Chapter 2

Exponential functions

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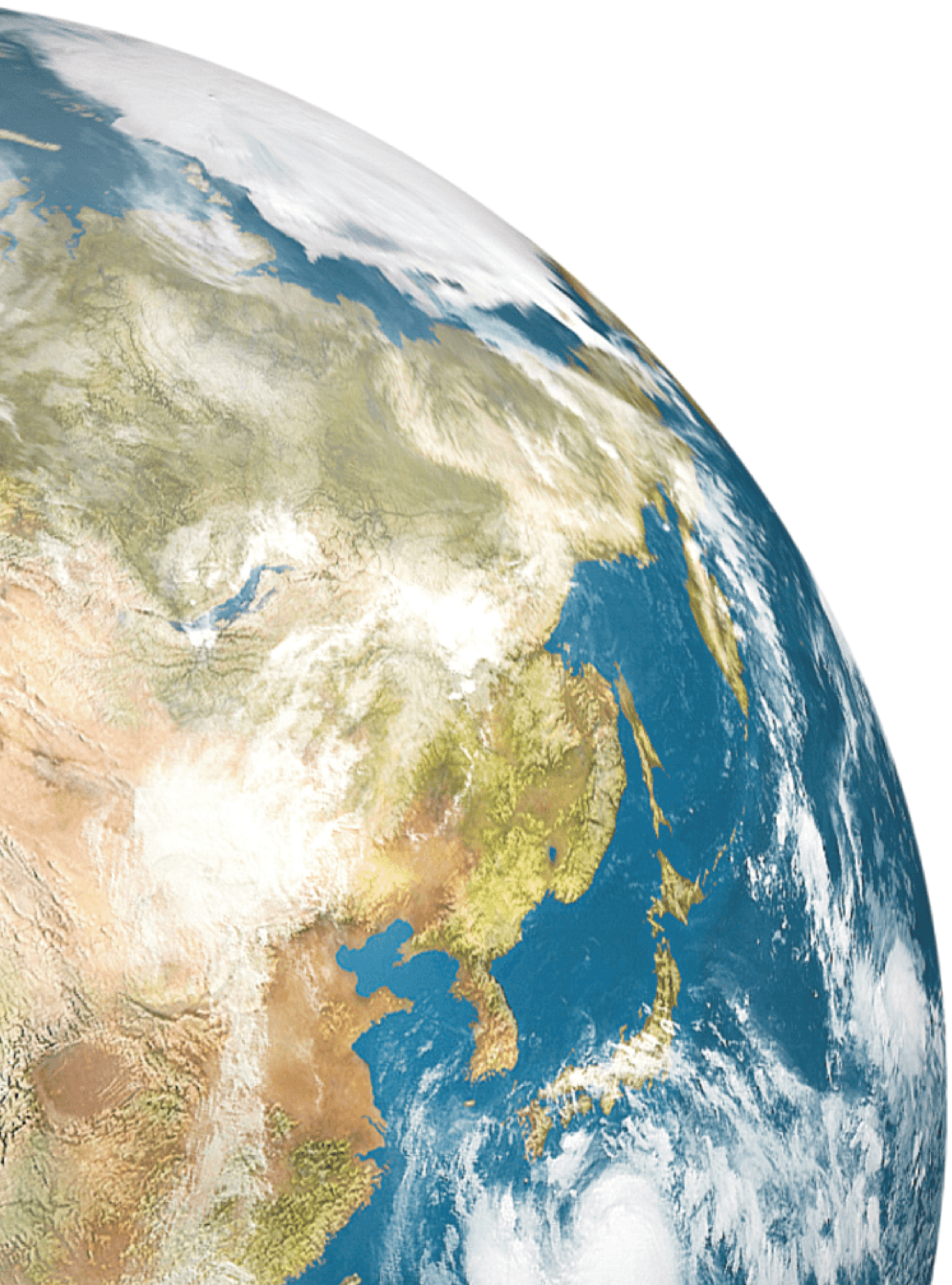


OPENING PROBLEM

At an antiques fair, Bernard purchases a clock for £500 and a vase for £400. The clock increases in value by 5% each year, and the vase increases in value by 7% each year.

Things to think about:

- **a** What is the value of each item 1 year after purchase?
- **b** Can you write a formula for the value of each item t years after purchase?
- **c** Which item is more valuable 15 years after purchase?
- **d** How can we determine when the items are equal in value?

We have seen previously how exponents are used to indicate when a number is raised to a power.

For a positive integer exponent, the exponent tells us how many of the base are multiplied together.

Any non-zero base to the power 0 is defined as 1, to give consistency to the exponent laws.

For a negative integer exponent, we take the reciprocal of the corresponding positive integer power.

$$\begin{cases} 2^3 = 2\times 2\times 2 & = 8 \\ 2^2 = 2\times 2 & = 4 \\ 2^1 = 2 & = 2 \end{cases}$$

$$2^0 = 1 \qquad = 1$$

$$\begin{cases} 2^{-1} = \frac{1}{2} & = \frac{1}{2} \\ 2^{-2} = \dfrac{1}{2\times 2} & = \frac{1}{4} \\ 2^{-3} = \dfrac{1}{2\times 2\times 2} & = \frac{1}{8} \end{cases}$$

In this Chapter we give meaning to exponents which are **rational**, allowing us to start filling in the gaps between the integer exponents. This will allow us to consider **exponential functions** for which the variable appears in an exponent.

A RATIONAL EXPONENTS

Using the definition $a^n = a \times a \times .... \times a$, the **laws of exponents** such as $a^n \times a^m = a^{n+m}$ can be proven for any integers n and m.

For a positive base a, we choose to define a raised to a rational exponent so that the laws of exponents still hold.

So, for any $a > 0$, notice that $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$ {exponent laws}

and $\sqrt{a} \times \sqrt{a} = a$ also.

Likewise, $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^1 = a$

and $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$ also.

By direct comparison, we conclude that $\boldsymbol{a^{\frac{1}{2}} = \sqrt{a}}$ and $\boldsymbol{a^{\frac{1}{3}} = \sqrt[3]{a}}$.

In general, $\boldsymbol{a^{\frac{1}{n}} = \sqrt[n]{a}}$ where $\sqrt[n]{a}$ reads "the nth root of a" for $n \in \mathbb{Z}^+$.

We can now determine that $\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}}$

$$= a^{\frac{m}{n}}$$

$\therefore$ $\boldsymbol{a^{\frac{m}{n}} = \sqrt[n]{a^m}}$ for $a > 0,\ n \in \mathbb{Z}^+,\ m \in \mathbb{Z}$

Example 1 Self Tutor

Write as a single power of 2:

a $\sqrt[3]{2}$ **b** $\dfrac{1}{\sqrt{2}}$ **c** $\sqrt[5]{4}$

a $\sqrt[3]{2}$
$= 2^{\frac{1}{3}}$

b $\dfrac{1}{\sqrt{2}}$
$= \dfrac{1}{2^{\frac{1}{2}}}$
$= 2^{-\frac{1}{2}}$

c $\sqrt[5]{4}$
$= (2^2)^{\frac{1}{5}}$
$= 2^{2\times\frac{1}{5}}$
$= 2^{\frac{2}{5}}$

EXERCISE 2A

1 Write as a single power of 2:

a $\sqrt[5]{2}$ **b** $\dfrac{1}{\sqrt[5]{2}}$ **c** $2\sqrt{2}$ **d** $4\sqrt{2}$ **e** $\dfrac{1}{\sqrt[3]{2}}$

f $2 \times \sqrt[3]{2}$ **g** $\dfrac{4}{\sqrt{2}}$ **h** $(\sqrt{2})^3$ **i** $\dfrac{1}{\sqrt[3]{16}}$ **j** $\dfrac{1}{\sqrt{8}}$

2 Write as a single power of 3:

a $\sqrt[3]{3}$ **b** $\dfrac{1}{\sqrt[3]{3}}$ **c** $\sqrt[4]{3}$ **d** $3\sqrt{3}$ **e** $\dfrac{1}{9\sqrt{3}}$

3 Write in the form a^k, where a is a prime number and k is rational:

a $\sqrt[3]{7}$ **b** $\sqrt[4]{27}$ **c** $\sqrt[5]{16}$ **d** $\sqrt[3]{32}$ **e** $\sqrt[7]{49}$

f $\dfrac{1}{\sqrt[3]{7}}$ **g** $\dfrac{1}{\sqrt[4]{27}}$ **h** $\dfrac{1}{\sqrt[5]{16}}$ **i** $\dfrac{1}{\sqrt[3]{32}}$ **j** $\dfrac{1}{\sqrt[7]{49}}$

4 Write in the form x^k, where k is rational:

a $\sqrt{x}$ **b** $x\sqrt{x}$ **c** $\dfrac{1}{\sqrt{x}}$ **d** $x^2\sqrt{x}$ **e** $\dfrac{1}{x\sqrt{x}}$

5 Use your calculator to find, correct to 3 significant figures:

a $3^{\frac{3}{4}}$ **b** $4^{-\frac{3}{5}}$ **c** $\sqrt[4]{8}$ **d** $\sqrt[5]{27}$ **e** $\dfrac{1}{\sqrt[3]{7}}$

GRAPHICS CALCULATOR INSTRUCTIONS

6 Write *without* rational exponents:

a $5^{\frac{1}{3}}$ **b** $3^{-\frac{1}{2}}$ **c** $3^{\frac{5}{2}}$ **d** $m^{\frac{3}{2}}$ **e** $x^{\frac{7}{2}}$

Example 2 Self Tutor

Without using a calculator, write in simplest rational form:

a $8^{\frac{4}{3}}$ **b** $27^{-\frac{2}{3}}$

a $8^{\frac{4}{3}}$
$= (2^3)^{\frac{4}{3}}$
$= 2^{3\times\frac{4}{3}}$ $\{(a^m)^n = a^{mn}\}$
$= 2^4$
$= 16$

b $27^{-\frac{2}{3}}$
$= (3^3)^{-\frac{2}{3}}$
$= 3^{3\times-\frac{2}{3}}$
$= 3^{-2}$
$= \frac{1}{9}$

7 Without using a calculator, write in simplest rational form:

a $4^{\frac{3}{2}}$ **b** $8^{\frac{5}{3}}$ **c** $16^{\frac{3}{4}}$ **d** $25^{\frac{3}{2}}$ **e** $32^{\frac{2}{5}}$

f $4^{-\frac{1}{2}}$ **g** $9^{-\frac{3}{2}}$ **h** $8^{-\frac{4}{3}}$ **i** $27^{-\frac{4}{3}}$ **j** $125^{-\frac{2}{3}}$

B ALGEBRAIC EXPANSION AND FACTORISATION

We can use the standard rules of algebra, together with the laws of exponents, to simplify expressions containing rational or variable exponents.

$$\begin{aligned} a(b+c) &= ab+ac \\ (a+b)(c+d) &= ac+ad+bc+bd \\ (a+b)(a-b) &= a^2-b^2 \\ (a+b)^2 &= a^2+2ab+b^2 \end{aligned}$$

Example 3 Self Tutor

Expand and simplify: $x^{-\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 3x^{-\frac{1}{2}})$

$x^{-\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 3x^{-\frac{1}{2}})$
$= x^{-\frac{1}{2}} \times x^{\frac{3}{2}} + x^{-\frac{1}{2}} \times 2x^{\frac{1}{2}} - x^{-\frac{1}{2}} \times 3x^{-\frac{1}{2}}$ {each term is multiplied by $x^{-\frac{1}{2}}$}
$= x^1 + 2x^0 - 3x^{-1}$ {adding indices}
$= x + 2 - \dfrac{3}{x}$

EXERCISE 2B

1 Simplify:

a $x^{\frac{1}{2}} \times x^{-\frac{1}{2}}$ **b** $x^{\frac{3}{2}} \times x^{-\frac{1}{2}}$ **c** $x^2 \times x^{-\frac{3}{2}}$

2 Expand and simplify:

a $x^2(x^3+2x^2+1)$
b $2^x(2^x+1)$
c $x^{\frac{1}{2}}(x^{\frac{1}{2}}+x^{-\frac{1}{2}})$
d $7^x(7^x+2)$
e $3^x(2-3^{-x})$
f $x^{\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}+3x^{-\frac{1}{2}})$
g $2^{-x}(2^x+5)$
h $5^{-x}(5^{2x}+5^x)$
i $x^{-\frac{1}{2}}(x^2+x+x^{\frac{1}{2}})$
j $3^x(3^x+5+3^{-x})$
k $x^{-\frac{1}{2}}(2x^2-x+5x^{\frac{1}{2}})$
l $2^{2x}(2^x-3-2^{-2x})$

Example 4 Self Tutor

Expand and simplify:

a $(2^x+3)(2^x+1)$
b $(7^x+7^{-x})^2$

a
$$\begin{aligned}&(2^x+3)(2^x+1)\\&=2^x\times 2^x+2^x+3\times 2^x+3\\&=2^{2x}+4\times 2^x+3\end{aligned}$$

b
$$\begin{aligned}&(7^x+7^{-x})^2\\&=(7^x)^2+2\times 7^x\times 7^{-x}+(7^{-x})^2\\&=7^{2x}+2\times 7^0+7^{-2x}\\&=7^{2x}+2+7^{-2x}\end{aligned}$$

3 Expand and simplify:

a $(2^x-1)(2^x+3)$
b $(3^x+2)(3^x+5)$
c $(5^x-2)(5^x-4)$
d $(2^x+3)^2$
e $(3^x-1)^2$
f $(4^x+7)^2$
g $(x^{\frac{1}{2}}+2)(x^{\frac{1}{2}}-2)$
h $(2^x+3)(2^x-3)$
i $(x^{\frac{1}{2}}+x^{-\frac{1}{2}})(x^{\frac{1}{2}}-x^{-\frac{1}{2}})$
j $\left(x+\frac{2}{x}\right)^2$
k $(7^x-7^{-x})^2$
l $(5-2^{-x})^2$

Example 5 Self Tutor

Factorise:

a $2^{n+3}+2^n$
b $2^{n+3}+8$
c $2^{3n}+2^{2n}$

a
$$\begin{aligned}&2^{n+3}+2^n\\&=2^n2^3+2^n\\&=2^n(2^3+1)\\&=2^n\times 9\end{aligned}$$

b
$$\begin{aligned}&2^{n+3}+8\\&=2^n2^3+8\\&=8(2^n)+8\\&=8(2^n+1)\end{aligned}$$

c
$$\begin{aligned}&2^{3n}+2^{2n}\\&=2^{2n}2^n+2^{2n}\\&=2^{2n}(2^n+1)\end{aligned}$$

4 Factorise:

a $5^{2x}+5^x$
b $3^{n+2}+3^n$
c 7^n+7^{3n}
d $5^{n+1}-5$
e $6^{n+2}-6$
f $4^{n+2}-16$
g $2^{2n}-2^{n+3}$
h $2^{n+1}+2^{n-1}$
i $4^{n+1}+2^{2n-1}$

Example 6 Self Tutor

Factorise:

a $4^x - 9$ **b** $9^x + 4(3^x) + 4$

a $4^x - 9$
$= (2^x)^2 - 3^2$ {compare $a^2 - b^2 = (a+b)(a-b)$}
$= (2^x + 3)(2^x - 3)$

b $9^x + 4(3^x) + 4$
$= (3^x)^2 + 4(3^x) + 4$ {compare $a^2 + 4a + 4$}
$= (3^x + 2)^2$ {as $a^2 + 4a + 4 = (a+2)^2$}

5 Factorise:

a $9^x - 4$ **b** $4^x - 25$ **c** $16 - 9^x$
d $25 - 4^x$ **e** $9^x - 4^x$ **f** $4^x + 6(2^x) + 9$
g $9^x + 10(3^x) + 25$ **h** $4^x - 14(2^x) + 49$ **i** $25^x - 4(5^x) + 4$

6 Factorise:

a $(2^x)^2 - 2^x - 2$ **b** $(3^x)^2 + 3^x - 6$ **c** $4^x - 7(2^x) + 12$
d $4^x + 9(2^x) + 18$ **e** $4^x - 2^x - 20$ **f** $9^x + 9(3^x) + 14$
g $9^x + 4(3^x) - 5$ **h** $25^x + 5^x - 2$ **i** $49^x - 7^{x+1} + 12$

Example 7 Self Tutor

Simplify:

a $\dfrac{6^n}{3^n}$ **b** $\dfrac{4^n}{6^n}$

a $\dfrac{6^n}{3^n}$ $= \dfrac{2^n \cancel{3^n}}{\cancel{3^n}_1}$ $= 2^n$ **or** $\dfrac{6^n}{3^n}$ $= \left(\frac{6}{3}\right)^n$ $= 2^n$

b $\dfrac{4^n}{6^n}$ $= \dfrac{\cancel{2^n} 2^n}{\cancel{2^n} 3^n}$ $= \dfrac{2^n}{3^n}$ **or** $\dfrac{4^n}{6^n}$ $= \left(\frac{4}{6}\right)^n$ $= \left(\frac{2}{3}\right)^n$

7 Simplify:

a $\dfrac{12^n}{6^n}$ **b** $\dfrac{20^a}{2^a}$ **c** $\dfrac{6^b}{2^b}$ **d** $\dfrac{4^n}{20^n}$ **e** $\dfrac{35^x}{7^x}$
f $\dfrac{6^a}{8^a}$ **g** $\dfrac{24^k}{9^k}$ **h** $\dfrac{5^{n+1}}{5^n}$ **i** $\dfrac{5^{n+1}}{5}$

Example 8 Self Tutor

Simplify:

a $\dfrac{3^n + 6^n}{3^n}$ **b** $\dfrac{2^{m+2} - 2^m}{2^m}$ **c** $\dfrac{2^{m+3} + 2^m}{9}$

a $\dfrac{3^n+6^n}{3^n}$
$= \dfrac{3^n+2^n3^n}{3^n}$
$= \dfrac{\cancel{3^n}(1+2^n)}{\cancel{3^n}_1}$
$= 1+2^n$

b $\dfrac{2^{m+2}-2^m}{2^m}$
$= \dfrac{2^m2^2-2^m}{2^m}$
$= \dfrac{\cancel{2^m}(4-1)}{\cancel{2^m}_1}$
$= 3$

c $\dfrac{2^{m+3}+2^m}{9}$
$= \dfrac{2^m2^3+2^m}{9}$
$= \dfrac{2^m\cancel{(8+1)}}{\cancel{9}_1}$
$= 2^m$

8 Simplify:

a $\dfrac{6^m+2^m}{2^m}$ **b** $\dfrac{2^n+12^n}{2^n}$ **c** $\dfrac{8^n+4^n}{2^n}$

d $\dfrac{12^x-3^x}{3^x}$ **e** $\dfrac{6^n+12^n}{1+2^n}$ **f** $\dfrac{5^{n+1}-5^n}{4}$

g $\dfrac{5^{n+1}-5^n}{5^n}$ **h** $\dfrac{4^n-2^n}{2^n}$ **i** $\dfrac{2^n-2^{n-1}}{2^n}$

9 Simplify:

a $2^n(n+1)+2^n(n-1)$ **b** $3^n\left(\dfrac{n-1}{6}\right)-3^n\left(\dfrac{n+1}{6}\right)$

C EXPONENTIAL EQUATIONS

An **exponential equation** is an equation in which the unknown occurs as part of the index or exponent.

