# Chapter 15

# Functions

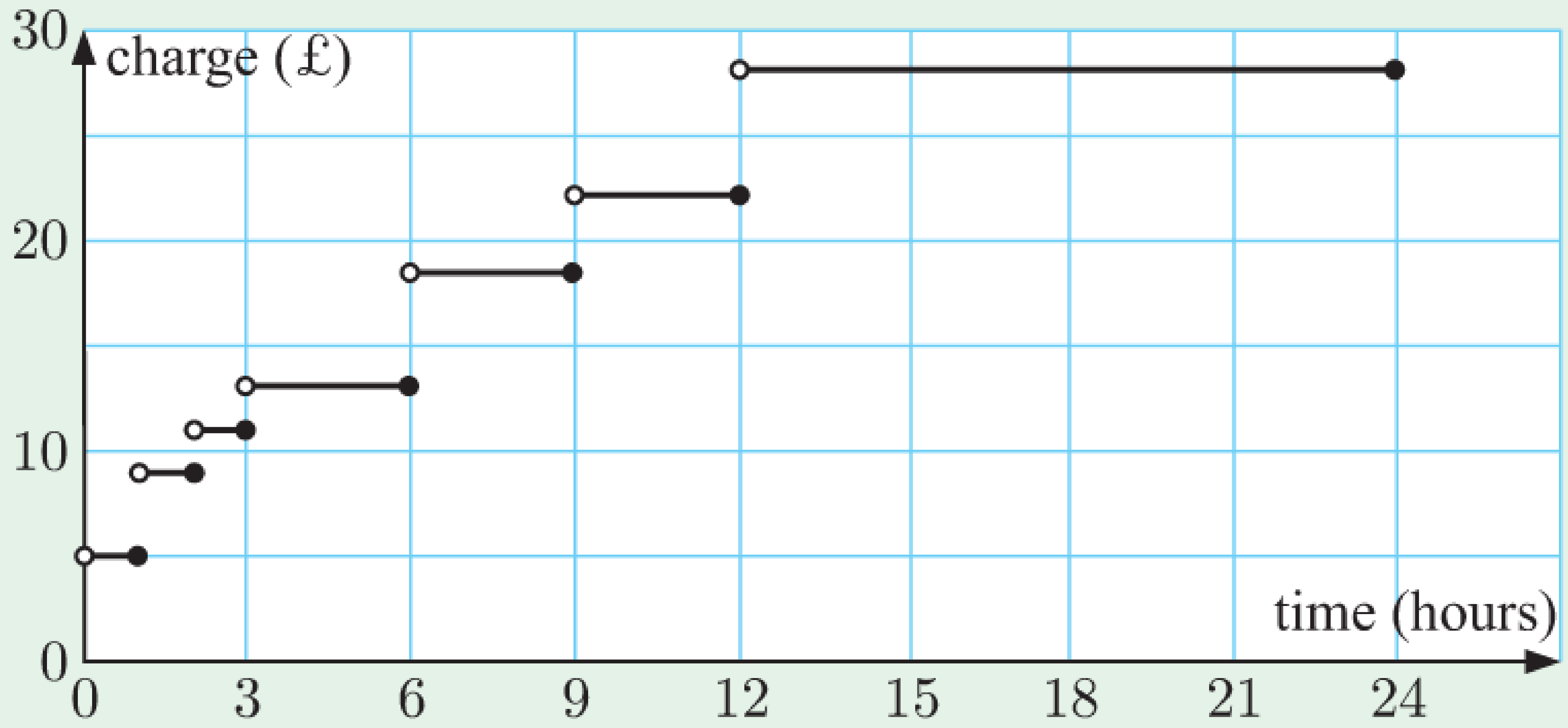
**Contents:**



## OPENING PROBLEM

The charges for parking a car in a short-term car park at an airport are shown in the table and graph below. The total charge is *dependent* on the length of time $t$ the car is parked.

| Car park charges | |
|---|---|
| *Time* $t$ (hours) | *Charge* |
| $0 < t \leqslant 1$ | £5.00 |
| $1 < t \leqslant 2$ | £9.00 |
| $2 < t \leqslant 3$ | £11.00 |
| $3 < t \leqslant 6$ | £13.00 |
| $6 < t \leqslant 9$ | £18.00 |
| $9 < t \leqslant 12$ | £22.00 |
| $12 < t \leqslant 24$ | £28.00 |

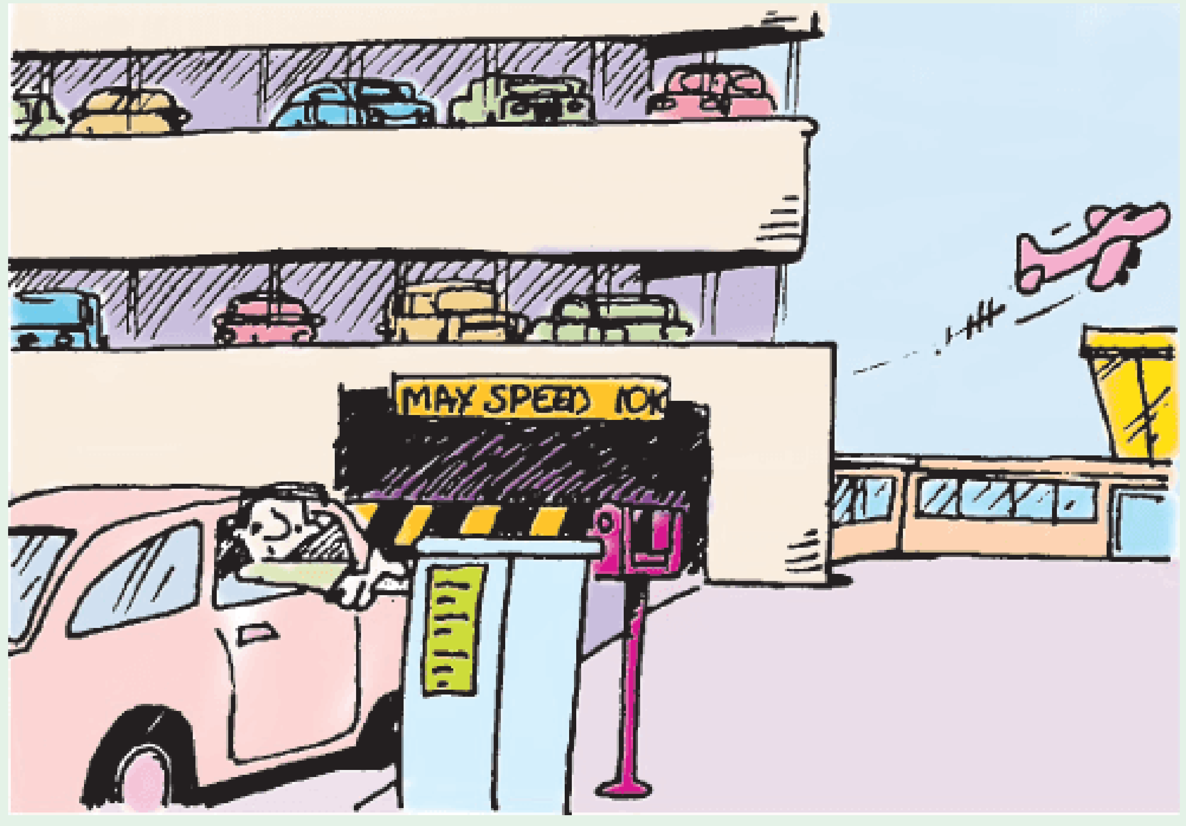



**Things to think about:**

**a** What values of *time* are illustrated in the graph?

**b** What are the possible charges?

**c** What feature of the graph ensures that there is only one charge for any given time?



In the course so far, we have studied several different relationships between variables. In particular, for two variables $x$ and $y$:

- A **linear function** is a relationship which can be expressed in the form $y = ax + b$ where $a$, $b$ are constants, $a \neq 0$.
- A **quadratic function** is a relationship which can be expressed in the form $y = ax^2 + bx + c$ where $a$, $b$, $c$ are constants, $a \neq 0$.

