5$

i $-15x^2 - x + 2$

5 Draw a sign diagram for:

a $x^2 + 10x + 25$

b $x^2 - 2x + 1$

c $-x^2 + 4x - 4$

d $4x^2 - 4x + 1$

e $-x^2 - 6x - 9$

f $-4x^2 + 12x - 9$

QUADRATIC INEQUALITIES

To solve quadratic inequalities we use the following procedure:

- Make the RHS zero by shifting all terms to the LHS.
- Fully factorise the LHS.
- Draw a sign diagram for the LHS.
- Determine the values required from the sign diagram.

Example 22

Self Tutor

Solve for x :

a $3x^2 + 5x \geq 2$

b $x^2 + 9 < 6x$

a $3x^2 + 5x \geq 2$

$\therefore 3x^2 + 5x - 2 \geq 0$

$\therefore (3x - 1)(x + 2) \geq 0$

Sign diagram of LHS is



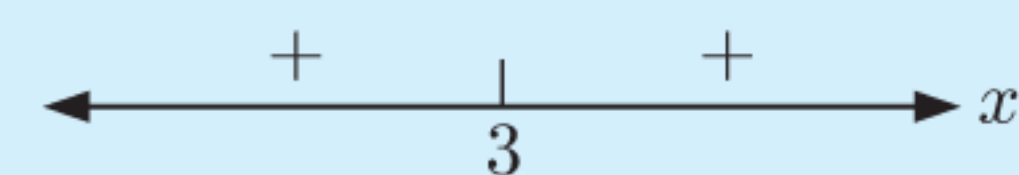
$x \leq -2$ or $x \geq \frac{1}{3}$

b $x^2 + 9 < 6x$

$\therefore x^2 - 6x + 9 < 0$

$\therefore (x - 3)^2 < 0$

Sign diagram of LHS is



So, the inequality is not true for any real x .

EXERCISE 14H.2

1 Solve for x :

a $(x - 2)(x + 5) \leq 0$

b $(2 - x)(x + 3) \geq 0$

c $(x - 1)^2 < 0$

d $(x + 5)^2 \geq 0$

e $(2x + 1)(3 - x) > 0$

f $(x - 4)(2x + 3) < 0$

2 Solve for x :

a $x^2 - x \geq 0$

b $3x^2 + 2x < 0$

c $x^2 + 4x + 4 > 0$

d $x^2 + 2x - 15 \leq 0$

e $x^2 - 4x - 12 > 0$

f $3x^2 + 9x - 12 < 0$

3 Solve for x :

- | | | |
|-----------------------------|------------------------------|-----------------------------|
| a $x^2 \geq 3x$ | b $x^2 < 4$ | c $2x^2 \geq 4$ |
| d $x^2 - 21 \leq 4x$ | e $x^2 + 30 > 11x$ | f $x + 42 < x^2$ |
| g $2x^2 \geq x + 3$ | h $4x^2 - 4x + 1 < 0$ | i $6x^2 + 7x < 3$ |
| j $3x^2 > 8(x + 2)$ | k $2x^2 - 4x + 2 > 0$ | l $6x^2 + 1 \leq 5x$ |
| m $1 + 5x < 6x^2$ | n $12x^2 \geq 5x + 2$ | o $2x^2 + 9 > 9x$ |

Example 23

Self Tutor

Find the value(s) of k for which the function $y = kx^2 + (k + 3)x - 1$:

- a** cuts the x -axis twice **b** touches the x -axis **c** misses the x -axis.

$$a = k, \quad b = k + 3, \quad c = -1$$

$$\therefore \Delta = b^2 - 4ac$$

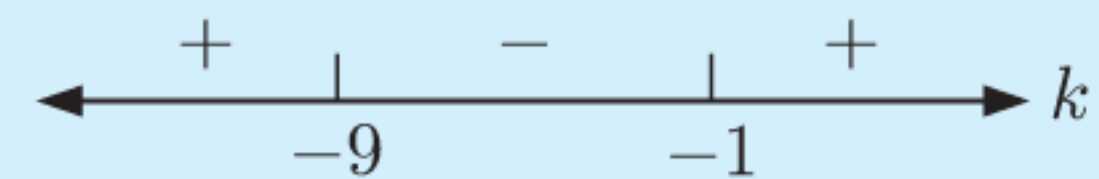
$$= (k + 3)^2 - 4(k)(-1)$$

$$= k^2 + 6k + 9 + 4k$$

$$= k^2 + 10k + 9$$

$$= (k + 9)(k + 1)$$

So, Δ has sign diagram:



- a** The graph cuts the x -axis twice if $\Delta > 0$

$$\therefore k < -9 \text{ or } k > -1, \quad k \neq 0.$$

- b** The graph touches the x -axis if $\Delta = 0$

$$\therefore k = -9 \text{ or } k = -1.$$

- c** The graph misses the x -axis if $\Delta < 0$

$$\therefore -9 < k < -1.$$

The discriminant Δ is a quadratic in k , so we must solve a quadratic inequality.



4 For each quadratic function, find the values of k for which the function:

- i** cuts the x -axis twice **ii** touches the x -axis **iii** misses the x -axis.

- a** $y = 2x^2 + kx - k$ **b** $y = kx^2 - 2x + k$ **c** $y = x^2 + (k + 2)x + 4$

5 For each quadratic equation, find the values of k for which the equation has:

- i** two real roots **ii** a repeated real root **iii** no real roots.

- a** $2x^2 + (k - 2)x + 2 = 0$ **b** $x^2 + (3k - 1)x + (2k + 10) = 0$

- c** $(k + 1)x^2 + kx + k = 0$

6 For what values of m is $y = (m - 2)x^2 + 6x + 3m$:

- a** positive definite **b** negative definite?

7 Consider the curve $y = -x^2 + 3x - 6$ and the line $y = mx - 2$. Find the values of m for which the line:

- a** meets the curve twice **b** is a tangent to the curve
c does not meet the curve.

DYNAMIC
GEOMETRY
PACKAGE



- 8 For what values of a do the curves $y = ax^2 + 2x + 1$ and $y = -x^2 + ax - 1$:
- a** meet twice **b** touch **c** never meet?

REVIEW SET 14A

1 Use the vertex, axis of symmetry, and y -intercept to graph:

a $y = (x - 2)^2 - 4$

b $y = -\frac{1}{2}(x + 4)^2 + 6$

2 Find the points of intersection of $y = x^2 - 3x$ and $y = 3x^2 - 5x - 24$.

3 For what values of k does the graph of $y = -2x^2 + 5x + k$ not cut the x -axis?

4 Find the values of m for which $2x^2 - 3x + m = 0$ has:

a a repeated root

b two distinct real roots

c no real roots.