For example: $2^x=8$ and $30\times 3^x=7$ are both exponential equations.

There are a number of methods we can use to solve exponential equations. These include graphing, using technology, and by using **logarithms**, which we will study in **Chapter 3**. However, in some cases we can solve the equation algebraically.

If both sides of an exponential equation are written as powers with the same base numbers, we can **equate indices**.

So, if $\boldsymbol{a^x=a^k}$ then $\boldsymbol{x=k}$.

For example, if $2^x=8$ then $2^x=2^3$. Thus $x=3$, and this is the only solution.

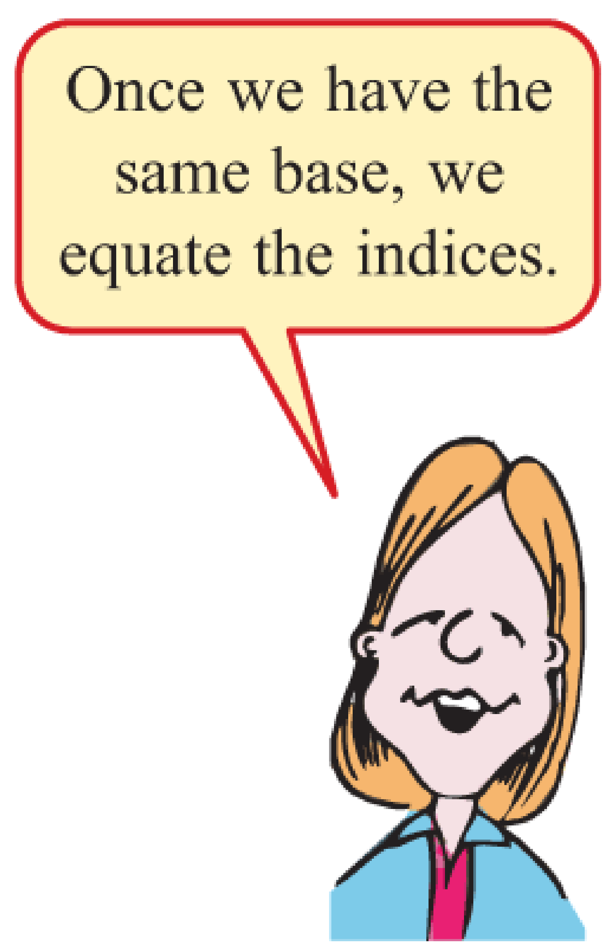
Example 9 Self Tutor

Solve for x:

a $2^x=16$ **b** $3^{x+2}=\frac{1}{27}$

a $2^x=16$
$\therefore\ 2^x=2^4$
$\therefore\ x=4$

b $3^{x+2}=\frac{1}{27}$
$\therefore\ 3^{x+2}=3^{-3}$
$\therefore\ x+2=-3$
$\therefore\ x=-5$



EXERCISE 2C

1 Solve for x:

a $2^x = 32$ **b** $5^x = 25$ **c** $3^x = 81$ **d** $7^x = 1$

e $3^x = \frac{1}{3}$ **f** $2^x = \sqrt{2}$ **g** $5^x = \frac{1}{125}$ **h** $4^{x+1} = 64$

i $2^{x-2} = \frac{1}{32}$ **j** $3^{x+1} = \frac{1}{27}$ **k** $7^{x+1} = 343$ **l** $5^{1-2x} = \frac{1}{\sqrt{5}}$

Example 10 **Self Tutor**

Solve for x:

a $4^x = 8$ **b** $9^{x-2} = \frac{1}{3}$

a
$$4^x = 8$$
$$\therefore\ (2^2)^x = 2^3$$
$$\therefore\ 2^{2x} = 2^3$$
$$\therefore\ 2x = 3$$
$$\therefore\ x = \tfrac{3}{2}$$

b
$$9^{x-2} = \tfrac{1}{3}$$
$$\therefore\ (3^2)^{x-2} = 3^{-1}$$
$$\therefore\ 3^{2(x-2)} = 3^{-1}$$
$$\therefore\ 2(x-2) = -1$$
$$\therefore\ 2x - 4 = -1$$
$$\therefore\ 2x = 3$$
$$\therefore\ x = \tfrac{3}{2}$$

2 Solve for x:

a $8^x = 32$ **b** $4^x = \frac{1}{8}$ **c** $9^x = \frac{1}{27}$ **d** $25^x = \frac{1}{5}$

e $27^x = \frac{1}{9}$ **f** $16^x = \frac{1}{32}$ **g** $4^{x+2} = 128$ **h** $25^{1-x} = \frac{1}{125}$

i $4^{4x-1} = \frac{1}{2}$ **j** $9^{x-3} = 27$ **k** $\left(\frac{1}{2}\right)^{x+1} = 8$ **l** $\left(\frac{1}{3}\right)^{x+2} = \sqrt{27}$

m $81^x = 27^{-x}$ **n** $\left(\frac{1}{4}\right)^{1-x} = 32$ **o** $\left(\frac{1}{7}\right)^x = \sqrt[3]{49}$ **p** $\left(\frac{1}{3}\right)^{x+1} = 243$

3 Solve for x, if possible:

a $4^{2x+1} = 8^{1-x}$ **b** $9^{2-x} = \left(\frac{1}{3}\right)^{2x+1}$ **c** $2^x \times 8^{1-x} = \frac{1}{4}$

d $3^{x+2} \times 9^x = 27$ **e** $\left(\frac{1}{2}\right)^{x-1} \times 8^x = 4^{-x}$ **f** $\left(\frac{1}{5}\right)^{x^2} \times 25^x = \frac{1}{125}$

4 Solve for x:

a $3 \times 2^x = 24$ **b** $7 \times 2^x = 28$ **c** $4 \times 3^{x+2} = 12$

d $12 \times 3^{-x} = \frac{4}{3}$ **e** $4 \times \left(\frac{1}{3}\right)^x = 36$ **f** $5 \times \left(\frac{1}{2}\right)^x = 20$

Example 11 **Self Tutor**

Solve for x: $4^x + 2^x - 20 = 0$

$$4^x + 2^x - 20 = 0$$
$$\therefore\ (2^x)^2 + 2^x - 20 = 0 \qquad \{\text{compare } a^2 + a - 20 = 0\}$$
$$\therefore\ (2^x - 4)(2^x + 5) = 0 \qquad \{a^2 + a - 20 = (a-4)(a+5)\}$$
$$\therefore\ 2^x = 4 \text{ or } 2^x = -5$$
$$\therefore\ 2^x = 2^2 \qquad \{2^x \text{ cannot be negative}\}$$
$$\therefore\ x = 2$$

5 Solve for x:

a $4^x - 6(2^x) + 8 = 0$ **b** $4^x - 2^x - 2 = 0$ **c** $9^x - 12(3^x) + 27 = 0$

d $9^x = 3^x + 6$ **e** $25^x - 23(5^x) - 50 = 0$ **f** $49^x + 1 = 2(7^x)$

g $3^x - 1 = 6(3^{-x})$ **h** $2(4^x) - 5(2^x) + 2 = 0$

i $4(9^x) - 35(3^x) = 9$ **j** $4^{x+1} + 2 = 9(2^x)$

k $3^{2x-1} = 3^x + 18$ **l** $4^x + 2^{x+\frac{1}{2}} = 4$

Check your answers using technology.

GRAPHICS CALCULATOR INSTRUCTIONS

6 Solve simultaneously: $4^x = 8^y$ and $9^y = \dfrac{243}{3^x}$.

D EXPONENTIAL FUNCTIONS

We have already seen how to evaluate a^n for any $n \in \mathbb{Q}$.

But how do we evaluate a^n when $n \in \mathbb{R}$, so n is real but not necessarily rational?

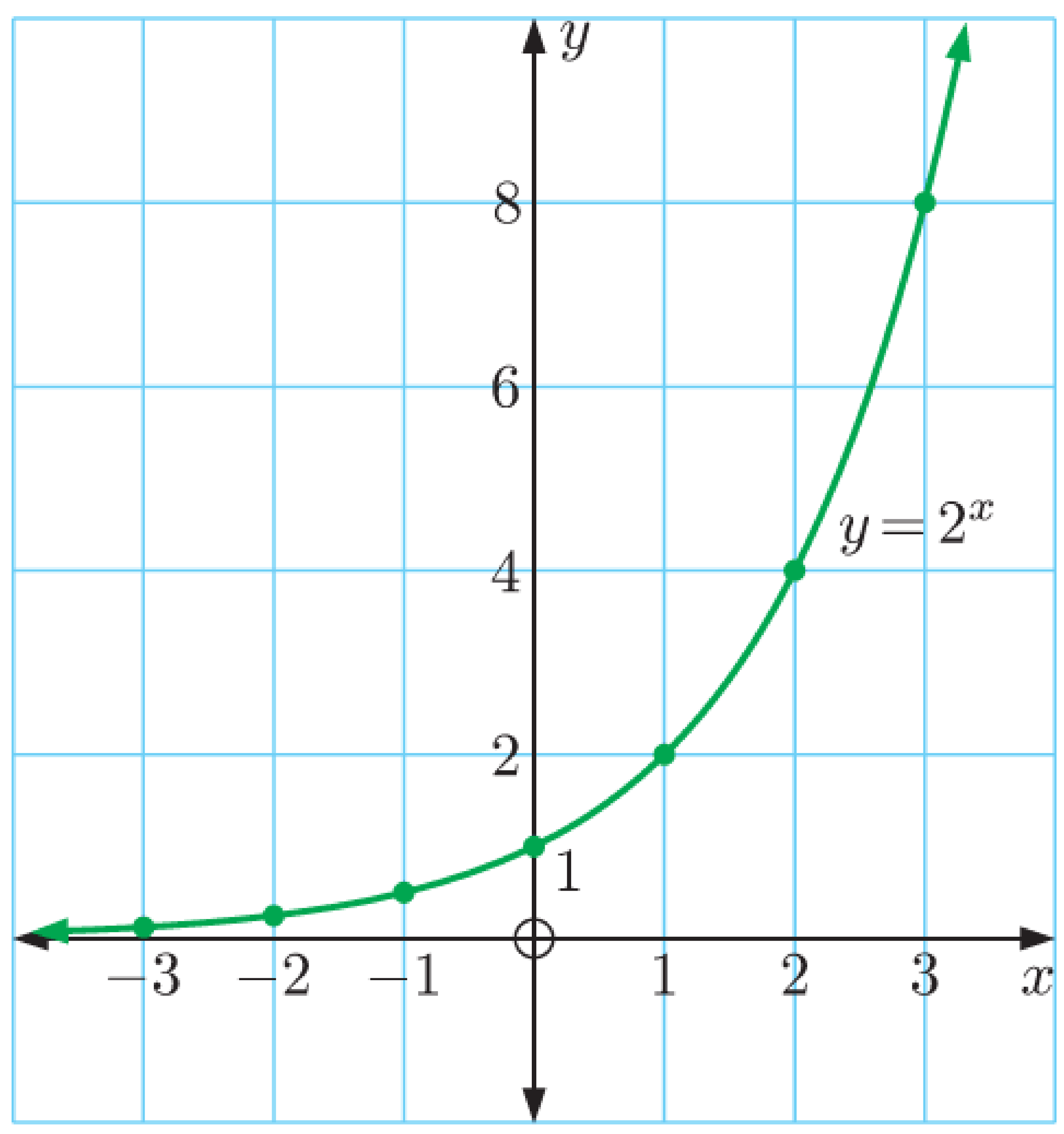
To answer this question, we can study the graphs of exponential functions.

> The most simple **exponential function** has the form $y = a^x$ where $a > 0$, $a \neq 1$.

For example, $y = 2^x$ is an exponential function.

We construct a table of values from which we graph the function:

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



As x becomes large and negative, the graph of $y = 2^x$ approaches the x-axis from above. However, it never touches the x-axis, since 2^x becomes very small but never zero.

So, as $x \to -\infty$, $y \to 0^+$.

$y = 0$ is therefore a **horizontal asymptote**.

Plotting $y = a^x$ for $x \in \mathbb{Q}$ suggests a smooth, continuous curve. This allows us to complete the curve for all $x \in \mathbb{R}$, giving meaning to a^x for irrational values of x.

INVESTIGATION 1 GRAPHS OF EXPONENTIAL FUNCTIONS

GRAPHING PACKAGE

In this Investigation we examine the graphs of various families of exponential functions. You can use the **graphing package** or your calculator.

What to do:

1 **a** State the transformation which maps $y = a^x$ to $y = a^x + k$.

b Predict the effect, if any, this transformation will have on:

i the shape of the graph **ii** the position of the graph

iii the horizontal asymptote.

c Check your predictions by graphing $y = 2^x$, $y = 2^x + 1$, and $y = 2^x - 2$ on the same set of axes.

2 **a** State the transformation which maps $y = a^x$ to $y = a^{x-h}$.

b Predict the effect, if any, this transformation will have on:

i the shape of the graph **ii** the position of the graph

iii the horizontal asymptote.

c Check your predictions by graphing $y = 2^x$, $y = 2^{x-1}$, $y = 2^{x+2}$, and $y = 2^{x-3}$ on the same set of axes.

3 **a** State the transformation which maps $y = a^x$ to $y = p \times a^x$, $p > 0$.

b Predict the effect, if any, this transformation will have on:

i the shape of the graph **ii** the position of the graph

iii the horizontal asymptote.

c Check your predictions by graphing $y = 2^x$, $y = 3 \times 2^x$, and $y = \frac{1}{2} \times 2^x$ on the same set of axes.

4 **a** State the transformation which maps $y = a^x$ to $y = -a^x$.

b Predict what the graph of $y = -2^x$ will look like, and check your answer using technology.

5 **a** State the transformation which maps $y = a^x$ to $y = a^{qx}$, $q > 0$.

b Predict the effect, if any, this transformation will have on:

i the shape of the graph **ii** the position of the graph

iii the horizontal asymptote.

c Notice that $2^{2x} = (2^2)^x = 4^x$ and $2^{3x} = (2^3)^x = 8^x$.

Check your predictions by graphing $y = 2^x$, $y = 4^x$, and $y = 8^x$ on the same set of axes.

6 **a** State the transformation which maps $y = a^x$ to $y = a^{-x}$.

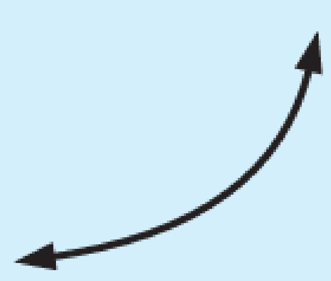
b Notice that $2^{-x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x$.

Predict what the graph of $y = \left(\frac{1}{2}\right)^x$ will look like, and check your answer using technology.

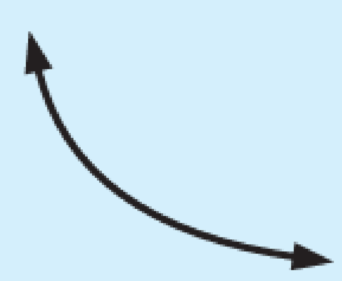
From your **Investigation** you should have discovered that:

For the general exponential function $\boldsymbol{y = p \times a^{x-h} + k}$ where $a > 0$, $a \neq 1$, $p \neq 0$:

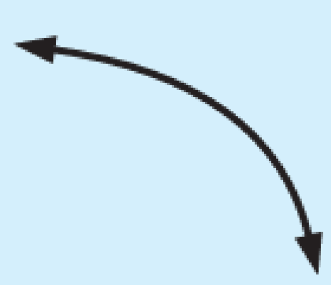
- a controls how steeply the graph increases or decreases.
- h controls horizontal translation.
- k controls vertical translation.
- The equation of the horizontal asymptote is $y = k$.
- If $p > 0$, $a > 1$ the function is increasing.



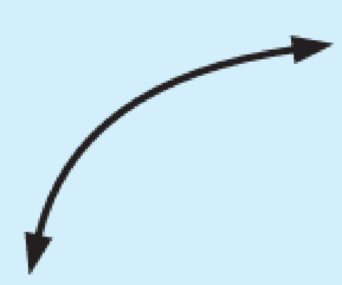
- If $p > 0$, $0 < a < 1$ the function is decreasing.



- If $p < 0$, $a > 1$ the function is decreasing.