In the **Opening Problem** we see another type of relationship, between the two variables *time* and *charge*. We call this a **piecewise function** because its graph has several sections.

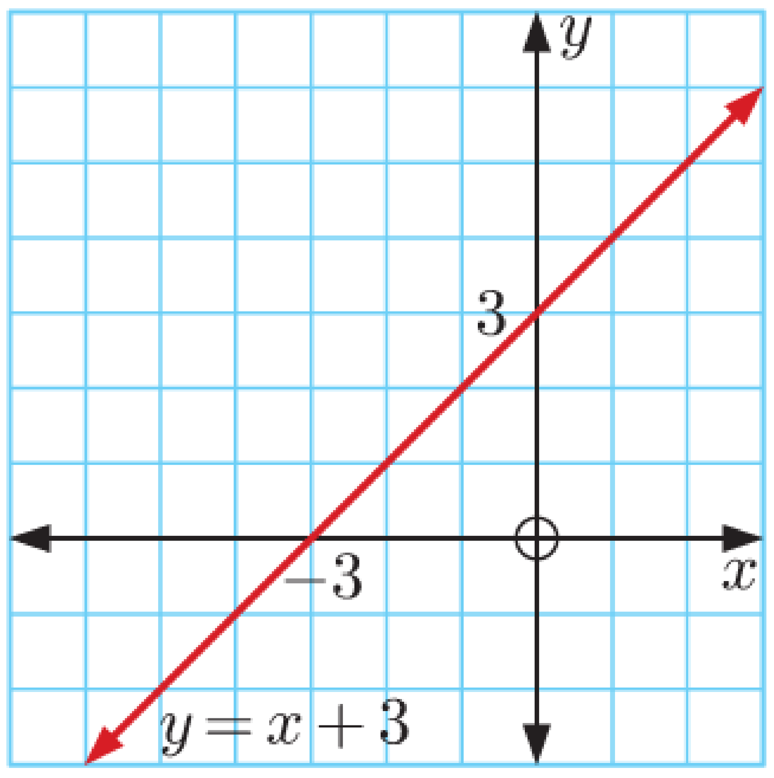
In this Chapter we explore what it really means for the relationship between two variables to be called a **function**. We will then explore properties of functions which will help us work with and understand them.

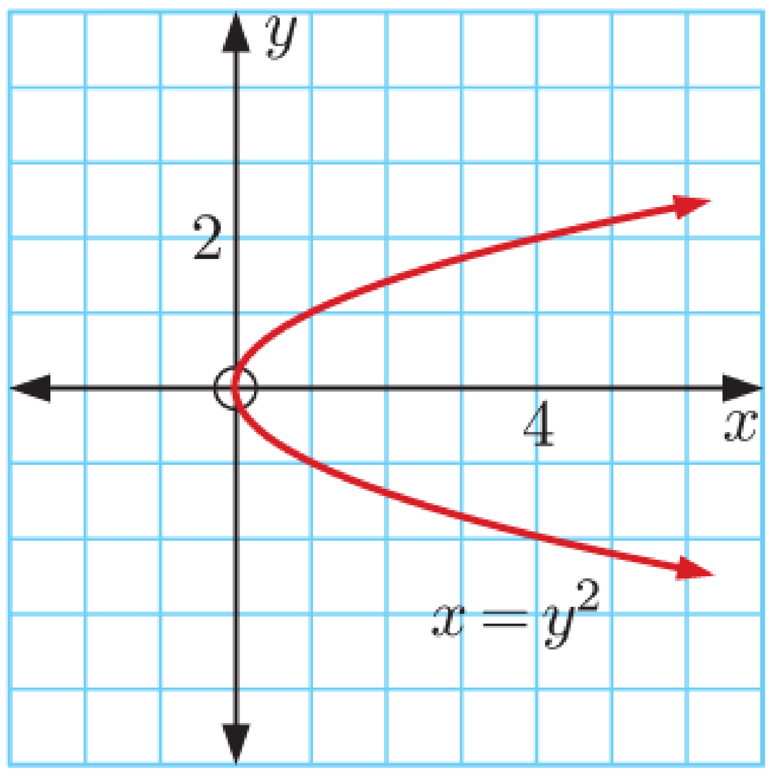
## A RELATIONS AND FUNCTIONS

A **relation** between variables $x$ and $y$ is any set of points in the $(x, y)$ plane. We say that the points *connect* the two variables.

A relation is often expressed in the form of an **equation** connecting the **variables** $x$ and $y$.

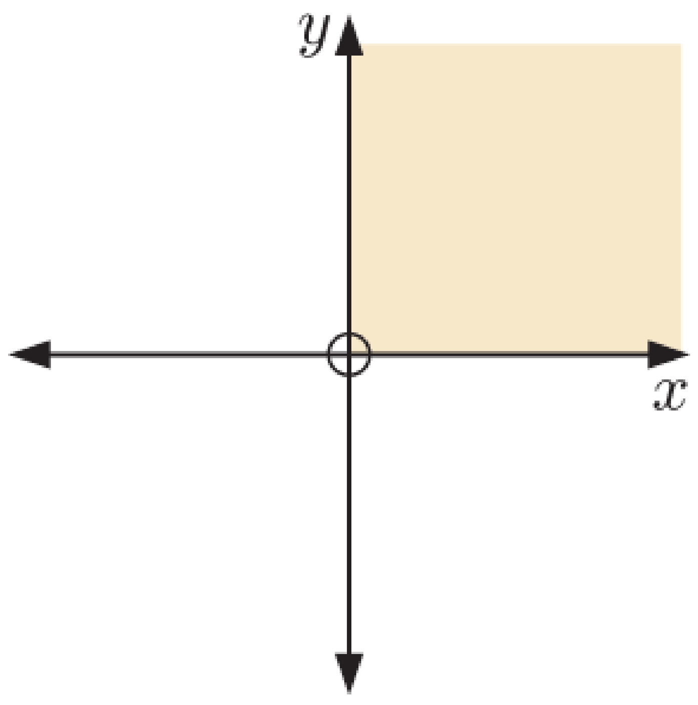
For example, $y = x + 3$ and $x = y^2$ are the equations of two relations. Each equation generates a set of ordered pairs, which we can graph:



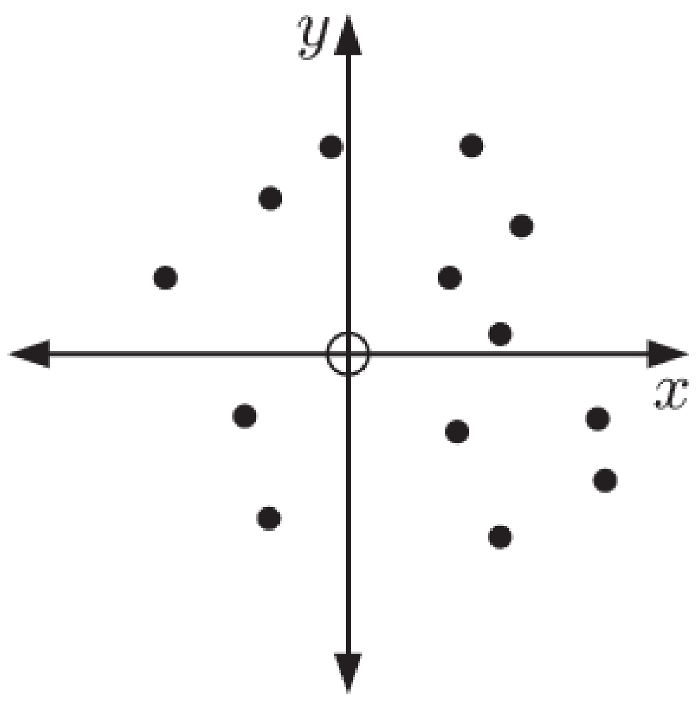

However, not all relations can be defined by an equation. Below are two examples:

**(1)**




The set of all points in the first quadrant is the relation $x > 0$, $y > 0$.