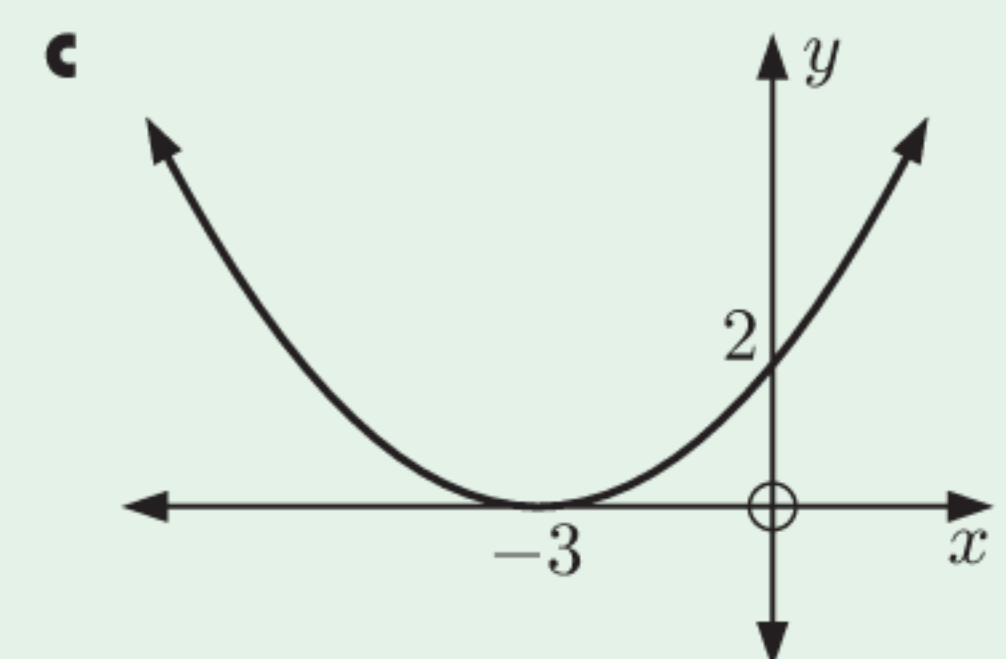
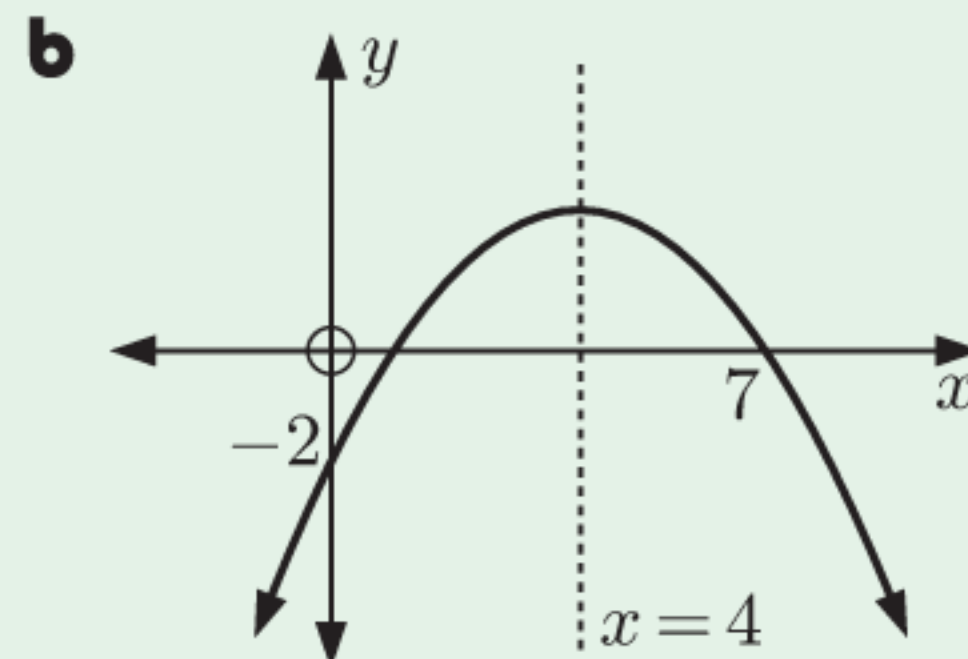
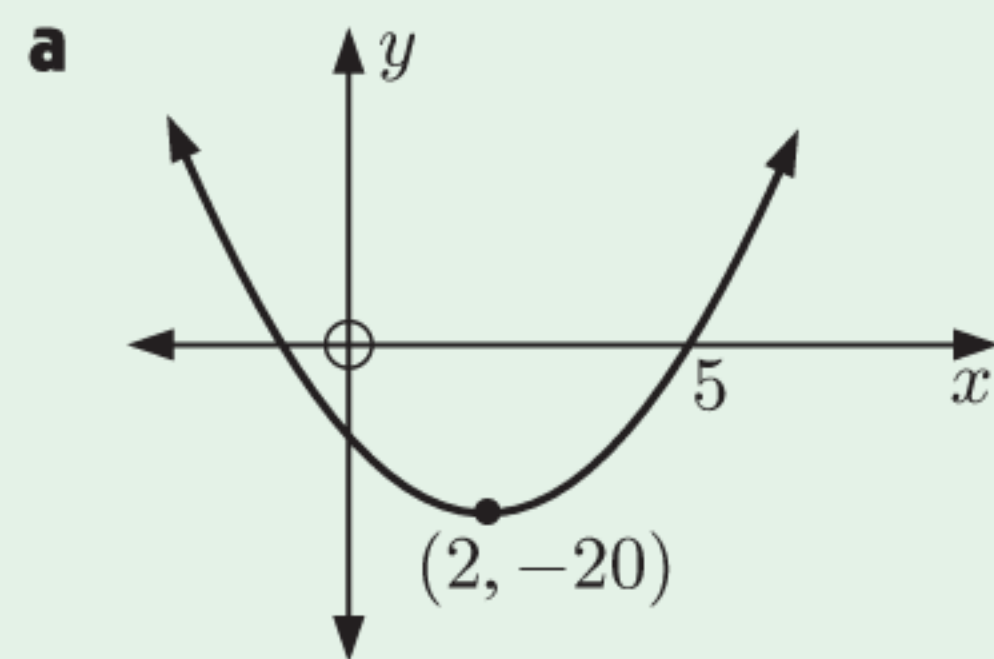
5 The sum of a number and its reciprocal is $2\frac{1}{30}$. Find the number.