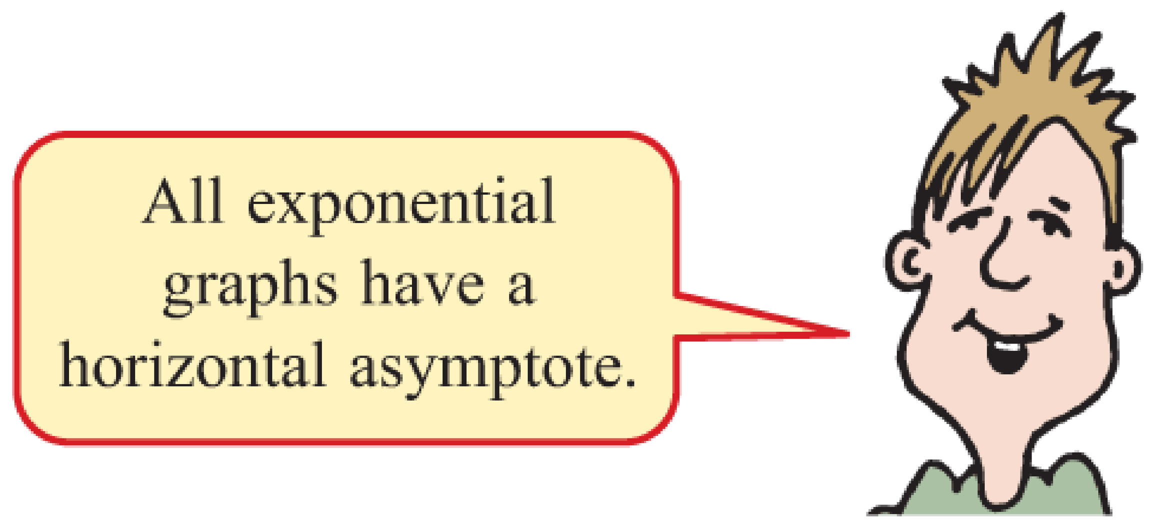


- If $p < 0$, $0 < a < 1$ the function is increasing.



We can sketch the graphs of exponential functions using:

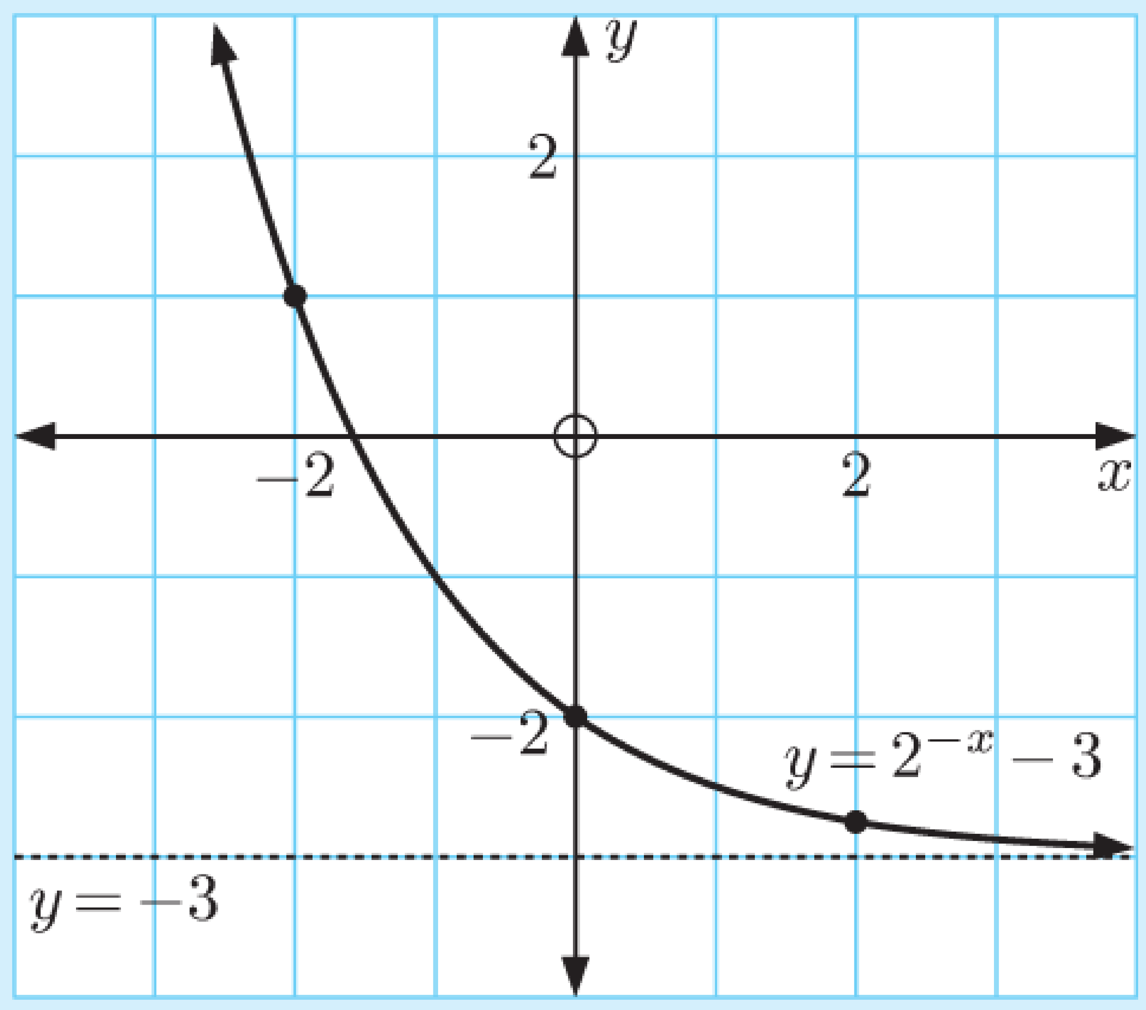
- the horizontal asymptote
- the y-intercept
- two other points.



Example 12 — Self Tutor

Sketch the graph of $y = 2^{-x} - 3$.
Hence state the domain and range of $f(x) = 2^{-x} - 3$.

For $y = 2^{-x} - 3$,
the horizontal asymptote is $y = -3$.



When $x = 0$, $y = 2^0 - 3$
$= 1 - 3$
$= -2$

$\therefore$ the y-intercept is -2.

When $x = 2$, $y = 2^{-2} - 3$
$= \frac{1}{4} - 3$
$= -2\frac{3}{4}$

When $x = -2$, $y = 2^2 - 3 = 1$

The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y > -3\}$.

EXERCISE 2D

1 Consider the graph of $y = 2^x$ alongside.

a Use the graph to estimate the value of:

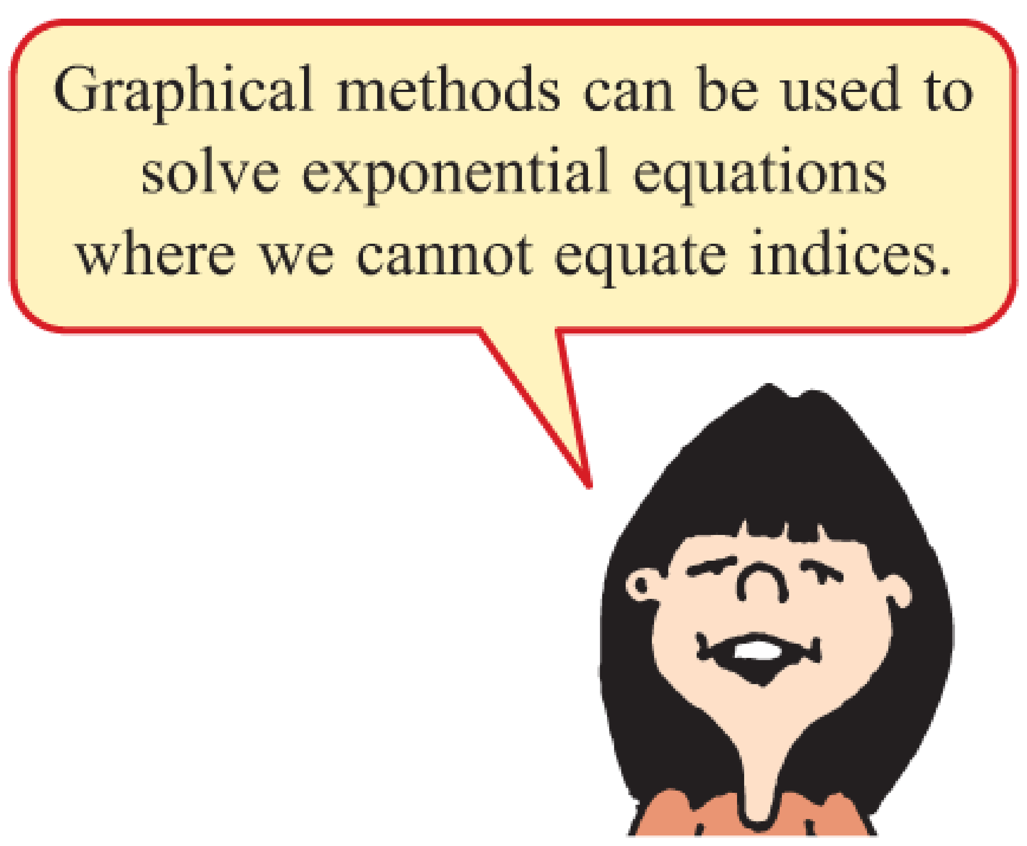
i $2^{\frac{1}{2}}$ or $\sqrt{2}$ **ii** $2^{0.8}$

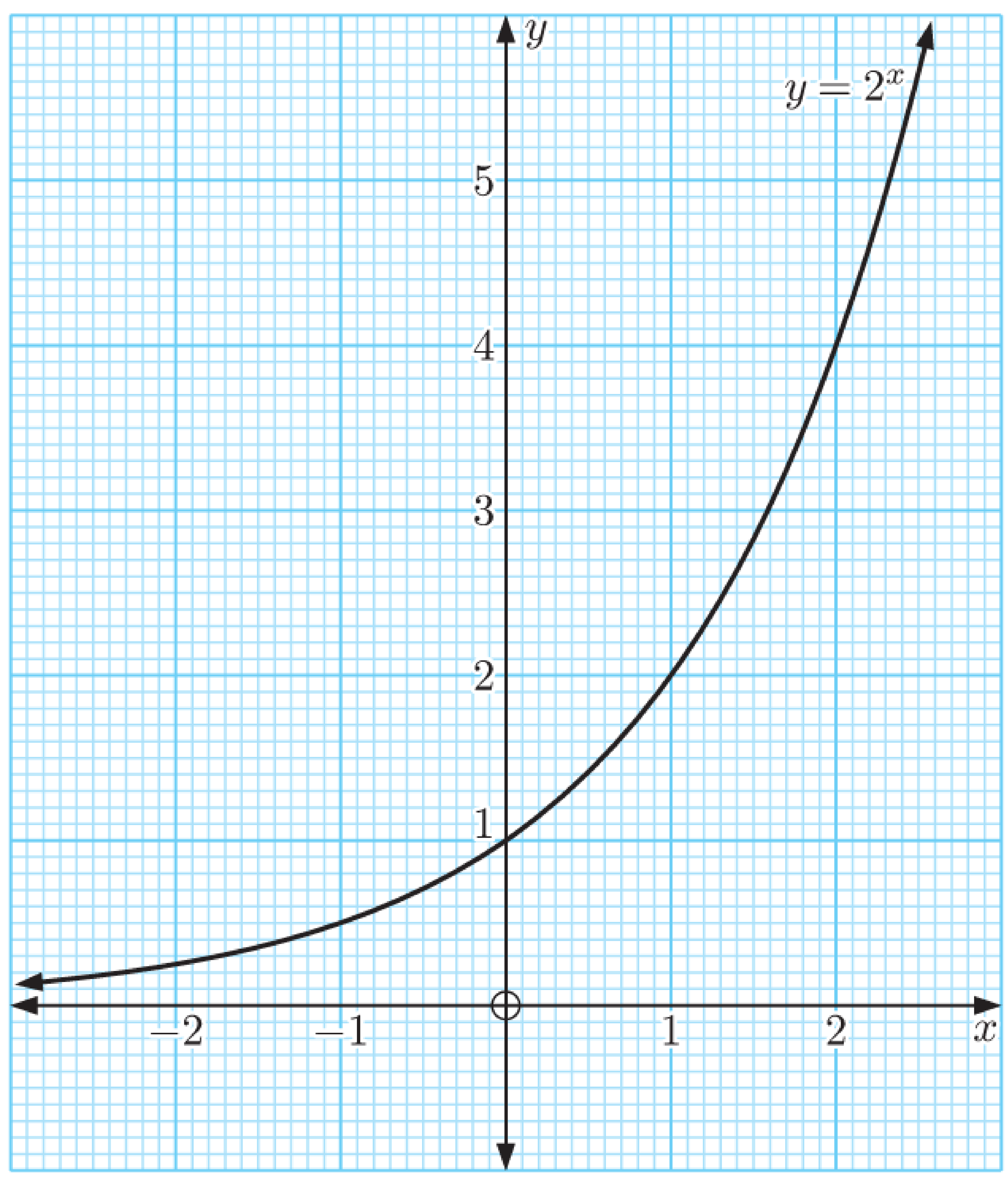
iii $2^{1.5}$ **iv** $2^{-\sqrt{2}}$

b Use the graph to estimate the solution to:

i $2^x = 3$ **ii** $2^x = 0.6$

c Use the graph to explain why $2^x = 0$ has no solutions.



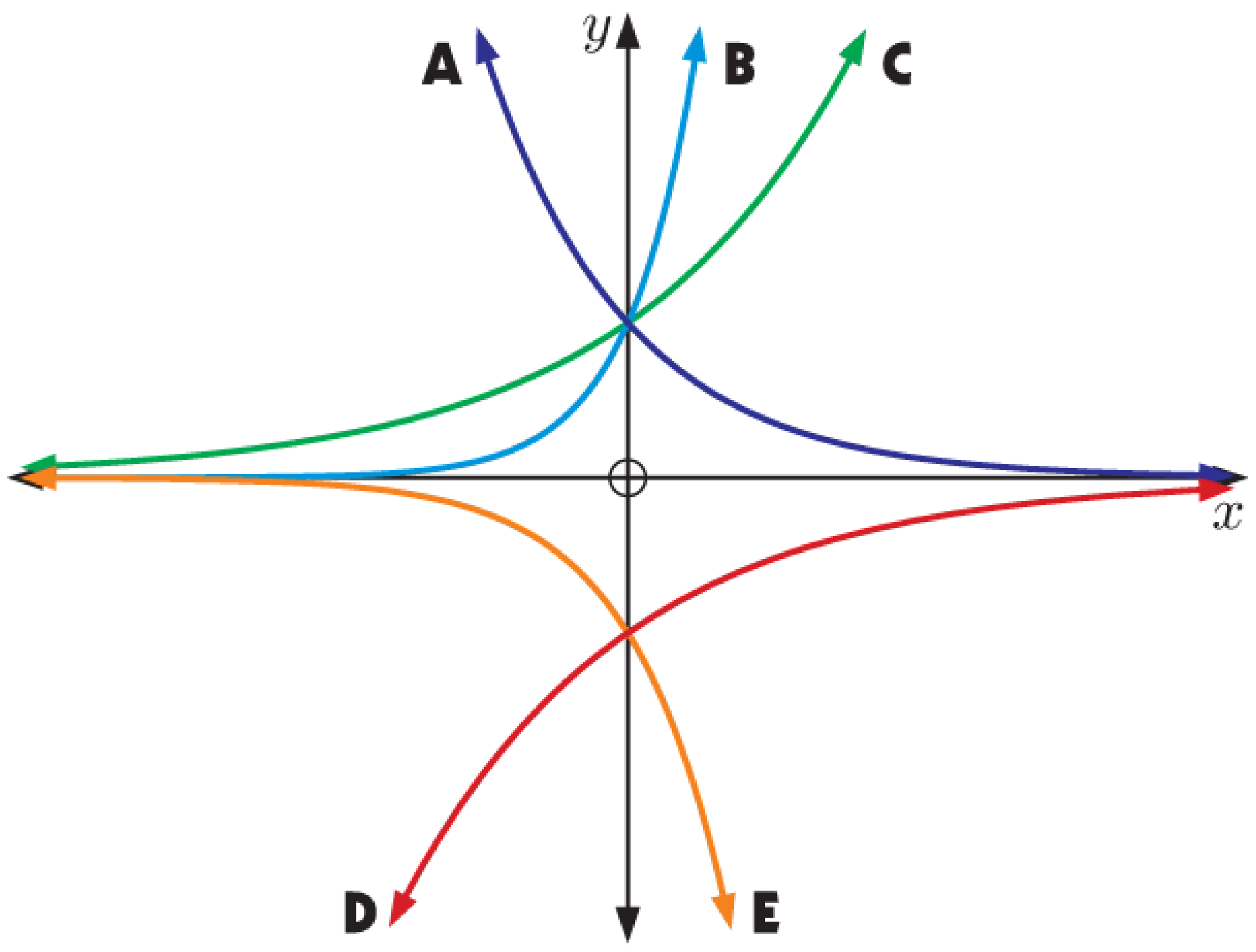


2 Match each function with its graph:

a $y = 2^x$ **b** $y = 10^x$

c $y = -5^x$ **d** $y = \left(\frac{1}{3}\right)^x$

e $y = -\left(\frac{1}{2}\right)^x$



3 Use a transformation to help sketch each pair of functions on the same set of axes:

a $y = 2^x$ and $y = 2^x - 2$ **b** $y = 2^x$ and $y = 2^{-x}$

c $y = 2^x$ and $y = 2^{x-2}$ **d** $y = 2^x$ and $y = 2(2^x)$

GRAPHING PACKAGE

4 Draw freehand sketches of the following pairs of graphs:

a $y = 3^x$ and $y = 3^{-x}$ **b** $y = 3^x$ and $y = 3^x + 1$

c $y = 3^x$ and $y = -3^x$ **d** $y = 3^x$ and $y = 3^{x-1}$

5 State the equation of the horizontal asymptote of:

a $y = 3^{-x}$ **b** $y = 2^x - 1$ **c** $y = 3 - 2^{-x}$

d $y = 4 \times 2^x + 2$ **e** $y = 5 \times 3^{x+2}$ **f** $y = -2 \times 3^{1-x} - 4$

6 Consider the exponential function $f(x) = 3^x - 2$.

a Find: **i** $f(0)$ **ii** $f(2)$ **iii** $f(-2)$

b State the equation of the horizontal asymptote.

c Sketch the graph of the function.

d State the domain and range of the function.