**(2)**




These 13 points form a relation. It can be described as a finite set of points, but not by an equation.

## FUNCTIONS

A **function** is a relation in which no two different ordered pairs have the same $x$-coordinate or first component.

We can see from this definition that a function is a special type of relation.

Every function is a relation, but not every relation is a function.

## ALGEBRAIC TEST FOR FUNCTIONS

Suppose a relation is given as an equation. If the substitution of any value for $x$ results in at most one value of $y$, then the relation is a function.

For example:

- $y = 3x - 1$ is a function, since for any value of $x$ there is only one corresponding value of $y$
- $x = y^2$ is not a function, since if $x = 4$ then $y = \pm 2$.

## GEOMETRIC TEST OR VERTICAL LINE TEST FOR FUNCTIONS

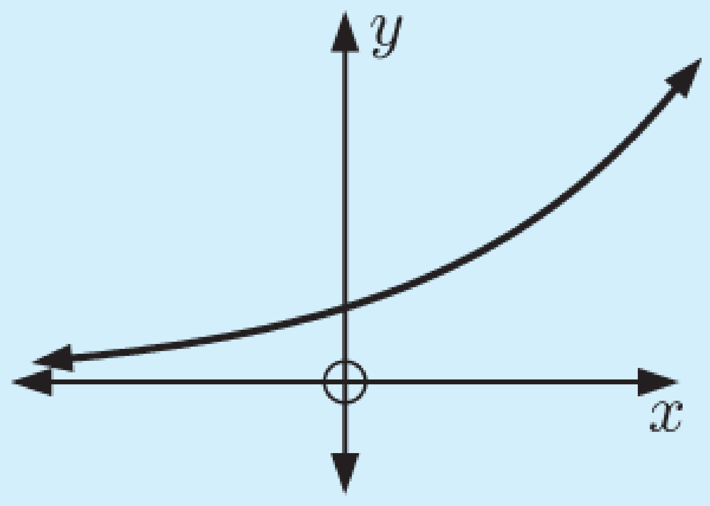
Suppose we draw all possible vertical lines on the graph of a relation.

- If each line cuts the graph at most once, then the relation is a function.
- If at least one line cuts the graph more than once, then the relation is not a function.

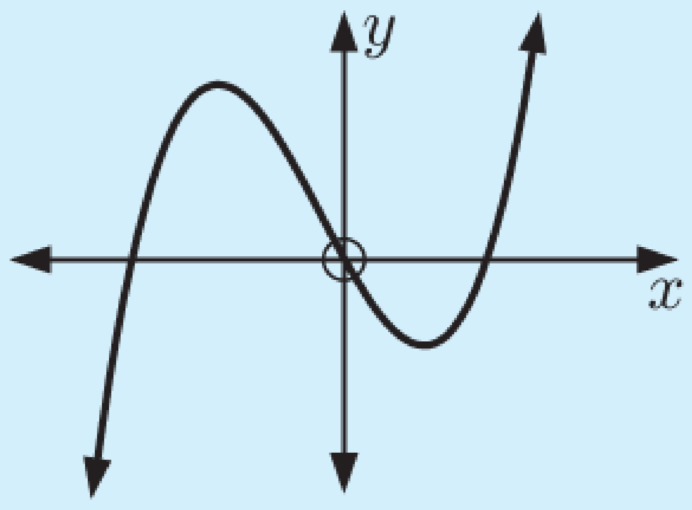
**Example 1** — Self Tutor

Which of the following relations are functions?