6 Show that no line with a y -intercept of 10 will ever be tangential to the curve with equation $y = 3x^2 + 7x - 2$.

7 **a** Write the quadratic $y = 2x^2 + 6x - 3$ in the form $y = a(x - h)^2 + k$.

b Hence sketch the graph of the quadratic.

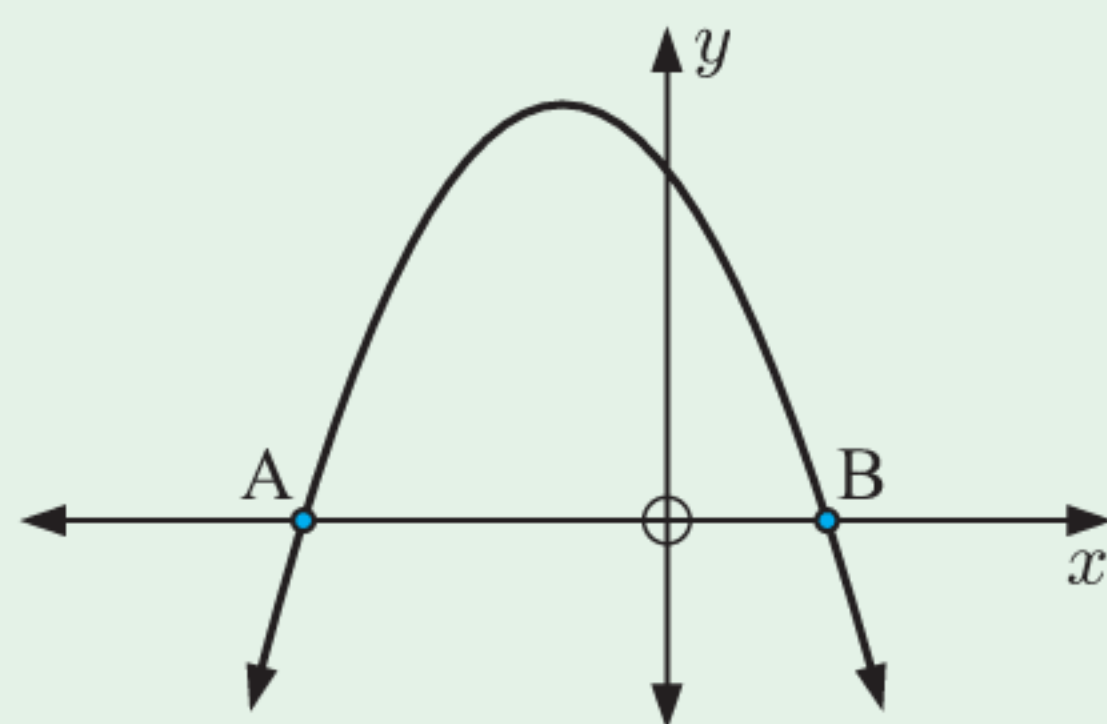
8 Find the equation of the quadratic with graph:



9 Draw the graph of $y = -x^2 + 2x$.

10 Find the y -intercept of the line with gradient -3 which is a tangent to the parabola $y = 2x^2 - 5x + 1$.

11 The graph shows the parabola $y = a(x + m)(x + n)$ where $m > n$.



a State the sign of:

i the discriminant Δ

ii a .

b Find, in terms of m and n , the:

i coordinates of the x -intercepts A and B

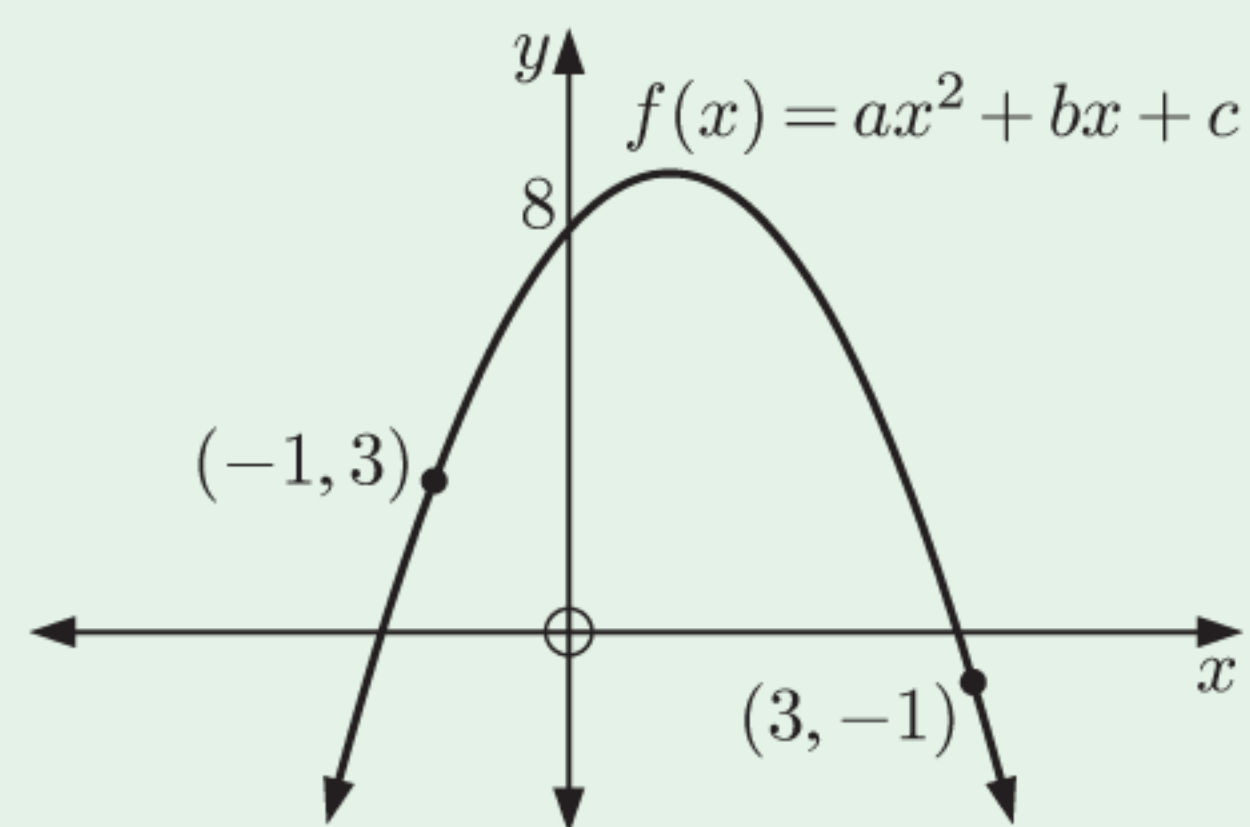
ii equation of the axis of symmetry.

12 For a quadratic function $y = ax^2 + bx + c$, suppose the constants a , b , and c are consecutive terms of a geometric sequence. Show that the function does not cut the x -axis.

13 Find the quadratic function which cuts the x -axis at 3 and -2 and which has y -intercept 24. Give your answer in the form $y = ax^2 + bx + c$.

14 Find the value of k for which the x -intercepts of $y = 3x^2 + 2kx + k - 1$ are closest together.

- 15** Consider the function $y = ax^2 + bx + c$ shown.
- State the value of c .
 - Use the other information to write two equations involving a and b .
 - Find a and b , and hence state the equation of the quadratic.



- 16** For what values of m are the lines $y = mx - 10$ tangents to the parabola $y = 3x^2 + 7x + 2$?
- 17** When Annie hits a softball, the height of the ball above the ground after t seconds is given by $h = -4.9t^2 + 19.6t + 1.4$ metres. Find the maximum height reached by the ball.