7 Consider the function $g(x) = 3 \times \left(\frac{1}{2}\right)^x + 4$.

a Find: **i** $g(0)$ **ii** $g(2)$ **iii** $g(-2)$

b State the equation of the horizontal asymptote.

c Sketch the graph of the function.

d State the domain and range of the function.

8 Consider the function $h(x) = -2^{x-3} + 1$.

a Find: **i** $h(0)$ **ii** $h(3)$ **iii** $h(6)$

b State the equation of the horizontal asymptote.

c Sketch the graph of the function.

d State the domain and range of the function.

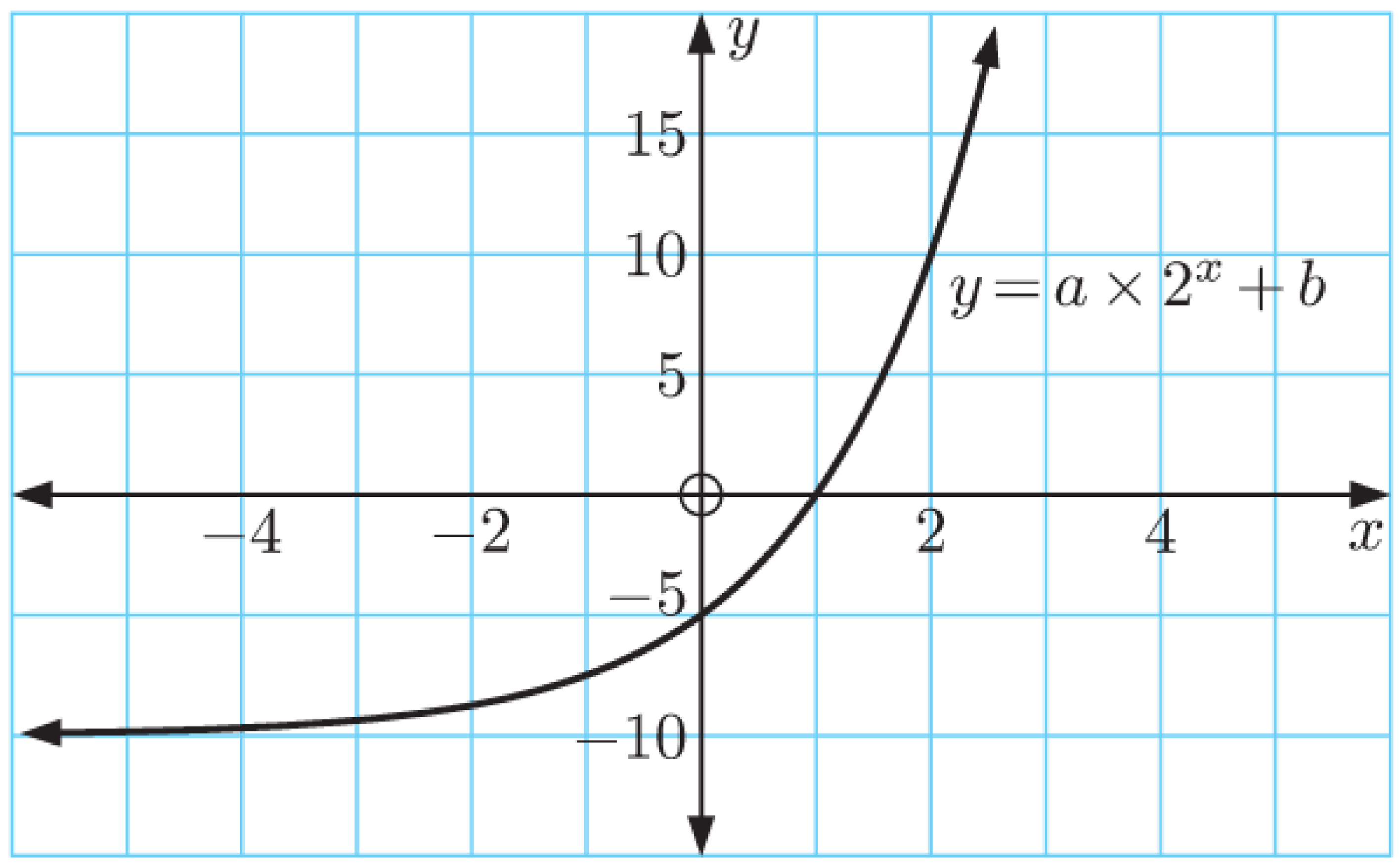
9 For each of the functions below:

i Sketch the graph of the function.
ii State the domain and range.
iii Use your calculator to find the value of y when $x = \sqrt{2}$.
iv Discuss the behaviour of y as $x \to \pm\infty$.
v Determine the horizontal asymptote.

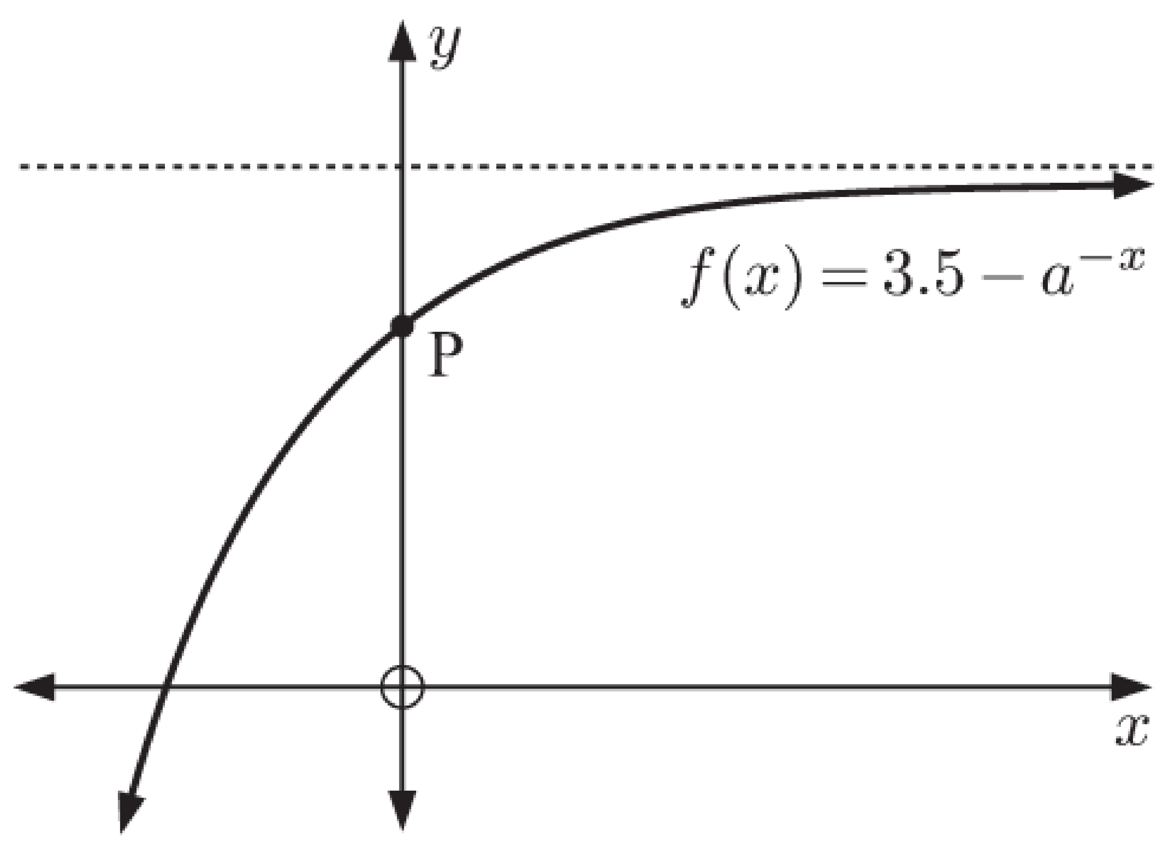
a $y = 2^x + 1$ **b** $y = 2 - 2^x$ **c** $y = 2^{-x} + 3$ **d** $y = 3 - 2^{-x}$

10 The graph alongside shows the curve $y = a \times 2^x + b$, where a and b are constants.

a Find the values of a and b.
b Find y when $x = 6$.



11



This graph shows the function $f(x) = 3.5 - a^{-x}$, where a is a positive constant.
The point $(-1, 2)$ lies on the graph.

a Write down the coordinates of P.
b Find the value of a.
c Find the equation of the horizontal asymptote.

12 Find the domain and range of:

a $y = 2^{x^2+1}$ **b** $y = \dfrac{1}{3^x - 1}$ **c** $y = \sqrt{5^x - 5}$

13 Let $f(x) = 3^x - 9$ and $g(x) = \sqrt{x}$.

a Find $(f \circ g)(x)$, and state its domain and range.
b Find $(g \circ f)(x)$, and state its domain and range.
c Solve:
i $(f \circ g)(x) = 0$ **ii** $(g \circ f)(x) = 3\sqrt{2}$

14 Suppose $f(x) = 2^x - 3$ and $g(x) = 1 + 2^{-x}$.

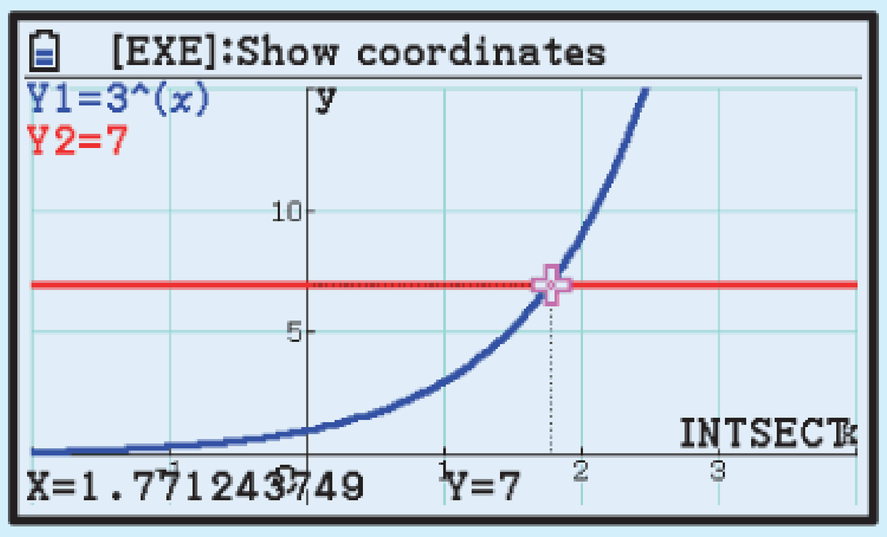
a For each function, find the:
i horizontal asymptote **ii** range **iii** y-intercept.
b Graph the functions on the same set of axes.
c Find the exact y-coordinate of the point where the graphs intersect.

Example 13 **Self Tutor**

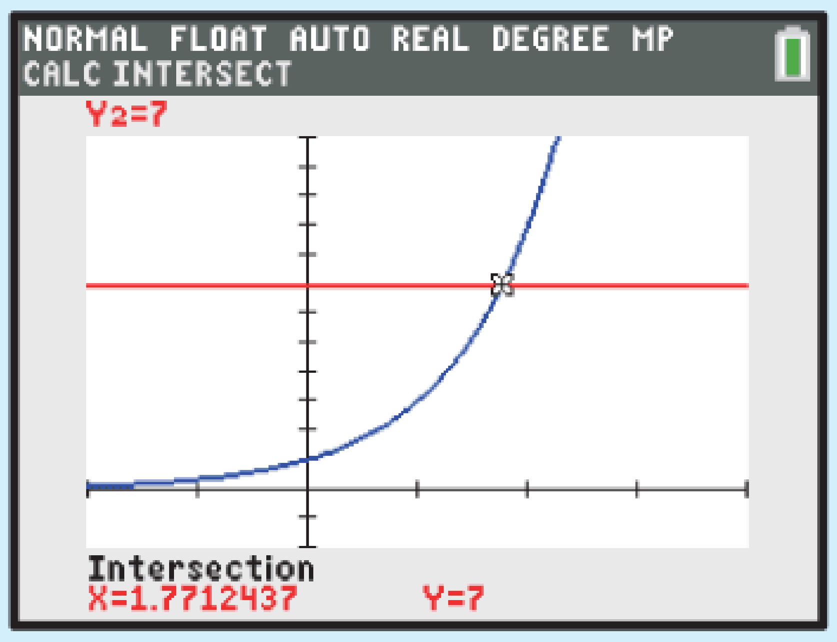
Use technology to solve the equation $3^x = 7$.

We graph $Y_1 = 3^X$ and $Y_2 = 7$ on the same set of axes, and find their point of intersection.

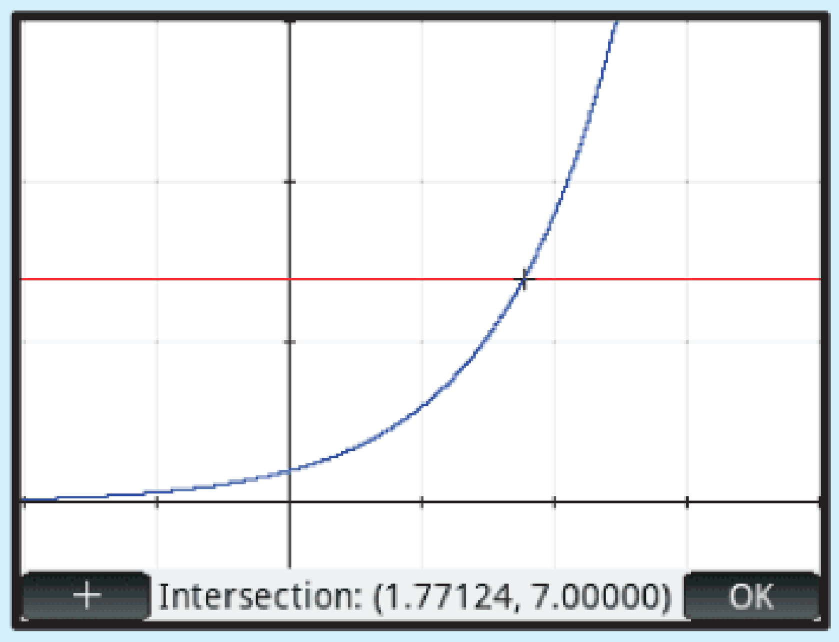
Casio fx-CG50



TI-84 Plus CE



HP Prime



The solution is $x \approx 1.77$.

15 Use technology to solve:

a $2^x = 11$ **b** $3^x = 15$ **c** $4^x + 5 = 10$

d $3^{x+2} = 4$ **e** $5 \times 2^x = 18$ **f** $3^{-x} = 0.9$

g $2 \times 3^{x-2} = 168$ **h** $26 \times (0.95)^x = 2$ **i** $2000 \times (1.03)^x = 5000$

DISCUSSION

For the exponential function $y = a^x$, why do we choose to specify $a > 0$?

What would the graph of $y = (-2)^x$ look like? What is its domain and range?

E GROWTH AND DECAY

In this Section we will examine situations where quantities are either increasing or decreasing exponentially. These situations are known as **growth** and **decay** modelling, and occur frequently in the world around us.

Populations of animals, people, and bacteria usually *grow* in an exponential way.

Radioactive substances, cooling, and items that depreciate in value, usually *decay* exponentially.

For the exponential function $y = p \times a^{x-h} + k$ where $a, p > 0$, $a \neq 1$, we see:
- growth if $a > 1$
- decay if $a < 1$.

GROWTH

Consider a population of 100 mice which under favourable conditions is increasing by 20% each week.

To increase a quantity by 20%, we multiply it by 1.2.