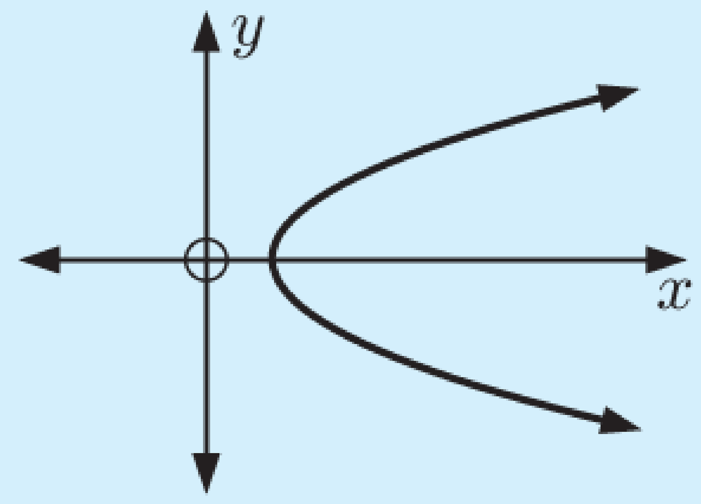
**a**



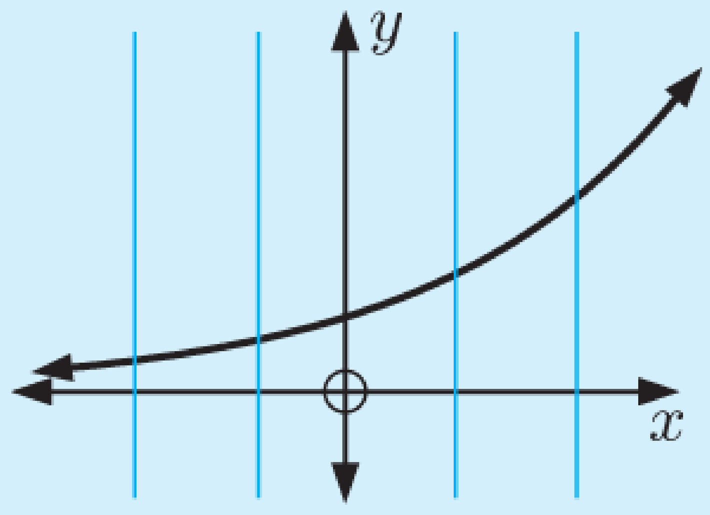

**b**




**c**

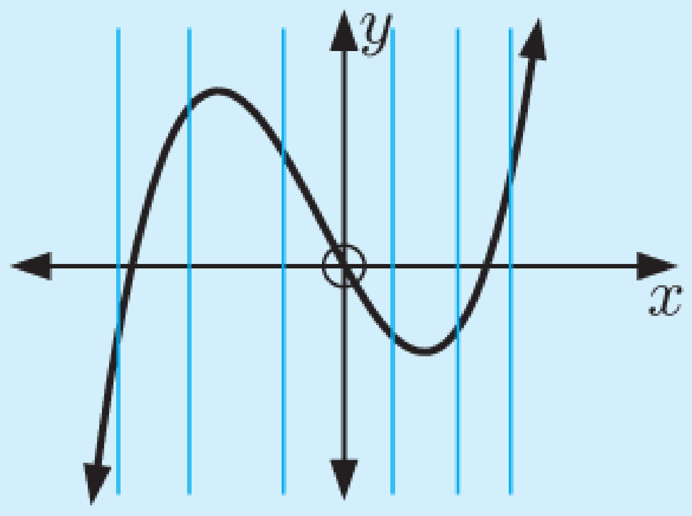



**a**



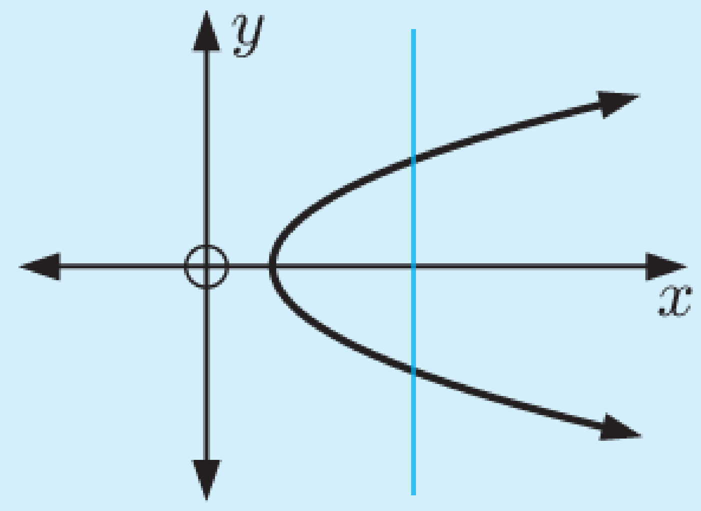

a function

**b**




a function

**c**




not a function

DEMO

## GRAPHICAL NOTE

- If a graph contains a small **open circle** such as —o—, this point is **not included**.
- If a graph contains a small **filled-in circle** such as ——•, this point is **included**.
- If a graph contains an **arrowhead** at an end such as ——▸, then the graph continues indefinitely in that general direction, or the shape may repeat as it has done previously.

## EXERCISE 15A

**1** Which of the following sets of ordered pairs are functions? Explain your answers.

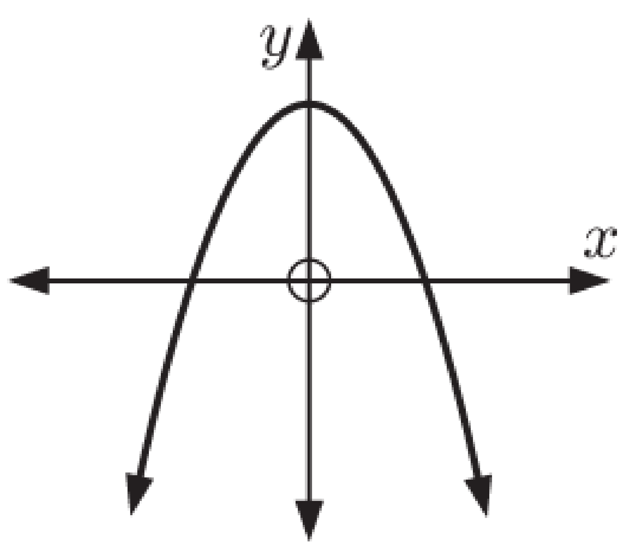
**a** $\{(1, 3), (2, 3), (3, 1), (4, 2)\}$  **b** $\{(2, 1), (3, 1), (-1, 2), (2, 0)\}$

**2** Use algebraic methods to decide whether these relations are functions. Explain your answers.

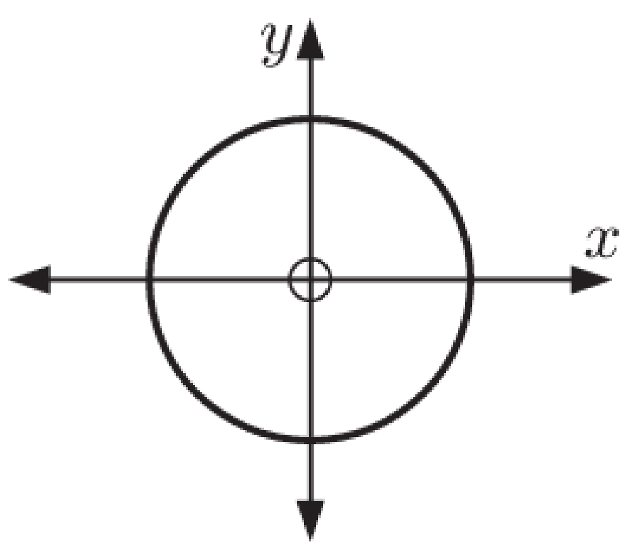
**a** $y = x^2 - 9$  **b** $x + y = 9$  **c** $x^2 + y^2 = 9$