- 18** Draw a sign diagram for:

a $(3x + 2)(4 - x)$

b $-x^2 + 3x + 18$

- 19** Solve for x :

a $(3 - x)(x + 2) < 0$

b $x^2 - 4x - 5 \leq 0$

c $2x^2 + x > 10$

- 20** Find the values of k for which the function $f(x) = x^2 + kx + (3k - 4)$:

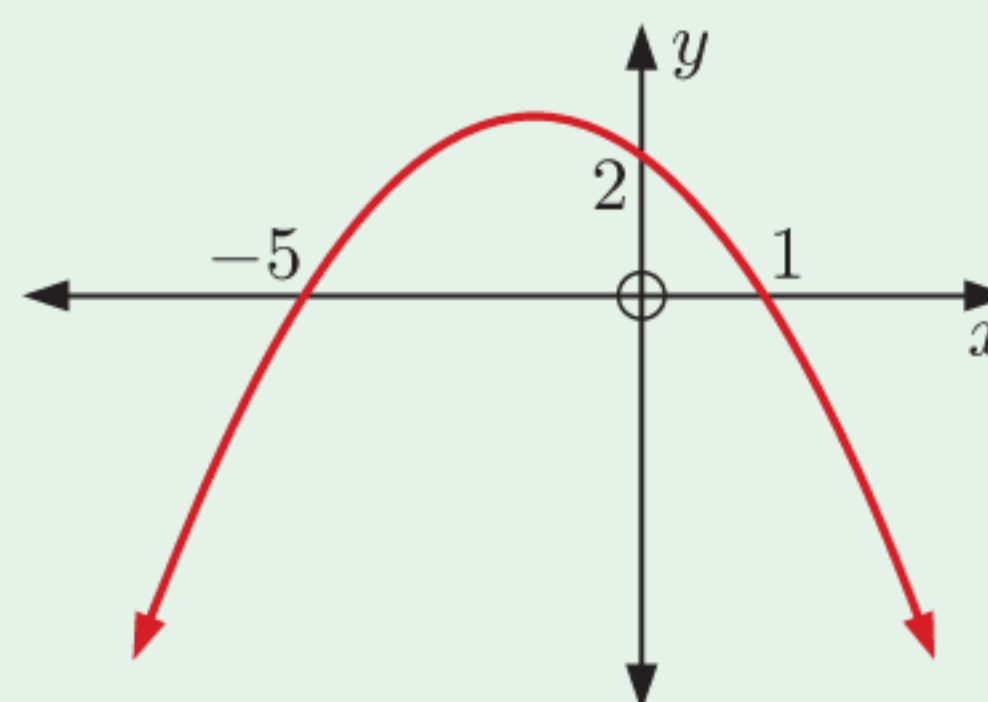
a cuts the x -axis twice

b touches the x -axis

c misses the x -axis.