If P_n is the population after n weeks, then:

$P_0 = 100$ {the *original* population}

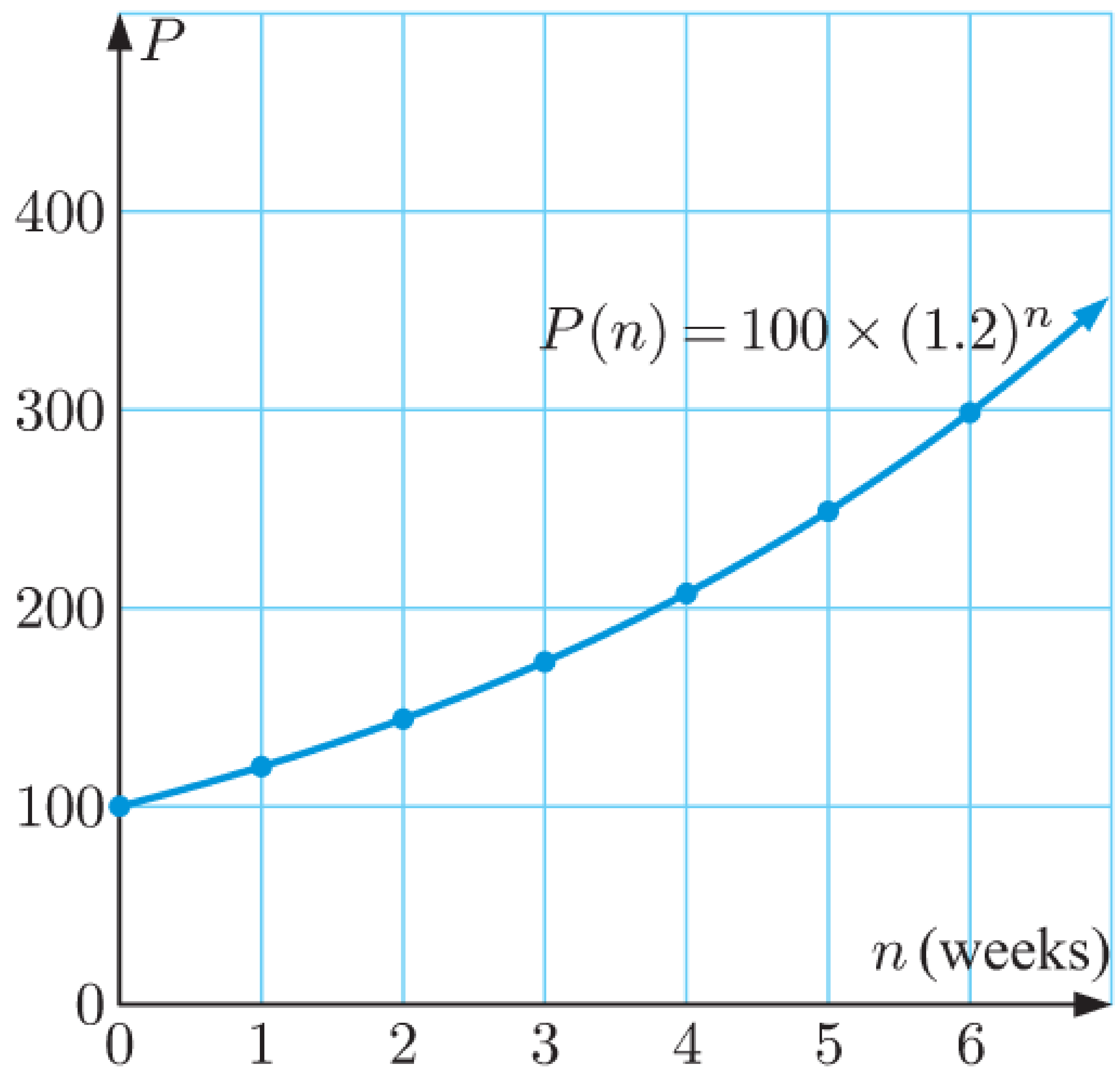
$P_1 = P_0 \times 1.2 = 100 \times 1.2$

$P_2 = P_1 \times 1.2 = 100 \times (1.2)^2$

$P_3 = P_2 \times 1.2 = 100 \times (1.2)^3$, and so on.

From this pattern we see that $P_n = 100 \times (1.2)^n$, $n \in \mathbb{Z}$, which is a geometric sequence.

However, while the population of mice must always be an integer, we expect that the population will grow continuously throughout the year, rather than in big, discrete jumps. We therefore expect it will be well approximated by the corresponding exponential function $P(n) = 100 \times (1.2)^n$, $n \in \mathbb{R}$.



Example 14 **Self Tutor**

A scientist is modelling a grasshopper plague. The area affected by the grasshoppers is given by $A(n) = 1000 \times (1.15)^n$ hectares, where n is the number of weeks after the initial observation.

a Find the original affected area.

b Find the affected area after: **i** 5 weeks **ii** 10 weeks.

c Draw the graph of the affected area over time.

d Use your graph or technology to find how long it will take for the affected area to reach 8000 hectares.

---

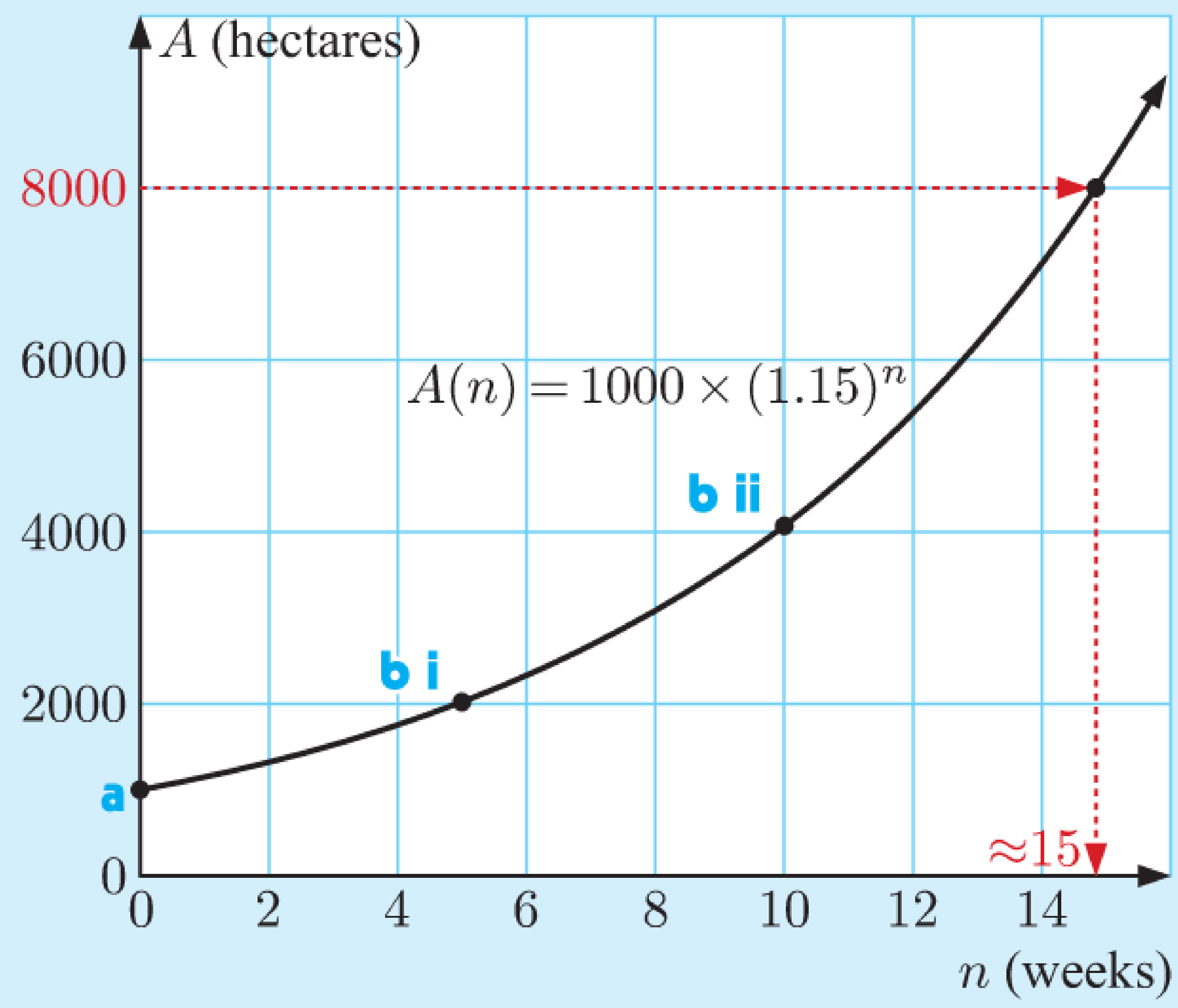
a $A(0) = 1000 \times 1.15^0 = 1000$

$\therefore$ the original affected area was 1000 hectares.

b **i** $A(5) = 1000 \times 1.15^5 \approx 2010$
The affected area is about 2010 hectares.

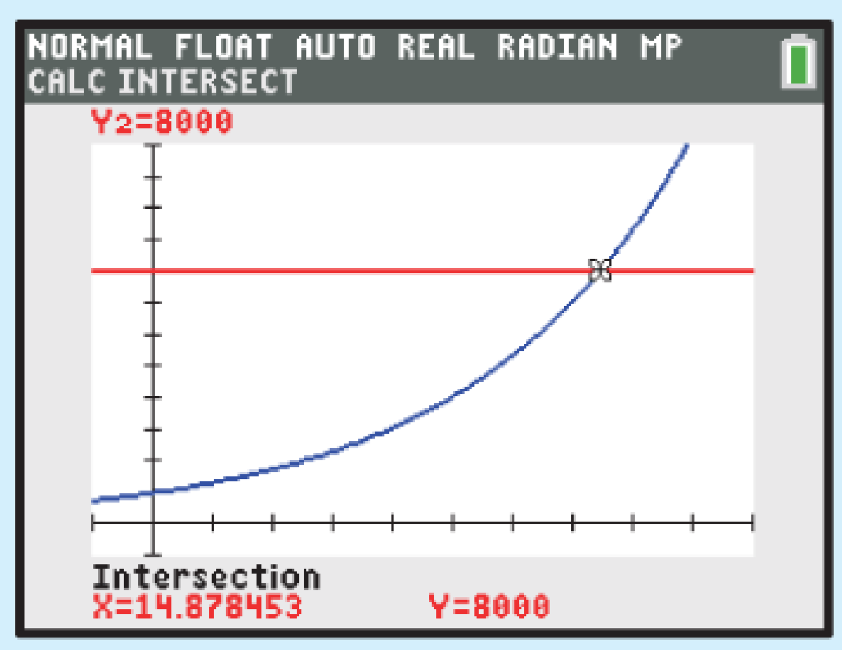
ii $A(10) = 1000 \times 1.15^{10} \approx 4050$
The affected area is about 4050 hectares.

c



d From the graph in **c**, it appears that it would take about 15 weeks for the affected area to reach 8000 hectares.

or Using technology, the solution is ≈ 14.9 weeks.



EXERCISE 2E.1

1 The weight W of bacteria in a culture t hours after establishment is given by $W(t) = 100 \times (1.07)^t$ grams.

- **a** Find the initial weight.
- **b** Find the weight after:
  i 4 hours **ii** 10 hours **iii** 24 hours.
- **c** Sketch the graph of the bacteria weight over time using the results of **a** and **b** only.

GRAPHING PACKAGE

Use technology to graph $\text{Y}_1 = 100 \times (1.07)^{\text{X}}$ and hence check your answers.

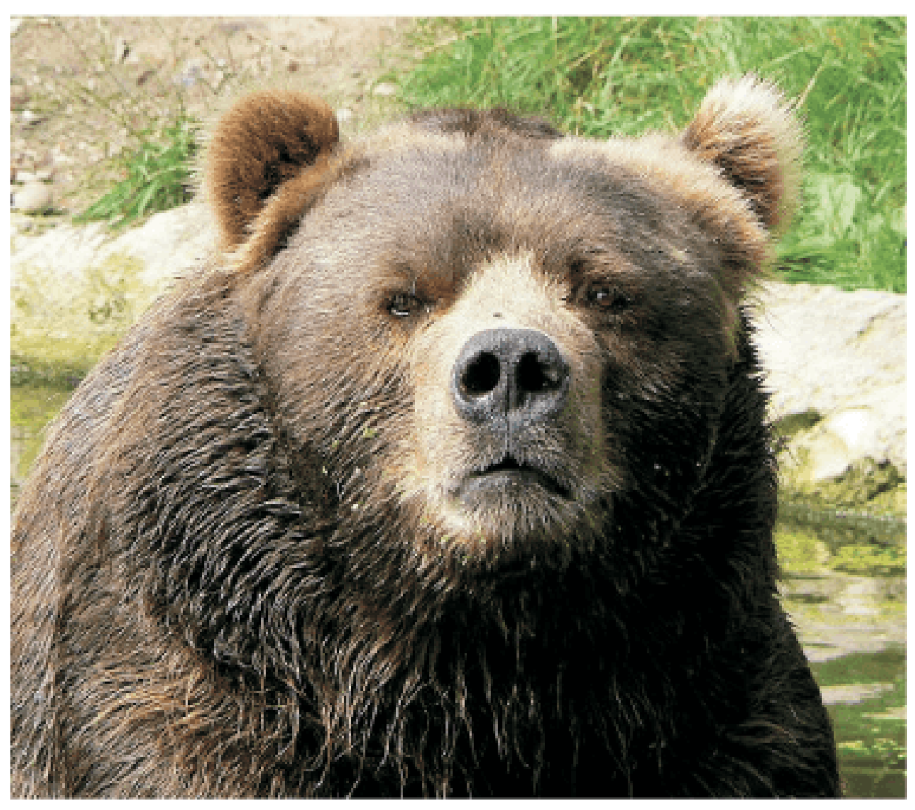
2 A breeding program to ensure the survival of pygmy possums is established with an initial population of 50 (25 pairs). From a previous program, the expected population P in n years' time is given by $P(n) = P_0 \times (1.23)^n$.

- **a** What is the value of P_0?
- **b** What is the expected population after:
  i 2 years **ii** 5 years **iii** 10 years?
- **c** Sketch the graph of the population over time using **a** and **b** only.
- **d** Hence estimate the time needed for the population to reach 500.
- **e** Use technology to graph $\text{Y}_1 = 50 \times (1.23)^{\text{X}}$. Hence check your answer to **d**.

3 A flu virus spreads in a school. The number of people N infected after t days is given by $N = 4 \times 1.332^t$, $t \geqslant 0$.

- **a** Find the number of people who were initially infected.
- **b** Calculate the number of people who were infected after 16 days.
- **c** There are 1200 people in the school. Estimate the time it will take for everybody in the school to catch the flu.

4 In 1998, 200 bears were introduced to a large island off Alaska where previously there were no bears. The population increased exponentially according to $B(t) = B_0 \times a^t$, where $a > 0$ is a constant and t is the time in years since the introduction.



- **a** Find B_0.
- **b** In 2000 there were 242 bears. Find a, and interpret your answer.
- **c** Find the expected bear population in 2018.
- **d** Find the expected percentage increase in population from 2008 to 2018.
- **e** How long will it take for the population to reach 2000?

5 The speed V of a chemical reaction is given by $V(t) = V_0 \times 2^{0.05t}$ where t is the temperature in °C.

- **a** Find the reaction speed at: **i** 0°C **ii** 20°C.
- **b** Find the percentage increase in reaction speed at 20°C compared with 0°C.
- **c** Find $\left(\dfrac{V(50) - V(20)}{V(20)}\right) \times 100\%$ and explain what this calculation means.

6 Kayla deposited £5000 into an account. The amount in the account increases by 10% each year.

a Write a formula for the amount $A(t)$ in the account after t years.

b Find the amount in the account after: **i** 2 years **ii** 5 years.

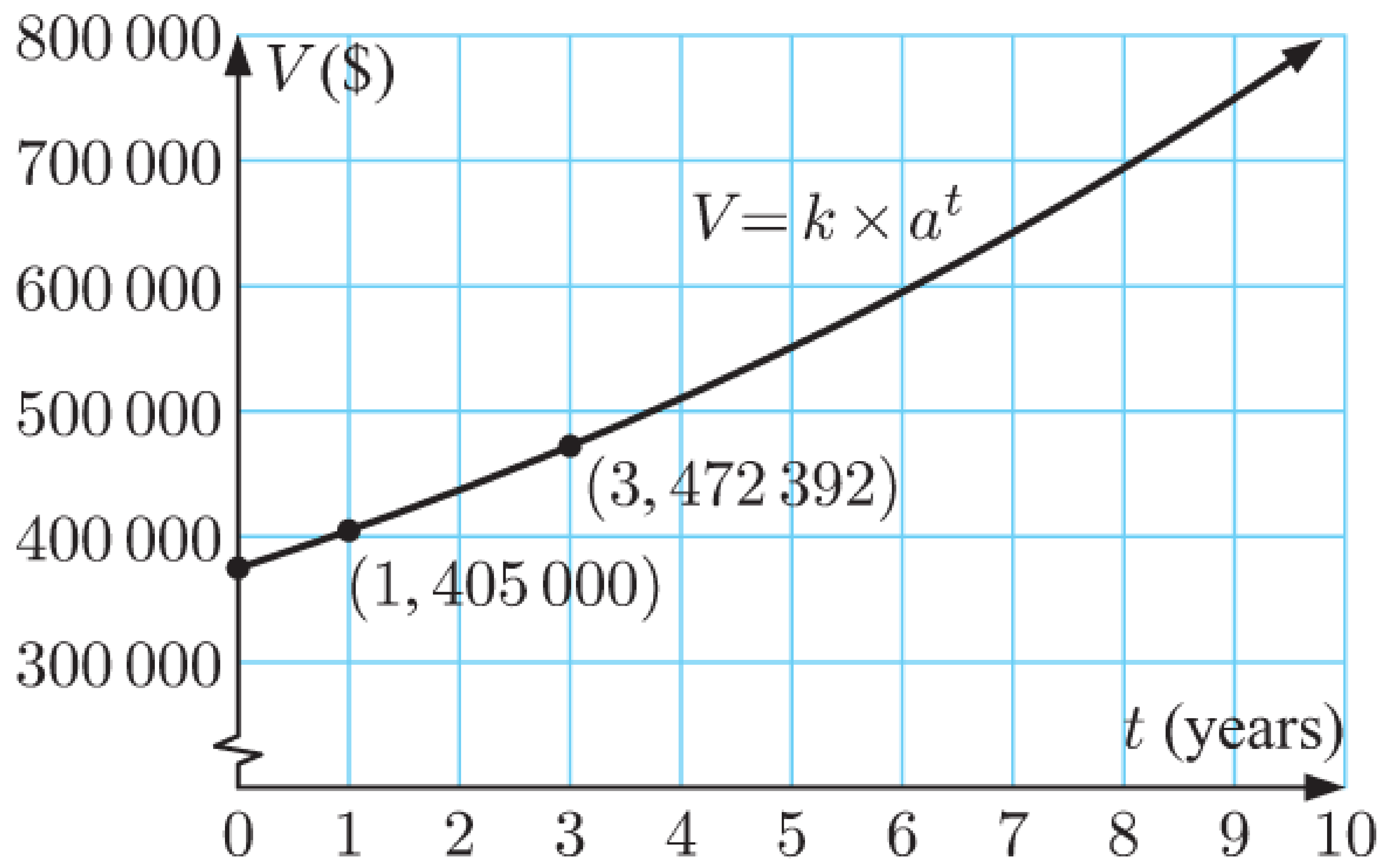
c Sketch the graph of $A(t)$.

d How long will it take for the amount in the account to reach £8000?