**3** Use the vertical line test to determine which of the following relations are functions:

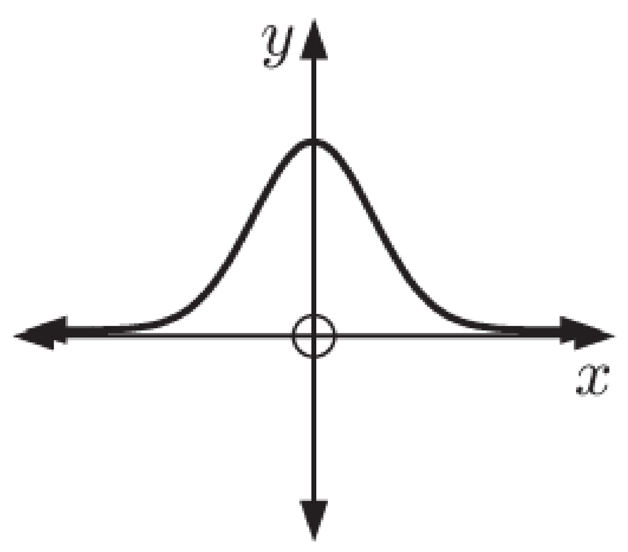
**a**



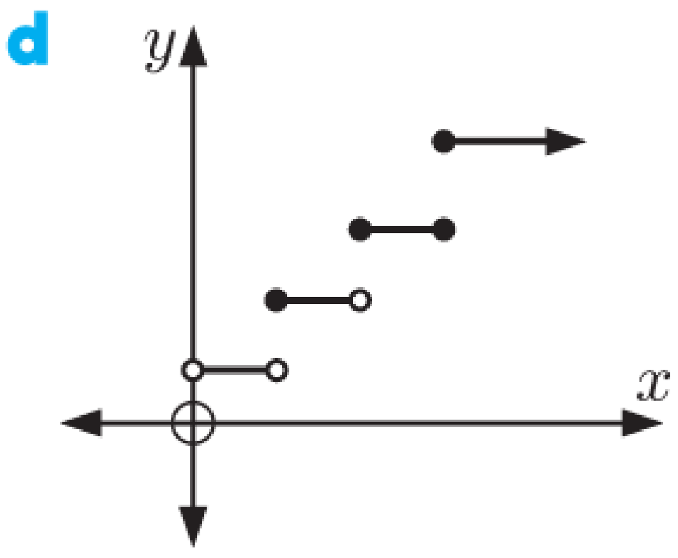

**b**




**c**




**d**



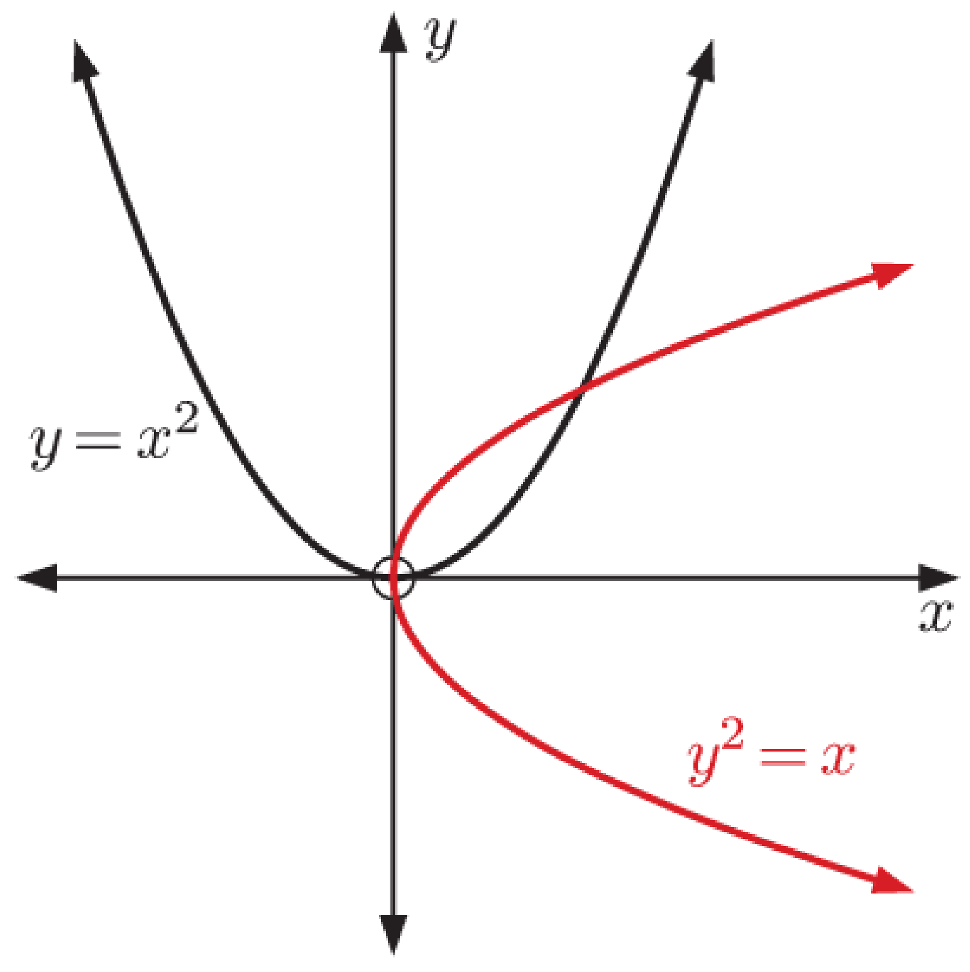

**4** The managers of a new amusement park are discussing the schedule of ticket prices. Maurice suggests the table alongside. Explain why this relation between *age* and *cost* is not a function, and discuss the problems that this will cause.

| *Age* | *Cost* |
|---|---|
| 0 - 2 years (infants) | $0 |
| 2 - 16 years (children) | $20 |
| 16+ years (adults) | $30 |

**5** Is it possible for a function to have more than one $y$-intercept? Explain your answer.

**6** Is the graph of a straight line always a function? Give evidence to support your answer.

**7** The graph alongside shows the curves $y = x^2$ and $y^2 = x$.



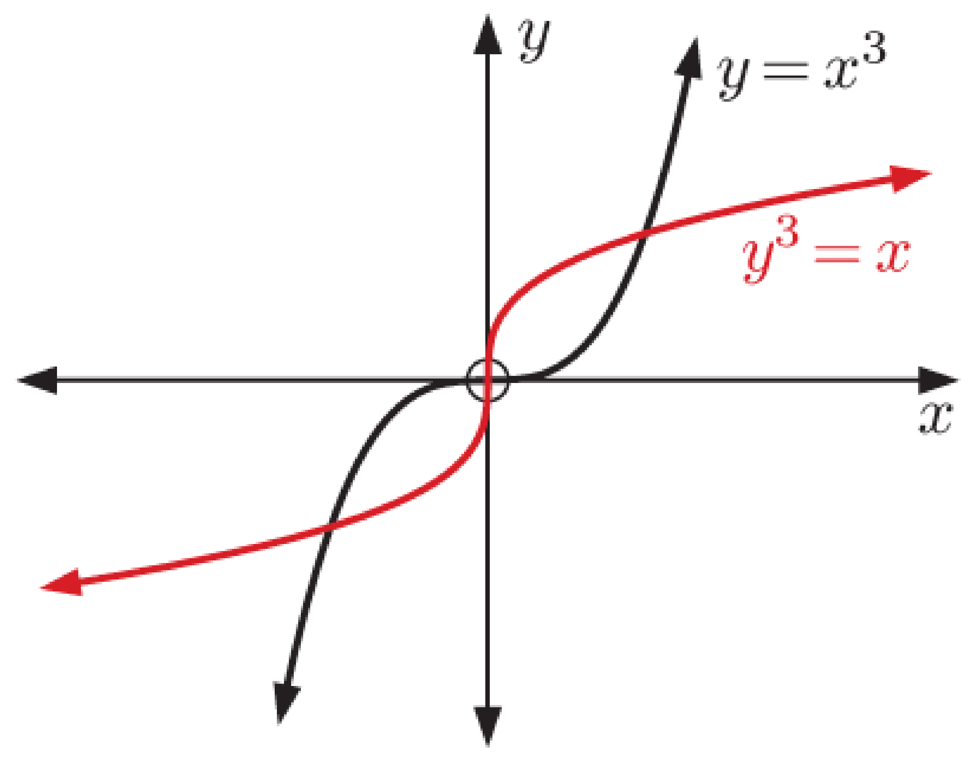

**a** Discuss the similarities and differences between the curves, including whether each curve is a function. You may also consider what transformation(s) map one curve onto the other.