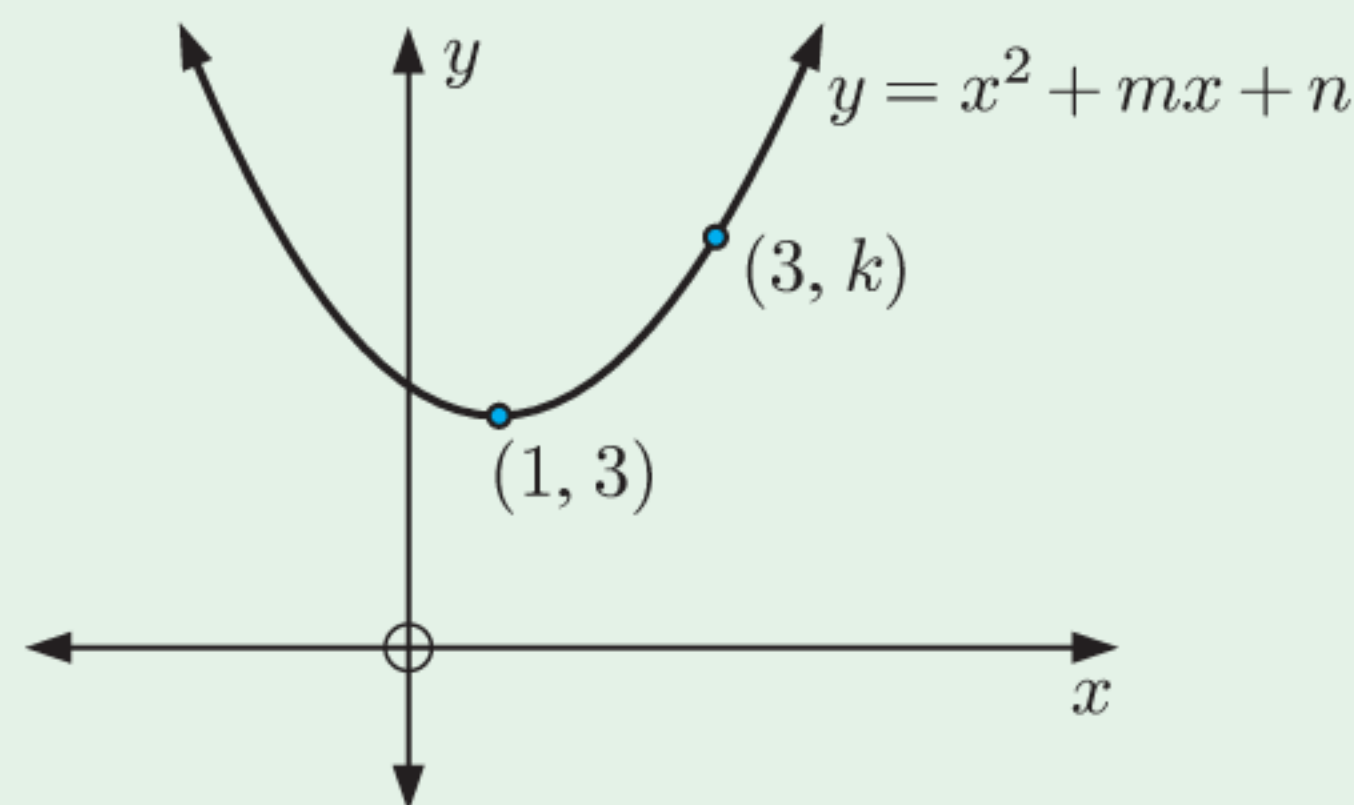
REVIEW SET 14B

- 1** Consider the quadratic $y = \frac{1}{2}(x - 2)^2 - 4$.
- State the equation of the axis of symmetry.
 - Find the coordinates of the vertex.
 - Find the y -intercept.
 - Sketch the function.
- 2** Consider the quadratic $y = -3x^2 + 8x + 7$. Find the equation of the axis of symmetry, and the coordinates of the vertex.
- 3** Use the discriminant only to find the relationship between the graph and the x -axis for:
- $y = 2x^2 + 3x - 7$
 - $y = -3x^2 - 7x + 4$
- 4** Find the equation of the quadratic with vertex $(2, 25)$ and y -intercept 1.
- 5**
- Find the equation of the quadratic illustrated.
 - Hence find its vertex and axis of symmetry.



- 6** Consider the quadratic $y = 2x^2 + 4x - 1$.
- State the axis of symmetry.
 - Find the coordinates of the vertex.
 - Find the axes intercepts.
 - Hence sketch the function.
- 7** Find, in the form $y = ax^2 + bx + c$, the quadratic function whose graph:
- touches the x -axis at 3 and passes through $(2, 2)$
 - has x -intercepts 3 and -2 , and y -intercept 3
 - passes through $(-1, -9)$, $(1, 5)$, and $(2, 15)$
 - has vertex $(3, 15)$ and passes through the point $(1, 7)$.
- 8**
- For what values of c do the lines with equations $y = 3x + c$ intersect the parabola $y = x^2 + x - 5$ in two distinct points?
 - Choose one such value of c and find the points of intersection in this case.
- 9** Find the maximum or minimum value of each quadratic, and the corresponding value of x :
- $y = 3x^2 + 4x + 7$
 - $y = -2x^2 - 5x + 2$
- 10** The graph of a quadratic function cuts the x -axis at -2 and 3 , and passes through $(-3, 18)$.
- Find the equation of the function in the form $y = ax^2 + bx + c$.
 - Write down the y -intercept of the function.
 - Find the coordinates of the vertex.

11



Consider the graph of $y = x^2 + mx + n$.

- Determine the values of m and n .
 - Hence find the value of k .
- 12** An open square-based box has capacity 120 mL. It is made from a square piece of tinfoil with 4 cm squares cut from each of its corners. Find the dimensions of the original piece of tinfoil.
- 13** Consider $y = -x^2 - 3x + 4$ and $y = x^2 + 5x + 4$.
- Solve for x : $-x^2 - 3x + 4 = x^2 + 5x + 4$.
 - Sketch the curves on the same set of axes.
 - Hence solve for x : $x^2 + 5x + 4 > -x^2 - 3x + 4$.
- 14** For each of the following quadratics:
- Write the quadratic in completed square form.
 - Write the quadratic in factored form.
 - Sketch the graph of the quadratic, identifying its axes intercepts, vertex, and axis of symmetry.
- $y = x^2 + 4x + 3$
 - $y = x^2 + 2x - 3$
 - $y = 2x^2 - 8x - 10$
 - $y = -x^2 + 6x + 7$