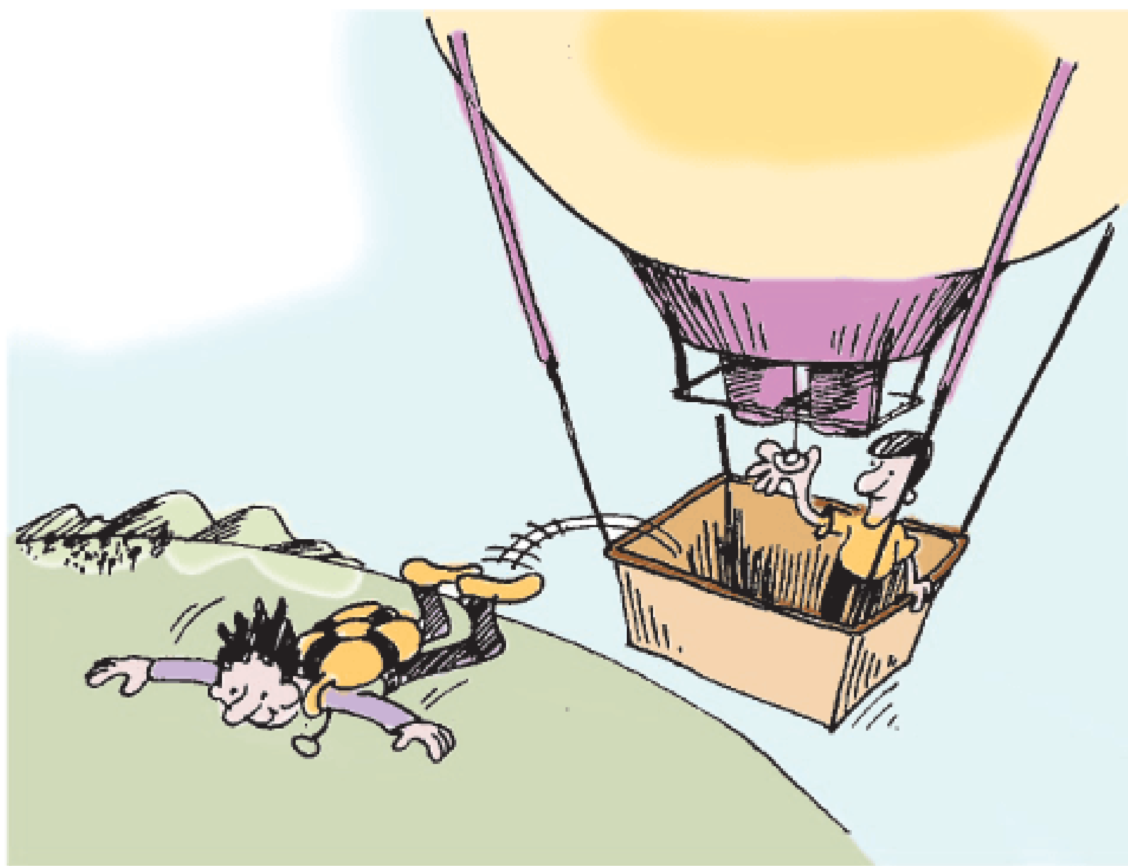
7 The expected value of a house in t years' time is given by the exponential function $V = k \times a^t$ dollars, where $t \geqslant 0$.
The function is graphed alongside.

a Find a and k, and interpret these values.

b How long will it take for the house's value to reach \$550 000?



8 A parachutist jumps from the basket of a stationary hot air balloon. His speed of descent is given by $V = c - 60 \times 2^{kt}$ m s^{-1} where c and k are constants, and t is the time in seconds.



a Explain why $c = 60$.

b After 5 seconds, the parachutist has speed 30 m s^{-1}. Find k.

c Find the speed of the parachutist after 12 seconds.

d Sketch the graph of V against t.

e Describe how the speed of the parachutist varies over time.

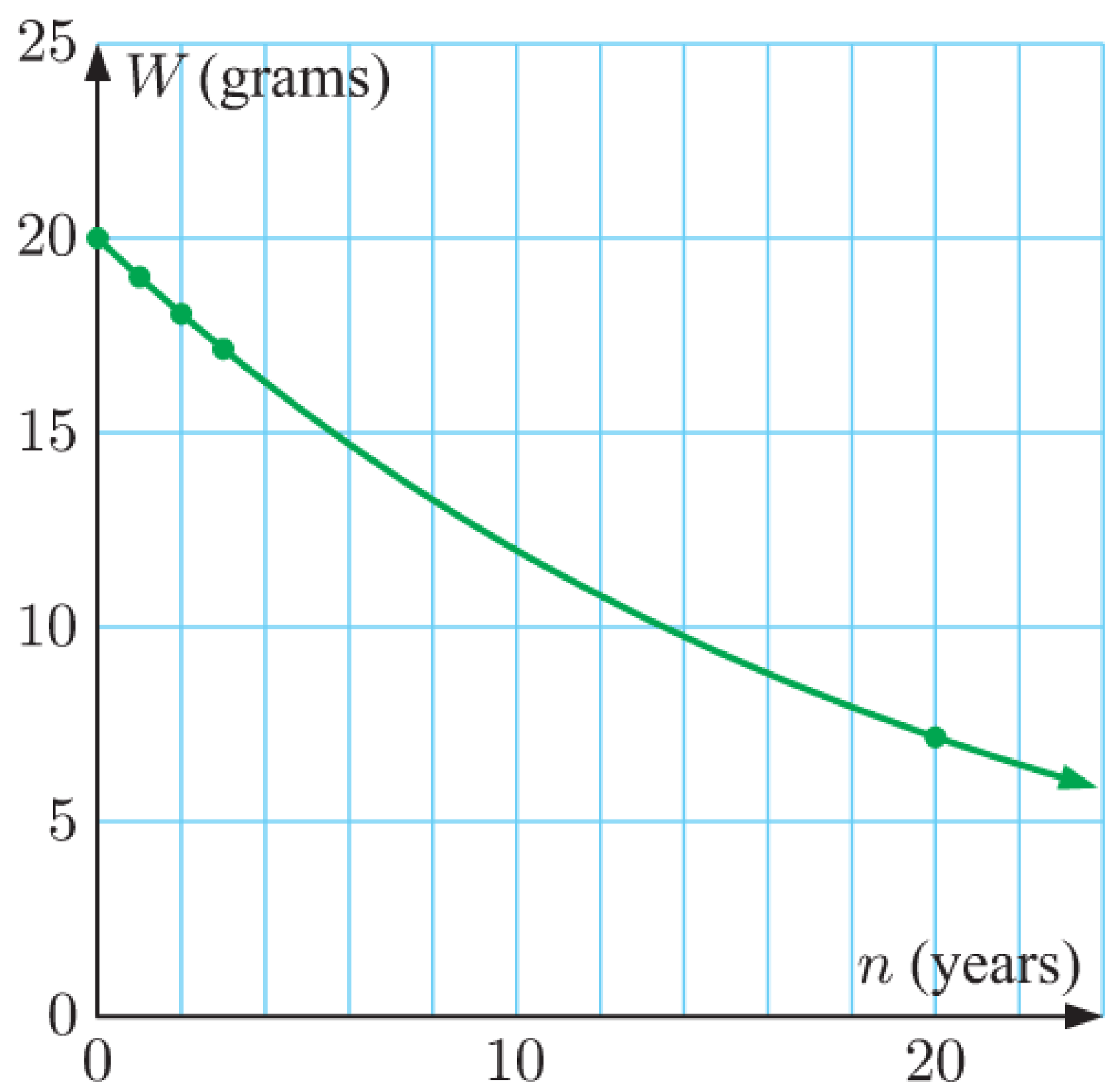
9 The number of microorganisms in a culture doubles every 6 hours. How long will it take for the number of microorganisms to increase by 30%?

DECAY

Consider a radioactive substance with original weight 20 grams. It *decays* or reduces by 5% each year. The multiplier for this is 95% or 0.95.

If W_n is the weight after n years, then:

$$W_0 = 20 \text{ grams}$$
$$W_1 = W_0 \times 0.95 = 20 \times 0.95 \text{ grams}$$
$$W_2 = W_1 \times 0.95 = 20 \times (0.95)^2 \text{ grams}$$
$$W_3 = W_2 \times 0.95 = 20 \times (0.95)^3 \text{ grams}$$
$$\vdots$$
$$W_{20} = 20 \times (0.95)^{20} \approx 7.2 \text{ grams}$$
$$\vdots$$



From this pattern we see that $W_n = 20 \times (0.95)^n$, $n \in \mathbb{Z}$, which is again a geometric sequence.

However, we know that radioactive decay is a continuous process, so the weight remaining will actually be given by the smooth exponential curve $W(n) = 20 \times (0.95)^n$, $n \in \mathbb{R}$.

Example 15 **Self Tutor**

When a diesel-electric generator is switched off, the current dies away according to the formula $I(t) = 24 \times (0.25)^t$ amps, where t is the time in seconds after the power is cut.

a Find $I(t)$ when $t = 0$, 1, 2, and 3.

b What current flowed in the generator at the instant it was switched off?

c Plot the graph of $I(t)$ for $t \geqslant 0$ using the information above.

d Use your graph or technology to find how long it takes for the current to reach 4 amps.

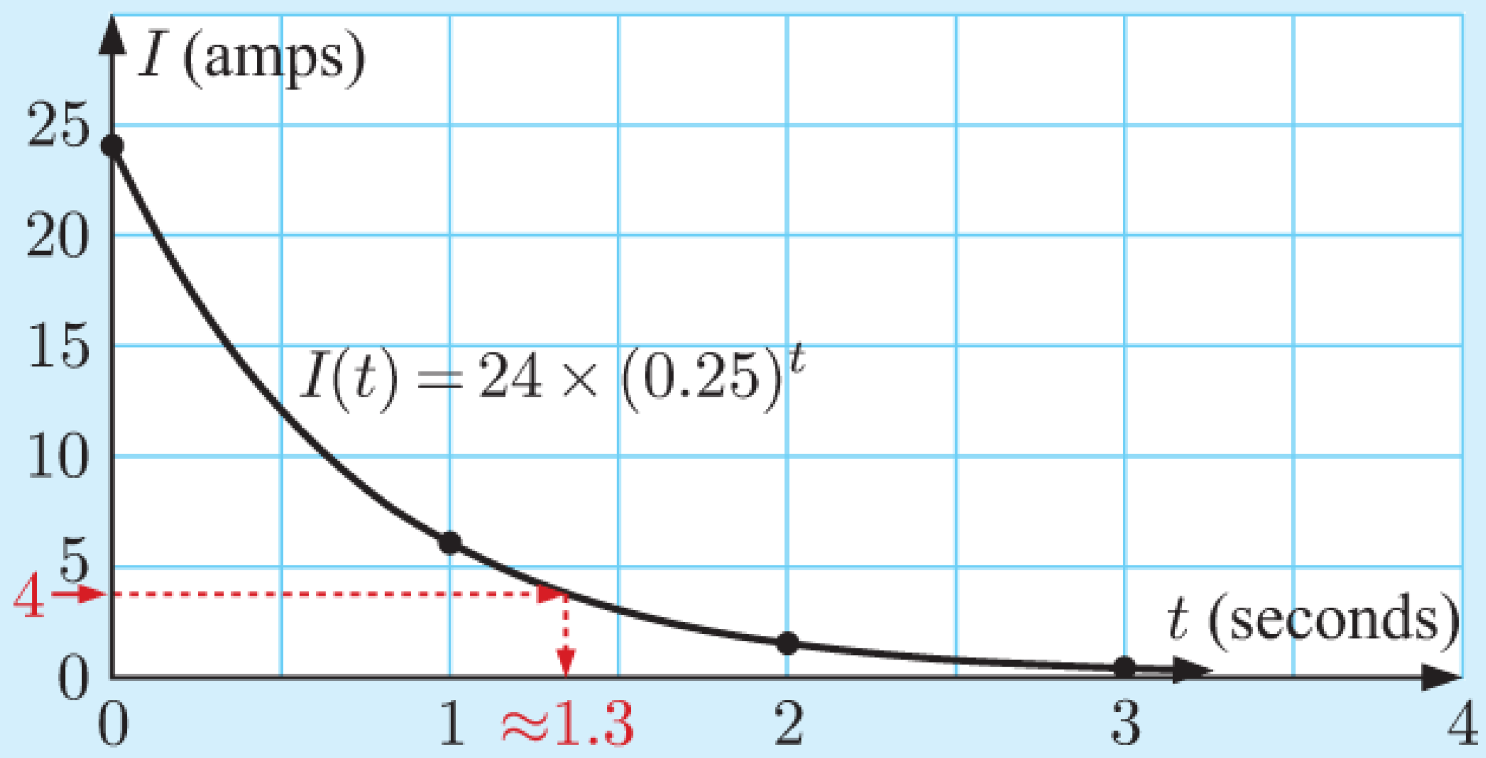
a $I(t) = 24 \times (0.25)^t$ amps

$I(0)$	$I(1)$	$I(2)$	$I(3)$
$= 24 \times (0.25)^0$	$= 24 \times (0.25)^1$	$= 24 \times (0.25)^2$	$= 24 \times (0.25)^3$
$= 24$ amps	$= 6$ amps	$= 1.5$ amps	$= 0.375$ amps

b $I(0) = 24$

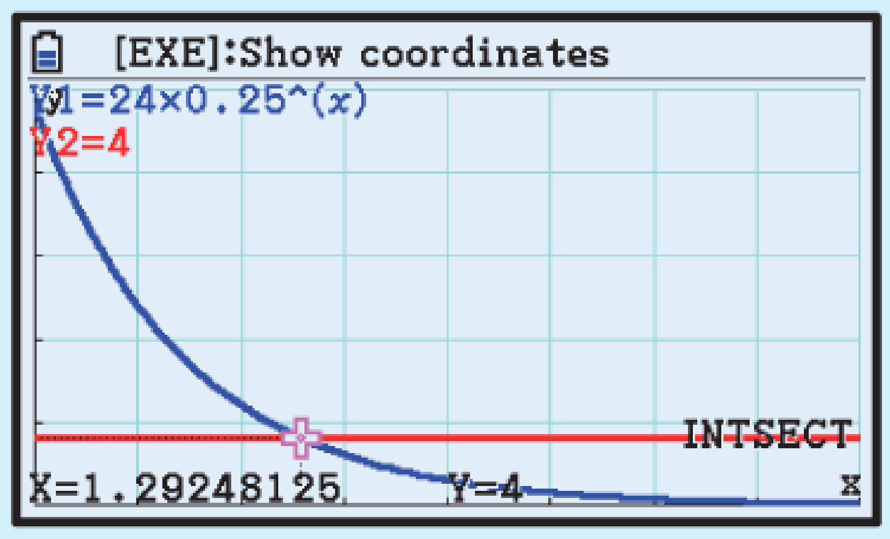
When the generator was switched off, 24 amps of current flowed in the circuit.

c



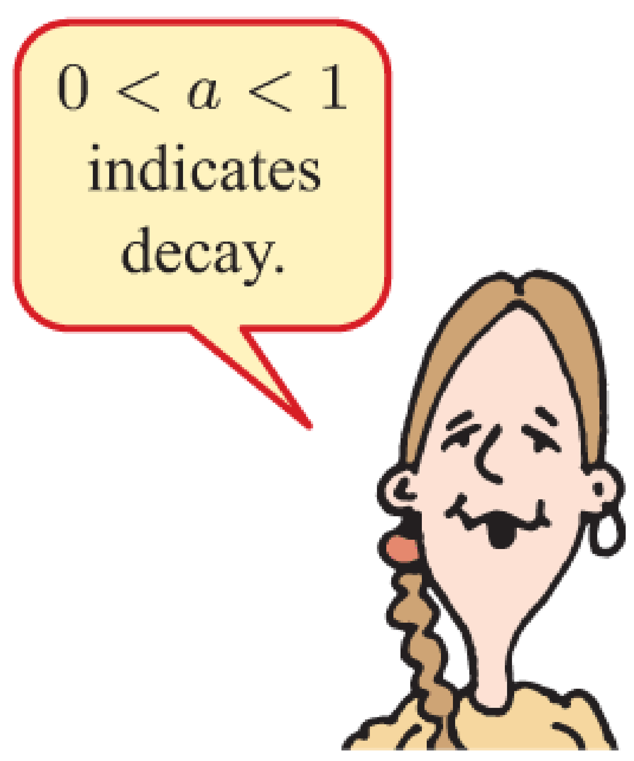
d From the graph above, the time to reach 4 amps is about 1.3 seconds.

or Using technology, the solution is ≈ 1.29 seconds.



EXERCISE 2E.2

1 The weight of a radioactive substance t years after being set aside is given by $W(t) = 250 \times (0.998)^t$ grams.



a How much radioactive substance was initially set aside?

b Determine the weight of the substance after:

i 400 years **ii** 800 years **iii** 1200 years.

c Sketch the graph of $W(t)$ for $t \geqslant 0$ using **a** and **b** only.

d Use your graph or graphics calculator to find how long it takes for the substance to decay to 125 grams.

2 The temperature T of a liquid which has been placed in a refrigerator is given by $T(t) = 100 \times (0.986)^t$ °C where t is the time in minutes.

a Find the initial temperature of the liquid.

b Find the temperature after:

i 15 minutes **ii** 20 minutes **iii** 78 minutes.

c Sketch the graph of $T(t)$ for $t \geqslant 0$ using **a** and **b** only.

3 The weight W of radioactive substance remaining after t years is given by $W(t) = 1000 \times 2^{-0.03t}$ grams.

- **a** Find the initial weight of the radioactive substance.
- **b** Find the weight remaining after:
  - **i** 10 years **ii** 100 years **iii** 1000 years.
- **c** Graph the weight remaining over time using **a** and **b** only.
- **d** Use your graph or graphics calculator to find the time when 10 grams of the substance remains.
- **e** Write an expression for the amount of substance that has decayed after t years.