**b** Using $y^2 = x$, we can write $y = \pm\sqrt{x}$.

**i** What part of the graph of $y^2 = x$ corresponds to $y = \sqrt{x}$?

**ii** Is $y = \sqrt{x}$ a function? Explain your answer.

**8** The graph alongside shows the curves $y = x^3$ and $y^3 = x$.




**a** Explain why both of these curves are functions.

**b** For the curve $y^3 = x$, write $y$ as a function of $x$.

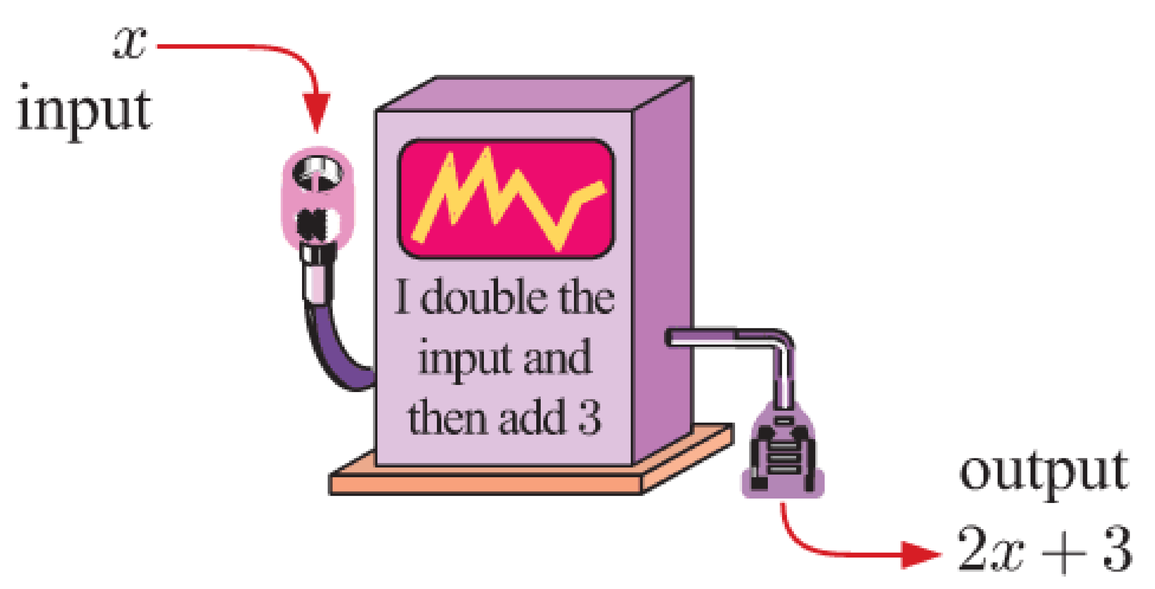
## DISCUSSION

In the **Opening Problem**:

- Is the relation describing the car park charges a function?
- If we know the *time* somebody parked for, can we determine the exact *charge* they need to pay?
- If we know the *charge* somebody pays, can we determine the exact *time* they have parked for?

## B FUNCTION NOTATION

**Function machines** are sometimes used to illustrate how functions behave.




If 4 is the input fed into the machine, the output is $2(4) + 3 = 11$.

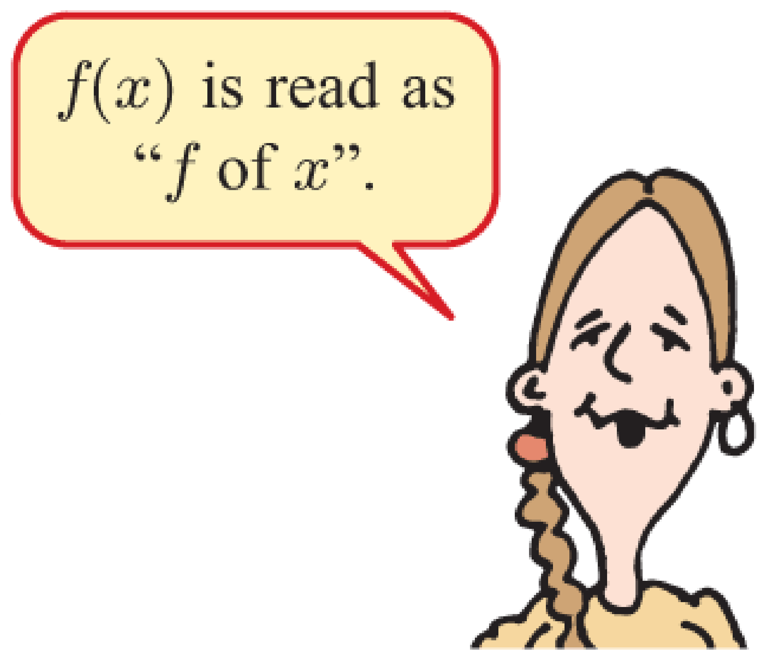
The above "machine" has been programmed to perform a particular function. If we use $f$ to represent that particular function, we can write "$f$ is the function that will convert $x$ into $2x + 3$."

So, $f$ would convert $2$ into $2(2) + 3 = 7$ and
$-4$ into $2(-4) + 3 = -5$.

This function can be written as:

$$\boldsymbol{f : x \mapsto 2x + 3}$$

function $f$ — such that — $x$ is converted into $2x+3$




Two other equivalent forms we use are $f(x) = 2x + 3$ and $y = 2x + 3$.

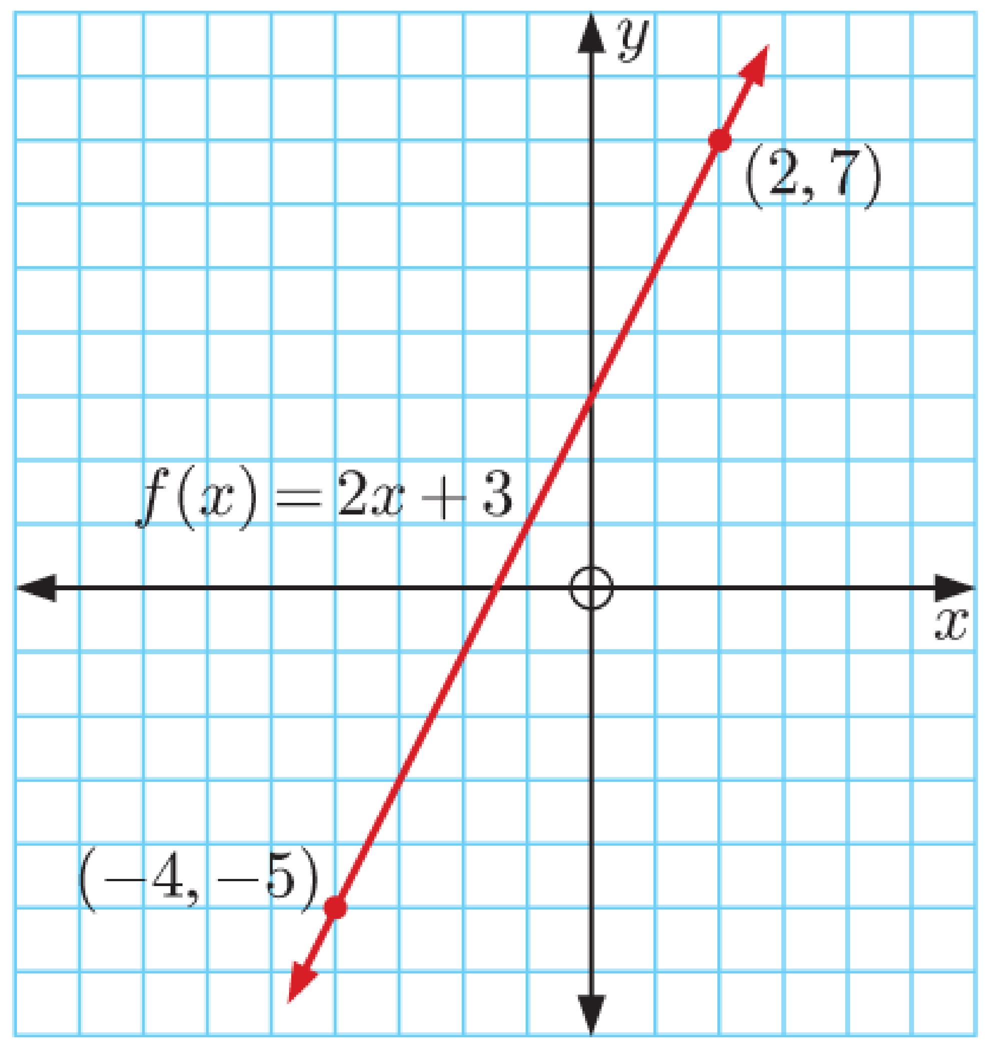
> $f(x)$ is the value of $y$ for a given value of $x$, so $y = f(x)$.

$f$ is the function which converts $x$ into $f(x)$, so we write $f : x \mapsto f(x)$.

$y = f(x)$ is sometimes called the **function value** or **image** of $x$.