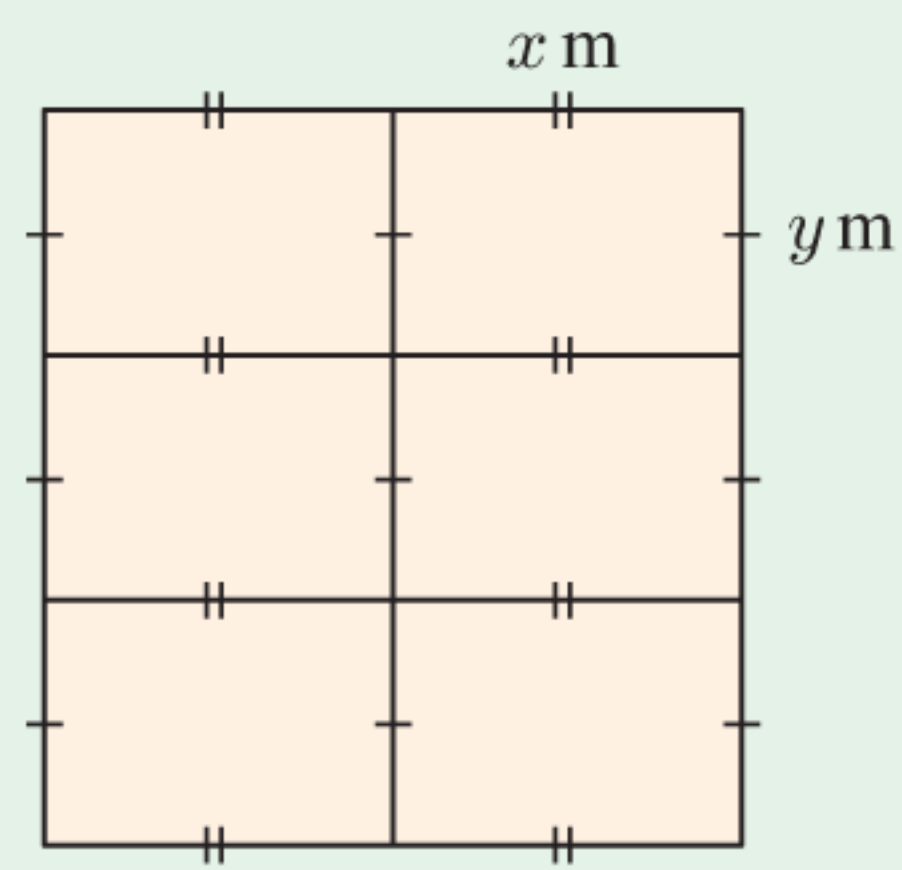
- 15** Two different quadratic functions of the form $y = 9x^2 - kx + 4$ both *touch* the x -axis.
- Find the two values of k .
 - Find the point of intersection of the two quadratic functions.

- 16** 600 m of fencing is used to construct 6 rectangular animal pens as shown.

- a** Show that the area A of each pen is

$$A = x \left(\frac{600 - 8x}{9} \right) \text{ m}^2.$$

- Find the dimensions of each pen so that it has the maximum possible area.
- What is the area of each pen in this case?



- 17** A retailer sells sunglasses for \$45, and has 50 customers per day. From market research, the retailer discovers that for every \$1.50 increase in the price of the sunglasses, he will lose a customer per day.

Let $\$x$ be the price increase of the sunglasses.

- a** Show that the revenue collected by the retailer each day is

$$R = (45 + x) \left(50 - \frac{x}{1.5} \right) \text{ dollars.}$$

- b** Find the price the retailer should set for his sunglasses in order to maximise his daily revenue. How much revenue is made per day at this price?

- 18** Draw a sign diagram for:

a $x^2 - 3x - 10$

b $-(x + 3)^2$

- 19** Solve for x :

a $4x^2 - 3x < 0$

b $2x^2 - 3x - 5 \geq 0$

c $\frac{11}{3}x \leq 2x^2 + 1$

- 20** Find the values of m for which the function $y = mx^2 + 5x + (m + 12)$:

a cuts the x -axis twice

b touches the x -axis

c misses the x -axis.