4 An initial count of orangutans in a forest found that the forest contained 400 orangutans. Since then, the destruction of their habitat has caused the population to fall by 8% each year.

- **a** Write a formula for the population P of orangutans t years after the initial count.
- **b** Find the population of orangutans after:
  - **i** 1 year **ii** 5 years.
- **c** Sketch the graph of the population over time.
- **d** How long will it take for the population to fall to 200?

5 The intensity of light L diminishes below the surface of the sea according to the formula $L = L_0 \times (0.95)^d$ units, where d is the depth in metres measured from the surface of the sea.

- **a** If the intensity of light at the surface is 10 units, find the value of L_0.
- **b** Find the intensity of light 25 m below the surface.
- **c** A light intensity of 4 units is considered adequate for divers to be able to see clearly. Calculate the depth corresponding to this intensity of light.
- **d** Calculate the range of depths for which the light intensity is between 1 and 3 units.

6 The value of a car after t years is $V = 24\,000 \times r^t$ dollars, $t \geqslant 0$.

- **a** Write down the value of the car when it was first purchased.
- **b** The value of the car after 2 years was \$17 340. Find the value of r.
- **c** How long will it take for the value of the car to reduce to \$8000? Give your answer to the nearest year.

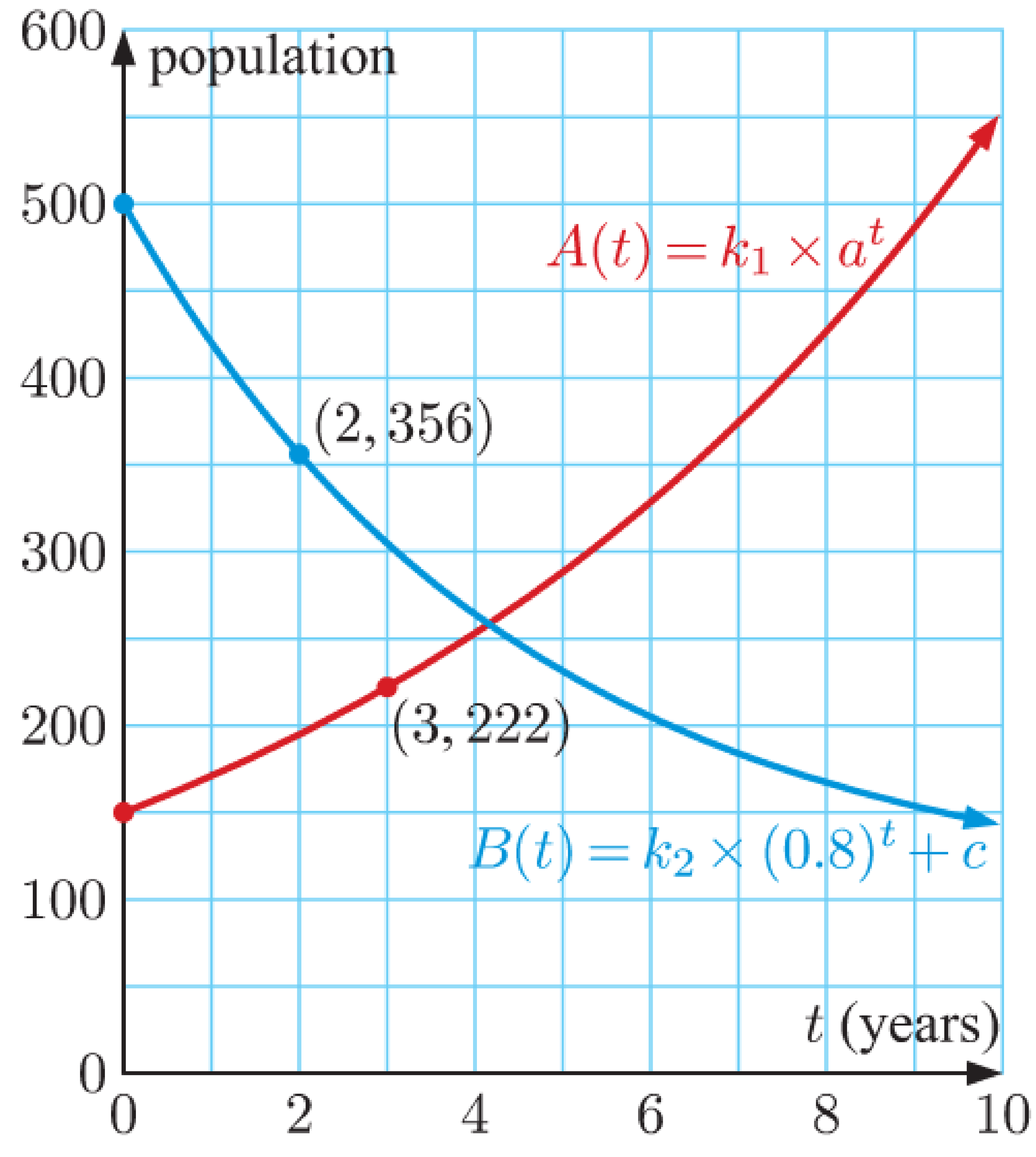
7 The interior of a freezer has temperature $-10°$C. When a packet of peas is placed in the freezer, its temperature after t minutes is given by $T(t) = -10 + 32 \times 2^{-0.2t}$ °C.

- **a** What was the temperature of the packet of peas:
  - **i** when placed in the freezer **ii** after 5 minutes **iii** after 10 minutes?
- **b** Sketch the graph of $T(t)$.
- **c** How long does it take for the temperature of the packet of peas to fall to 0°C?
- **d** Will the temperature of the packet of peas ever reach $-10°$C? Explain your answer.

8 The weight W_t of a radioactive uranium-235 sample remaining after t years is given by the formula $W_t = W_0 \times 2^{-0.0002t}$ grams, $t \geqslant 0$.

- **a** Find the original weight. **b** Find the percentage weight loss after 1000 years.
- **c** How long will it take until $\frac{1}{512}$ of the sample remains?

9 When scientists first observe a population of endangered marsupials, they notice two distinct groups. Group A is smaller in number, but appear to be larger and stronger individuals. Group B are more numerous, but smaller animals. The number of animals in each group over time are given by $A(t)$ and $B(t)$ respectively.



a Use the graph to find the exponential function for each animal group.

b Determine the time at which:

i there are the same number of group A and group B animals

ii there are 50 less group A than group B animals

iii there are twice as many group B animals compared to group A animals.

10 The **half-life** of a radioactive substance is the time it takes for the substance's weight to fall to half of its original value.
The radioactive isotope fermium-253 has a half-life of 3 days. The weight of fermium-253 detected t days after an explosion is $W(t) = 10 \times a^t$ mg.

a Interpret the value 10 in this model.

b Calculate the value of a, correct to 4 decimal places, and interpret this value.

c Find the weight of fermium-253 after 2 days.

d How long will it take for the weight of fermium-253 to fall to:

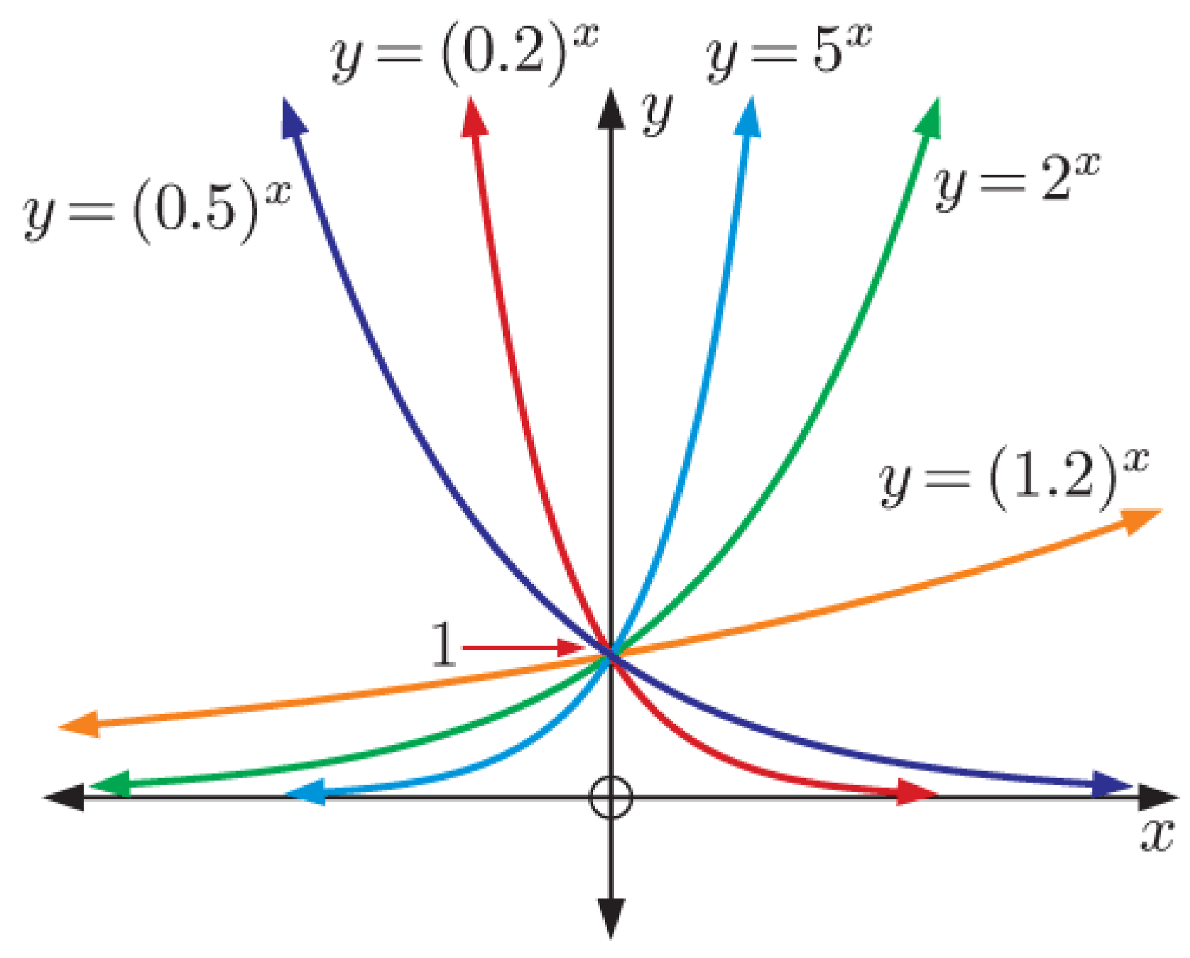
i 3 mg **ii** 1.25 mg?

11 The half-life of nitrogen-13 is 10 minutes. How long will it take for the mass of nitrogen-13 to fall to 10% of its original value?

F THE NATURAL EXPONENTIAL

We have seen that the simplest exponential functions have the form $f(x) = a^x$ where $a > 0$, $a \neq 1$.

Graphs of some of these functions are shown alongside.



We can see that for all positive values of the base a, the graph is always positive.

Hence $a^x > 0$ for all $a > 0$.

There are an infinite number of possible choices for the base number.

However, where exponential data is examined in science, engineering, and finance, the base $e \approx 2.7183$ is commonly used.

e is a special number in mathematics. It is irrational like π, and just as π is the ratio of a circle's circumference to its diameter, e also has a physical meaning. We explore this meaning in the following **Investigation**.

INVESTIGATION 2 CONTINUOUS COMPOUND INTEREST

A discrete formula for calculating the amount to which an investment grows under compound interest is $u_n = u_0(1+i)^n$ where:

u_n is the final amount, u_0 is the initial amount,
i is the interest rate per compounding period,
n is the number of periods, or times the interest is compounded.

We will investigate the final value of an investment for various values of n, and allow n to become extremely large.

What to do:

1 Suppose \$1000 is invested for one year at a fixed rate of 6% per annum. Use your calculator to find the final amount or *maturing value* if the interest is paid:

a annually $(n = 1,\ i = 6\% = 0.06)$ **b** quarterly $(n = 4,\ i = \frac{6\%}{4} = 0.015)$

c monthly **d** daily **e** by the second **f** by the millisecond.

Comment on your answers.

2 If r is the percentage rate per year, t is the number of years, and N is the number of interest payments per year, then $i = \frac{r}{N}$ and $n = Nt$.

If we let $a = \frac{N}{r}$, show that the growth formula becomes $u_n = u_0\left[\left(1+\frac{1}{a}\right)^a\right]^{rt}$.

3 For continuous compound growth, the number of interest payments per year N gets very large.

a Explain why a gets very large as N gets very large.

b Copy and complete the table, giving your answers as accurately as technology permits.

a	$\left(1+\frac{1}{a}\right)^a$
10	
100	
1000	
10 000	
100 000	
1 000 000	
10 000 000	

4 You should have found that for very large values of a, $\left(1+\frac{1}{a}\right)^a \approx 2.718\,281\,828\,459....$

Use the e^x key of your calculator to find the value of e^1. What do you notice?

5 For continuous growth, $u_n = u_0e^{rt}$ where u_0 is the initial amount, r is the annual percentage rate, and t is the number of years.

Use this formula to find the final amount if \$1000 is invested for 4 years at a fixed rate of 6% per annum, where the interest is paid continuously.

From **Investigation 2** we observe that:

If interest is paid *continuously* or *instantaneously* then the formula for calculating a compounding amount $u_n = u_0(1+i)^n$ can be replaced by $u_n = u_0e^{rt}$, where r is the percentage rate per annum and t is the number of years.

HISTORICAL NOTE

The natural exponential e was first described in 1683 by Swiss mathematician **Jacob Bernoulli**. He discovered the number while studying compound interest, just as we did in **Investigation 2**.

The natural exponential was first called e by Swiss mathematician and physicist **Leonhard Euler** in a letter to the German mathematician **Christian Goldbach** in 1731. The number was then published with this notation in 1736.

In 1748, Euler evaluated e correct to 18 decimal places.

Leonhard Euler

Euler also discovered some patterns in **continued fraction** expansions of e. He wrote that

$$\frac{e-1}{2} = \cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\ldots}}}}} \quad \text{and} \quad e-1 = 1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}}}}}$$

One may think that e was chosen because it was the first letter of Euler's name or for the word exponential, but it is likely that it was just the next vowel available since he had already used a in his work.

EXERCISE 2F

GRAPHING PACKAGE

1 Sketch, on the same set of axes, the graphs of $y = 2^x$, $y = e^x$, and $y = 3^x$. Comment on any observations.

2 Sketch, on the same set of axes, the graphs of $y = e^x$ and $y = e^{-x}$. What is the geometric connection between these two graphs?

3 For the general exponential function $y = pe^{qx}$, what is the y-intercept?

4 Consider $y = 2e^x$.

a Explain why y can never be negative.

b Find y if: **i** $x = -20$ **ii** $x = 20$.

5 Find, to 3 significant figures, the value of:

a e^2 **b** e^3 **c** $e^{0.7}$ **d** $\sqrt{e}$ **e** e^{-1}

6 Write the following as powers of e:

a $\sqrt{e}$ **b** $\frac{1}{\sqrt{e}}$ **c** $\frac{1}{e^2}$ **d** $e\sqrt{e}$

7 Evaluate, to five significant figures:

a $e^{2.31}$ **b** $e^{-2.31}$ **c** $e^{4.829}$ **d** $e^{-4.829}$

e $50e^{-0.1764}$ **f** $80e^{-0.6342}$ **g** $1000e^{1.2642}$ **h** $0.25e^{-3.6742}$

8 Expand and simplify:

a $(e^x+1)^2$ b $(1+e^x)(1-e^x)$ c $e^x(e^{-x}-3)$

9 Factorise:

a $e^{2x}+e^x$ b $e^{2x}-16$ c $e^{2x}-8e^x+12$

10 a On the same set of axes, sketch and clearly label the graphs of:

$f: x \mapsto e^x$, $g: x \mapsto e^{x-2}$, $h: x \mapsto e^x+3$

b State the domain and range of each function.

11 a On the same set of axes, sketch and clearly label the graphs of:

$f: x \mapsto e^x$, $g: x \mapsto -e^x$, $h: x \mapsto 10-e^x$

b State the domain and range of each function.

c Describe the behaviour of each function as $x \to \pm\infty$.

12 Let $f(x)=e^x-1$ and $g(x)=\frac{1}{x}$.

a Find $(f \circ g)(x)$, and state its domain and range.

b Find $(g \circ f)(x)$, and state its domain and range.

13 The weight of bacteria in a culture is given by $W(t)=2e^{\frac{t}{2}}$ grams where t is the time in hours after the culture was set to grow.

a Find the weight of the culture:

i initially ii after 30 minutes iii after $1\frac{1}{2}$ hours iv after 6 hours.

b Hence sketch the graph of $W(t)=2e^{\frac{t}{2}}$.