For $f(x) = 2x + 3$:

- $f(2) = 2(2) + 3 = 7$
  $\therefore$ the point $(2, 7)$ lies on the graph of the function.
- $f(-4) = 2(-4) + 3 = -5$
  $\therefore$ the point $(-4, -5)$ also lies on the graph.




**Example 2** — **Self Tutor**

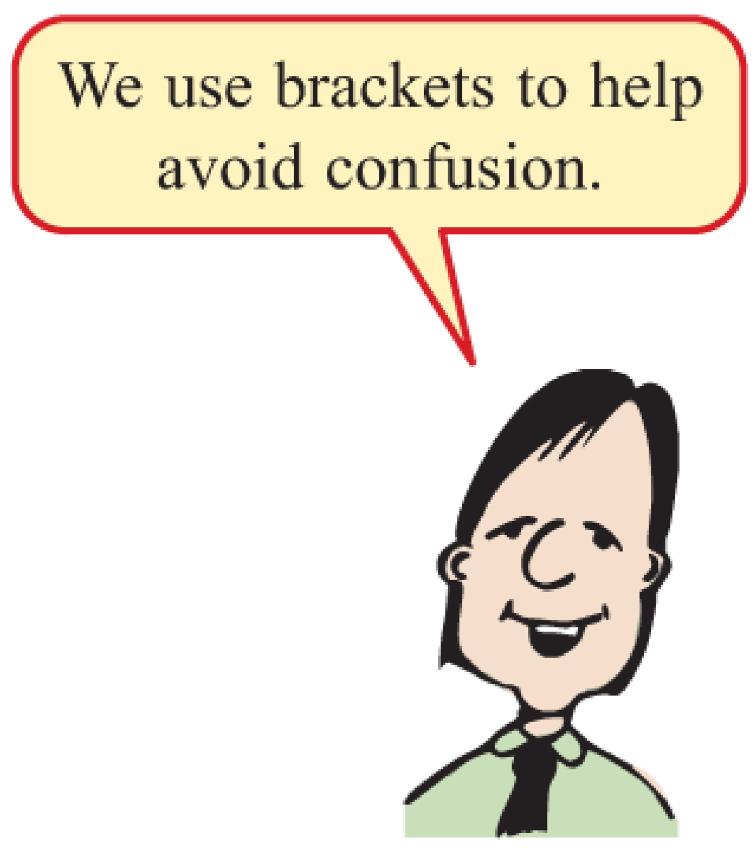
If $f : x \mapsto 2x^2 - 3x$, find the value of:

**a** $f(5)$ **b** $f(-4)$

$f(x) = 2x^2 - 3x$

**a** $f(5) = 2(5)^2 - 3(5)$ {replacing $x$ with $(5)$}
$= 2 \times 25 - 15$
$= 35$

**b** $f(-4) = 2(-4)^2 - 3(-4)$ {replacing $x$ with $(-4)$}
$= 2(16) + 12$
$= 44$




## EXERCISE 15B

**1** If $f(x) = 3x - x^2 + 2$, find the value of:

**a** $f(0)$ **b** $f(3)$ **c** $f(-3)$ **d** $f(-7)$ **e** $f(\frac{3}{2})$

**2** If $g : x \mapsto x - \dfrac{4}{x}$, find the value of:

**a** $g(1)$ **b** $g(4)$ **c** $g(-1)$ **d** $g(-4)$ **e** $g(-\frac{1}{2})$

**3** Suppose $G(x) = \dfrac{2x+3}{x-4}$.

**a** Evaluate: **i** $G(2)$ **ii** $G(0)$ **iii** $G(-\frac{1}{2})$

**b** Find a value of $x$ such that $G(x)$ does not exist.

**c** Find $x$ such that $G(x) = -3$.

**Example 3** **Self Tutor**

If $f(x) = 5 - x - x^2$, find in simplest form:

**a** $f(-x)$ **b** $f(x+2)$ **c** $f(x-1) - 5$

**a** $f(-x) = 5 - (-x) - (-x)^2$ {replacing $x$ with $(-x)$}
$= 5 + x - x^2$

**b** $f(x+2) = 5 - (x+2) - (x+2)^2$ {replacing $x$ with $(x+2)$}
$= 5 - x - 2 - [x^2 + 4x + 4]$
$= 3 - x - x^2 - 4x - 4$
$= -x^2 - 5x - 1$

**c** $f(x-1) - 5 = (5 - (x-1) - (x-1)^2) - 5$ {replacing $x$ with $(x-1)$}
$= 5 - x + 1 - (x^2 - 2x + 1) - 5$
$= -x^2 + x$

**4** If $f(x) = 7 - 3x$, find in simplest form:

**a** $f(a)$ **b** $f(-a)$ **c** $f(a+3)$
**d** $f(2a)$ **e** $f(x+2)$ **f** $f(x+h)$

**5** If $F(x) = 2x^2 + 3x - 1$, find in simplest form:

**a** $F(x+4)$ **b** $F(2-x)$ **c** $F(-x)$
**d** $F(x^2)$ **e** $F(3x)$ **f** $F(x+h)$

**6** If $f(x) = x^2$, find in simplest form:

**a** $f(3x)$ **b** $f\left(\frac{x}{2}\right)$ **c** $3\,f(x)$ **d** $2\,f(x-1) + 5$

**7** If $f(x) = \frac{1}{x}$, find in simplest form:

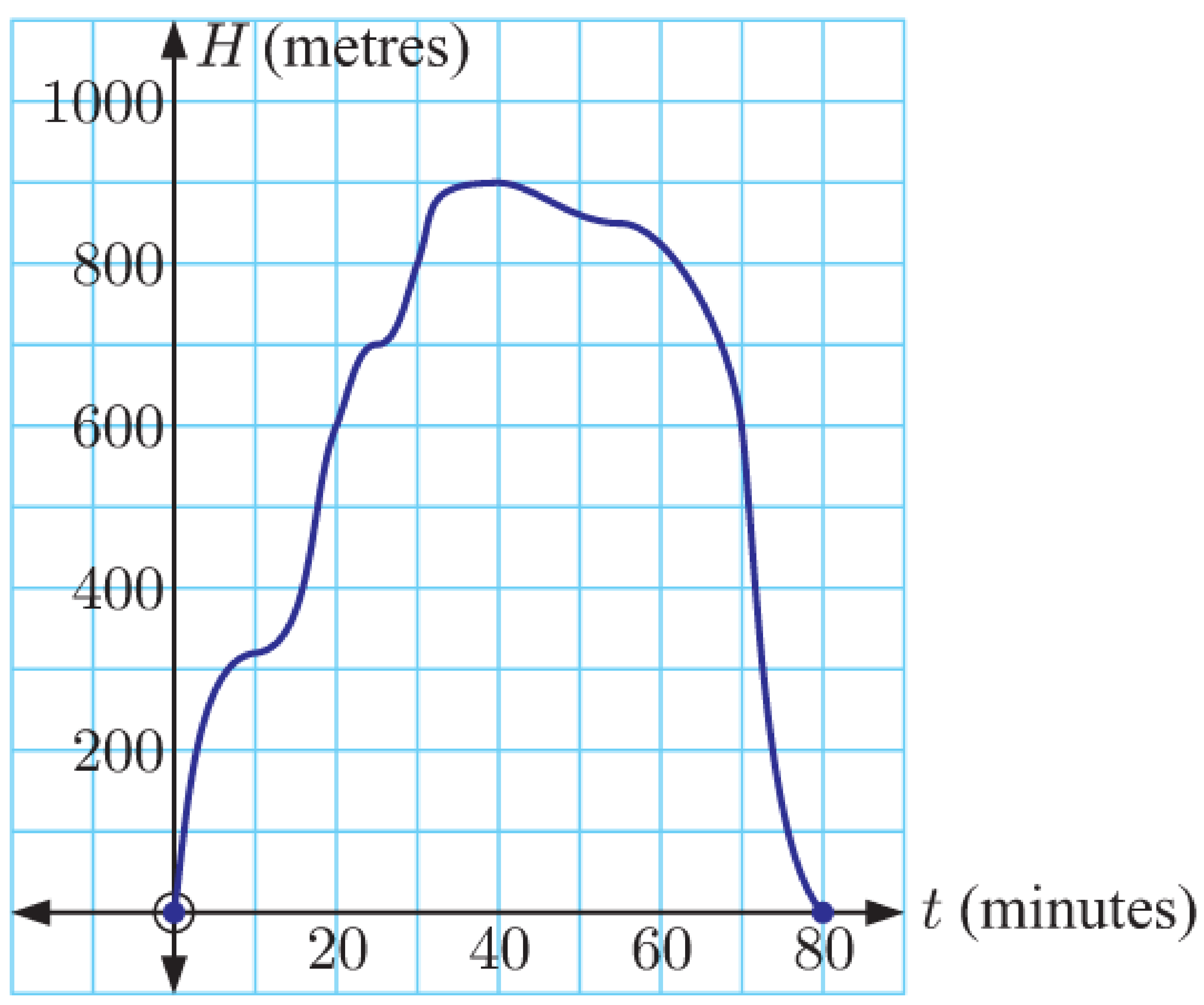
**a** $f(-x)$ **b** $f(\frac{1}{2}x)$ **c** $2\,f(x) + 3$ **d** $3\,f(x-1) + 2$