14 Solve for x:

a $e^x=\sqrt{e}$ b $e^{\frac{1}{2}x}=\frac{1}{e^2}$ c $e^{2x}+e^x=2$

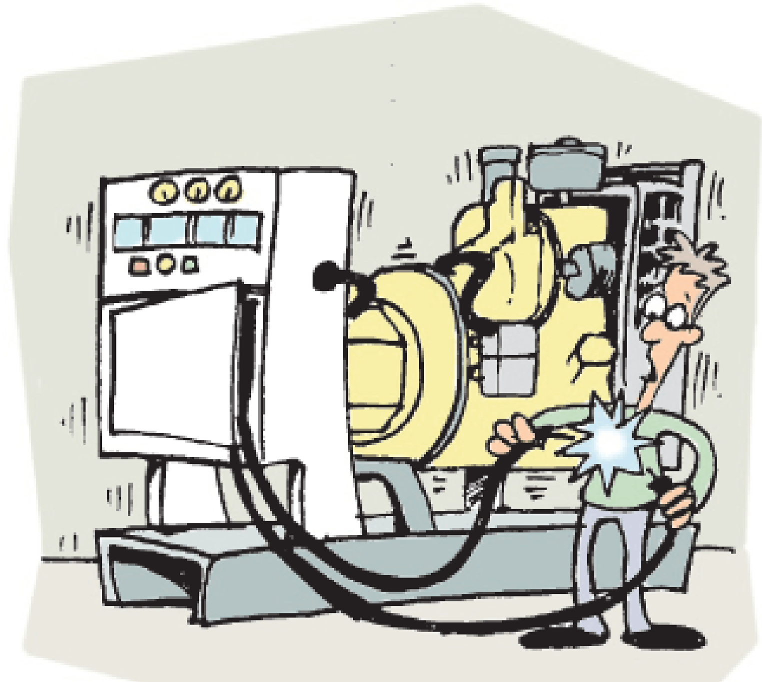
15 The current flowing in an electrical circuit t seconds after it is switched off is given by $I(t)=75e^{-0.15t}$ amps.

a What current is still flowing in the circuit after:

i 1 second ii 10 seconds?

b Use your graphics calculator to help sketch the graph of $I(t)=75e^{-0.15t}$.

c How long will it take for the current to fall to 1 amp?



16 The population P of trout in a lake is given by $P(t)=\frac{800}{1+ke^{-0.5t}}$, where t is the time in months.

a Given that there were initially 20 trout in the lake, find the value of k.

b Find the population after 6 months.

c Use technology to sketch the graph of $P(t)$.

d Describe what happens to the population as t increases.

e How long will it take for the population to reach 600?

17 Consider the function $f(x)=e^x$.

a On the same set of axes, sketch $y=f(x)$, $y=x$, and $y=f^{-1}(x)$.

b State the domain and range of f^{-1}.

18 It can be shown that $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{2\times3}x^3 + \frac{1}{2\times3\times4}x^4 + ... = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$ which is an infinite polynomial expansion.

Check this statement by using the first 20 terms of the series to find an approximation for e^1.

ACTIVITY

Click on the icon to run a card game for exponential functions.

CARD GAME

REVIEW SET 2A

1 Evaluate:

a $8^{\frac{2}{3}}$ **b** $27^{-\frac{2}{3}}$ **c** $81^{-\frac{1}{4}}$

2 Solve for x:

a $2^{x-3} = \frac{1}{32}$ **b** $9^x = 27^{2-2x}$ **c** $e^{2x} = \frac{1}{\sqrt{e}}$

3 Expand and simplify:

a $e^x(e^{-x} + e^x)$ **b** $(2^x + 5)^2$ **c** $(x^{\frac{1}{2}} - 7)(x^{\frac{1}{2}} + 7)$

4 Solve for x:

a $6 \times 2^x = 192$ **b** $9^{x-1} \times \left(\frac{1}{27}\right)^x = \sqrt{3}$ **c** $4^x - 32 = 4(2^x)$

5 The point $(1, \sqrt{8})$ lies on the graph of $y = 2^{kx}$. Find the value of k.

6 Consider the graph of $y = 3^x$ alongside.

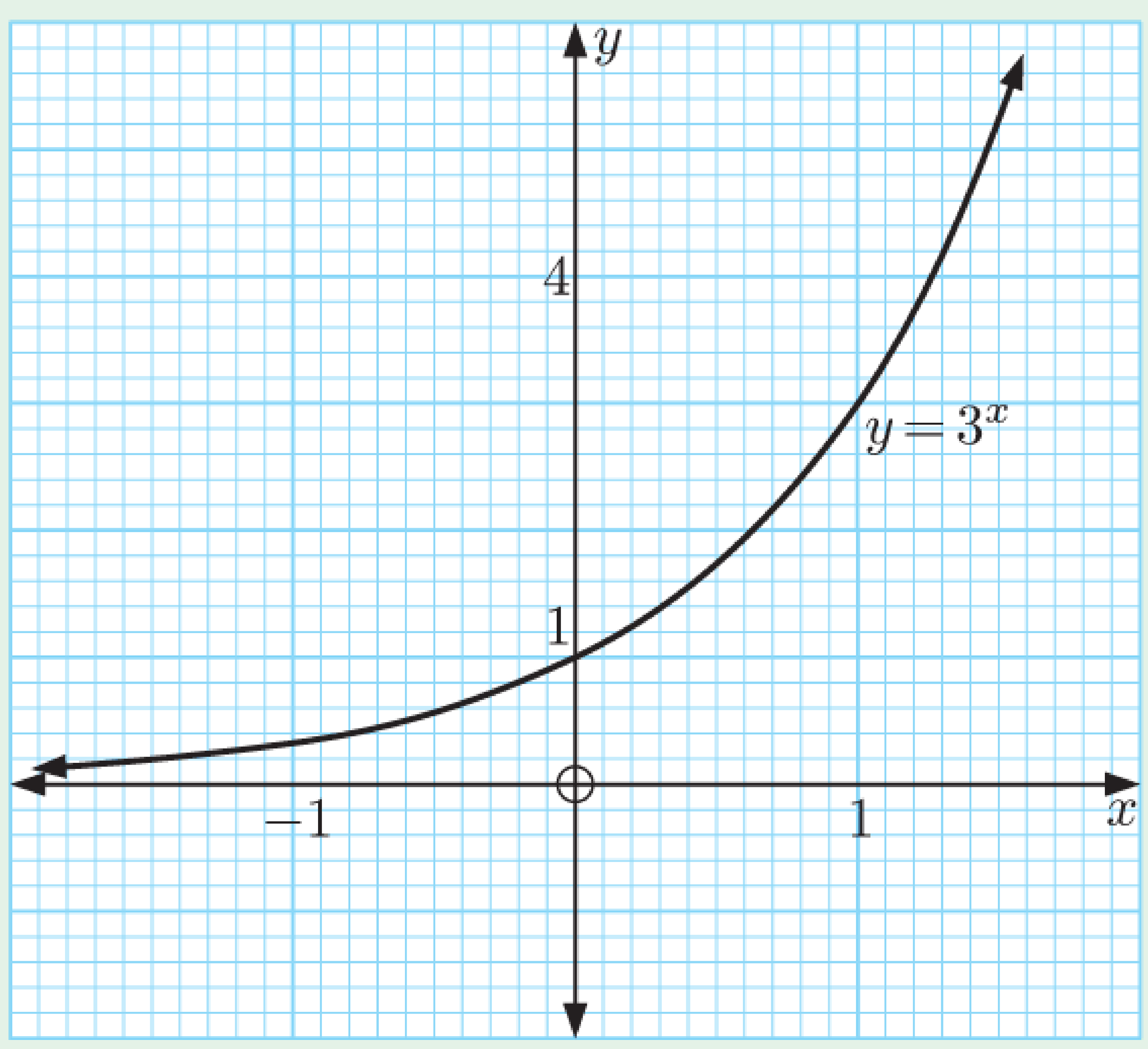
a Use the graph to estimate the value of:

i $3^{0.7}$ **ii** $3^{-0.5}$

b Use the graph to estimate the solution to:

i $3^x = 5$ **ii** $3^x = \frac{1}{2}$

iii $6 \times 3^x = 20$



7 If $f(x) = 3 \times 2^x$, find the value of:

a $f(0)$ **b** $f(3)$ **c** $f(-2)$

8 On the same set of axes, draw the graphs of $y = 2^x$ and $y = 2^x - 4$. Include on your graph the y-intercept and the equation of the horizontal asymptote of each function.

9 Consider $y = 3^x - 5$.

a Find y when $x = 0, \pm 1, \pm 2$.

b Discuss y as $x \to \pm\infty$.

c Sketch the graph of $y = 3^x - 5$.

d State the equation of any asymptote.

10 Consider $y = 3 - 2^{-x}$.

a Find y when $x = 0, \pm 1, \pm 2$.

b Discuss y as $x \to \pm\infty$.

c Sketch the graph of $y = 3 - 2^{-x}$.

d State the equation of any asymptote.

11 Let $f(x) = 2^x$ and $g(x) = 3 - x^2$.

a Find $(f \circ g)(x)$, and state its domain and range.

b Find $(g \circ f)(x)$, and state its domain and range.

c Solve for x:

i $(f \circ g)(x) = 2$ **ii** $(g \circ f)(x) = -13$

12 **a** On the same set of axes, sketch and clearly label the graphs of:

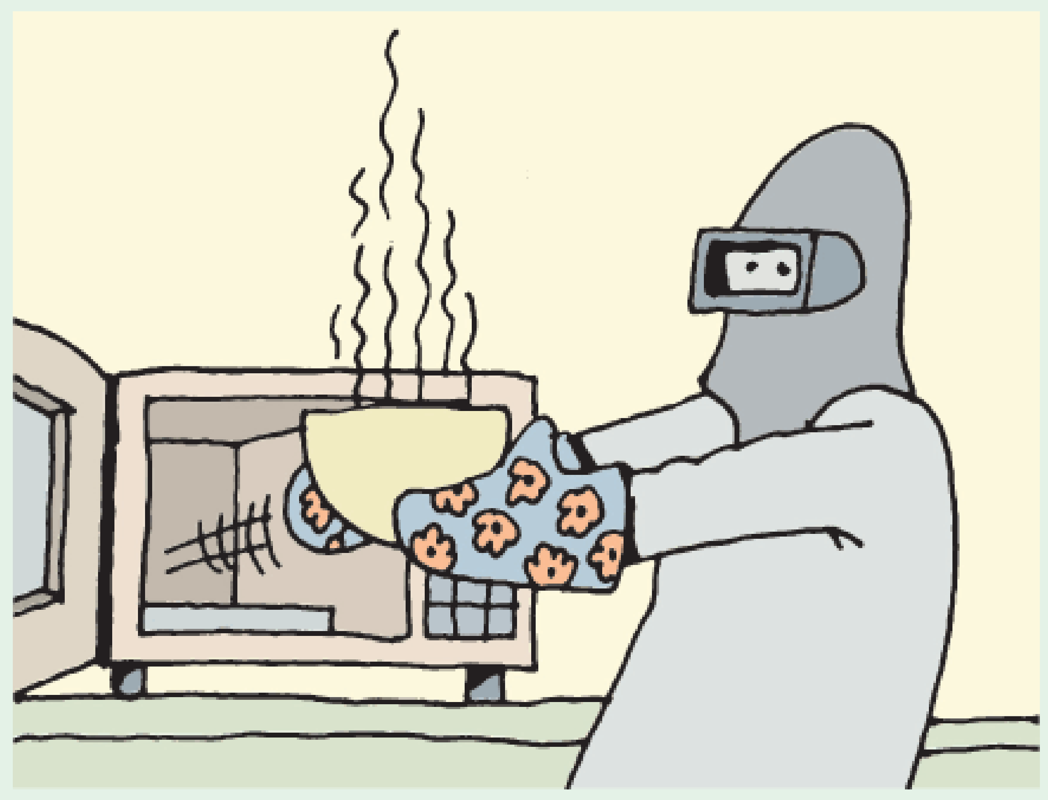
$f : x \mapsto e^x$, $g : x \mapsto e^{x-1}$, $h : x \mapsto 3 - e^x$

b State the domain and range of each function in **a**.

c Describe the behaviour of each function as $x \to \pm\infty$.

13 A plant doubles in size every 5 days. How often does it treble in size?

14 The temperature of a dish t minutes after it is removed from the microwave, is given by $T(t) = 80 \times (0.913)^t$ °C.



a Find the initial temperature of the dish.

b Find the temperature after:

i 12 minutes **ii** 24 minutes **iii** 36 minutes.

c Draw the graph of T against t for $t \geqslant 0$, using **a** and **b** or technology.

d Hence find the time taken for the temperature of the dish to fall to 25°C.

REVIEW SET 2B

1 Evaluate, correct to 3 significant figures:

a $3^{\frac{5}{4}}$ **b** $27^{-\frac{1}{5}}$ **c** $\sqrt[4]{100}$

2 Expand and simplify:

a $(3 - e^x)^2$ **b** $x^{-\frac{1}{2}}(x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - x^{-\frac{1}{2}})$ **c** $2^{-x}(2^{2x} + 2^x)$

3 Factorise:

a $3^{x+2} - 3^x$ **b** $4^x - 2^x - 12$ **c** $e^{2x} + 2e^x - 15$

4 Solve for x:

a $3 \times \left(\frac{1}{7}\right)^{x+1} = 1029$ **b** $9^x - 10(3^x) + 9 = 0$ **c** $2(4^{x+1}) + 1 = 6(2^x)$

5 Suppose $f(x) = 2^{-x} + 1$.

a Find $f\left(\frac{1}{2}\right)$.

b Find a such that $f(a) = 3$.

6 Consider $y = 2e^{-x} + 1$.

a Find y when $x = 0, \pm 1, \pm 2$.

b Discuss y as $x \to \pm\infty$.

c Sketch the graph of $y = 2e^{-x} + 1$.

d State the equation of any asymptote.

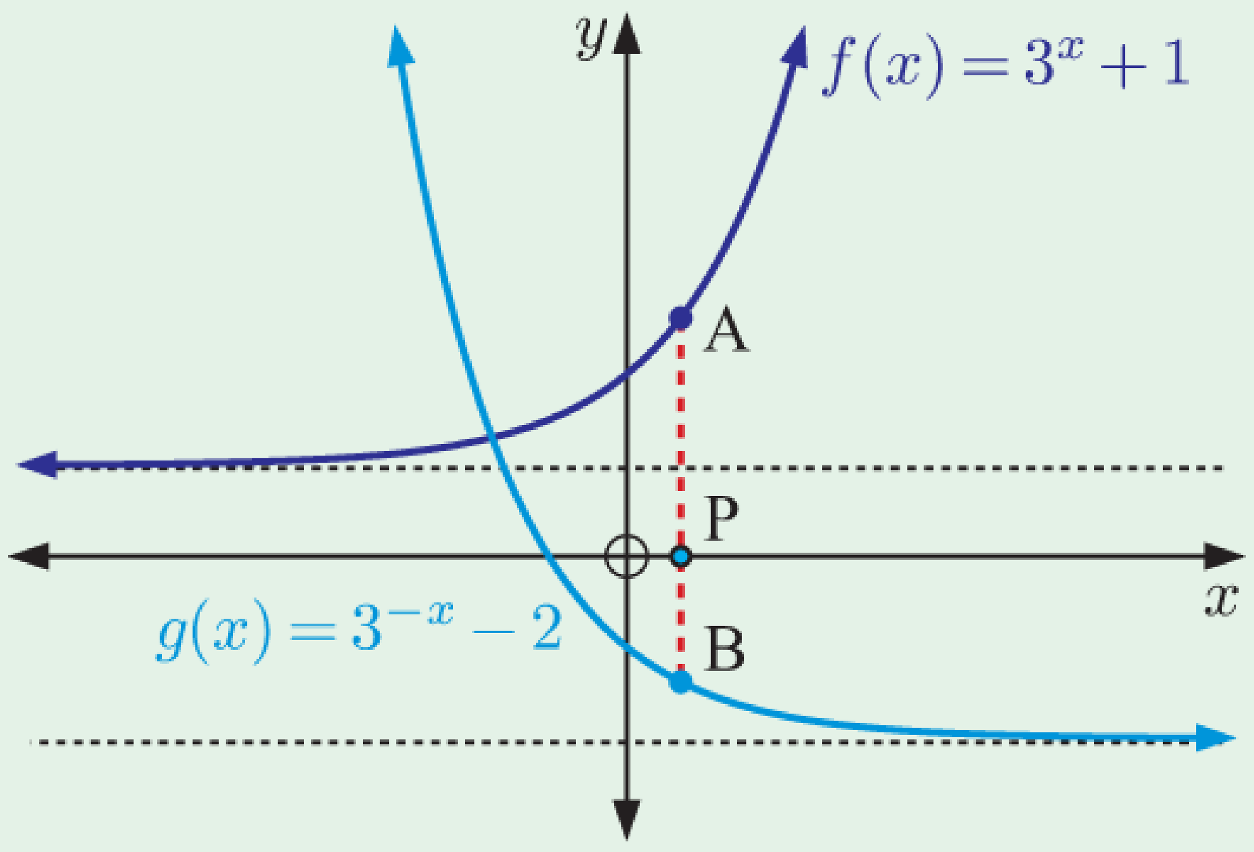
7 Answer the **Opening Problem** on page **42**.

8 Find the domain and range of $f(x) = 3^{\sqrt{x+1}}$.

9 The exponential functions $f(x) = 3^x + 1$ and $g(x) = 3^{-x} - 2$ are graphed alongside.

a Find the y-intercept of each function.

b Given that the vertical line segment [AB] has length 4 units, find the exact length of [PB]. Give your answer in the form $a + b\sqrt{5}$ units, where $a, b \in \mathbb{Q}$.



10 Let $f(x) = 3^x$.

a Write down the value of:

i $f(4)$

ii $f(-1)$

b Find the value of k such that $f(x+2) = k\,f(x)$, $k \in \mathbb{Z}$.

11 Suppose $y = a^x$. Express in terms of y:

a a^{2x}

b a^{-x}

c $\frac{1}{\sqrt{a^x}}$

12 The weight of a radioactive substance after t years is given by $W = 1500 \times (0.993)^t$ grams.

a Find the original amount of radioactive material.

b Find the amount of radioactive material remaining after:

i 400 years

ii 800 years.

c Sketch the graph of W against t for $t \geqslant 0$.

d Hence find the time taken for the weight to reduce to 100 grams.

13 A phycologist investigates an algal bloom in a lake. Initially it covers 10 square metres of water. Each day after it was discovered, the area covered increases by 15%.

a Write a formula for the area $A(t)$ covered after t days.

b Find the area covered after:

i 2 days

ii 5 days.

c Sketch the graph of $A(t)$.

d How long will it take for the affected area to reach 300 m^2?