**8** $f$ represents a function. Explain the difference in meaning between $f$ and $f(x)$.

**9** On the same set of axes, draw the graphs of three different functions $f(x)$ such that $f(2) = 1$ and $f(5) = 3$.

**10** Find a linear function $f(x) = ax + b$ for which $f(2) = 1$ and $f(-3) = 11$.

**11** For a hot air balloon ride, the function $H(t)$ gives the height of the balloon after $t$ minutes. Its graph is shown alongside.




**a** Find $H(30)$, and explain what your answer means.

**b** Find the values of $t$ such that $H(t) = 600$. Interpret your answer.

**c** For what values of $t$ was the height of the balloon recorded?

**d** What range of heights was recorded for the balloon?

**12** Given $f(x) = ax + \frac{b}{x}$, $f(1) = 1$, and $f(2) = 5$, find constants $a$ and $b$.

**13** The quadratic function $T(x) = ax^2 + bx + c$ has the values $T(0) = -4$, $T(1) = -2$, and $T(2) = 6$. Find $a$, $b$, and $c$.

**14** The value of a photocopier $t$ years after purchase is given by $V(t) = 9000 - 900t$ pounds.

**a** Find $V(4)$, and state what $V(4)$ means.

**b** Find $t$ when $V(t) = 3600$, and explain what this means.

**c** Find the original purchase price of the photocopier.

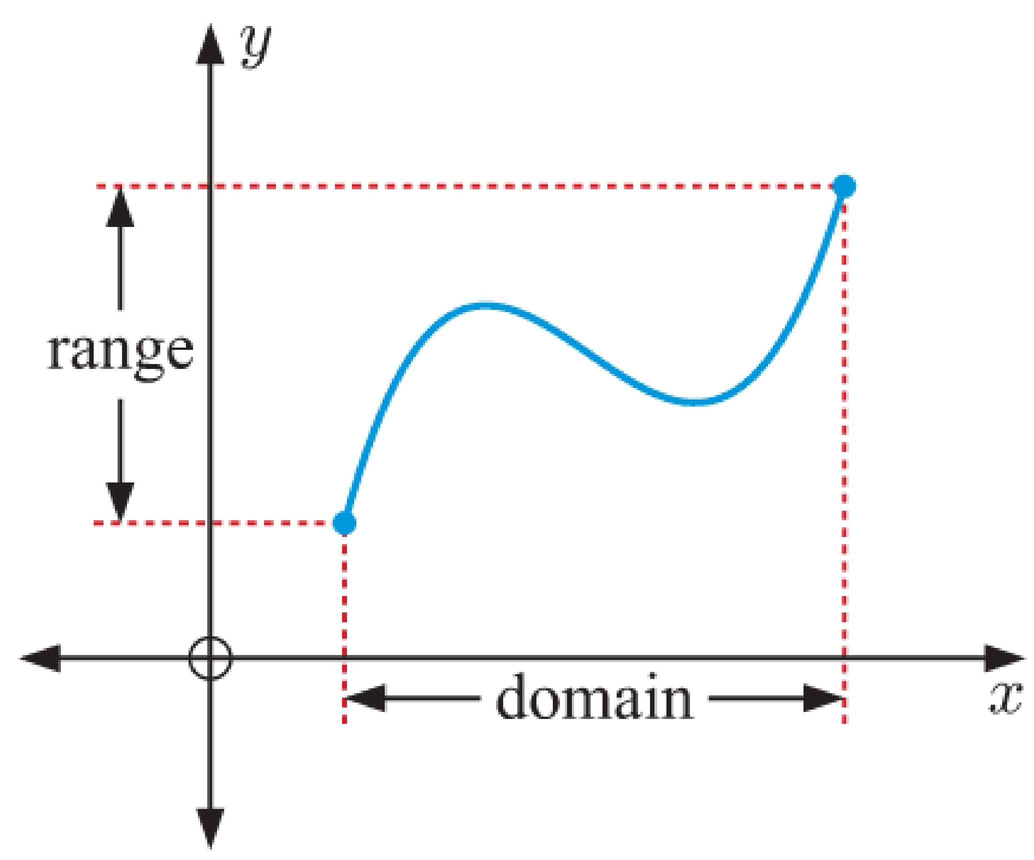
**d** For what values of $t$ is it reasonable to use this function?

# C DOMAIN AND RANGE

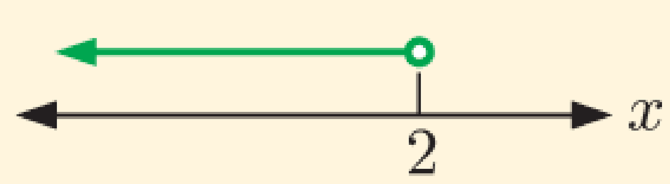
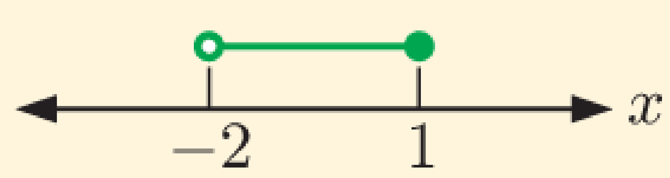
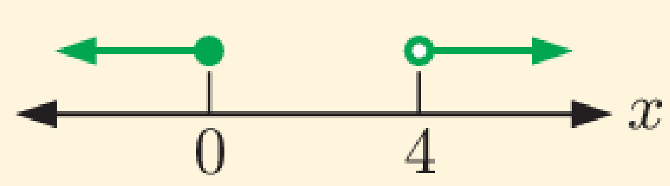
We have seen that a relation is a set of points which connects two variables.

The **domain** of a relation is the set of values which the variable on the horizontal axis can take. This variable is usually $x$.

The **range** of a relation is the set of values which the variable on the vertical axis can take. This variable is usually $y$.



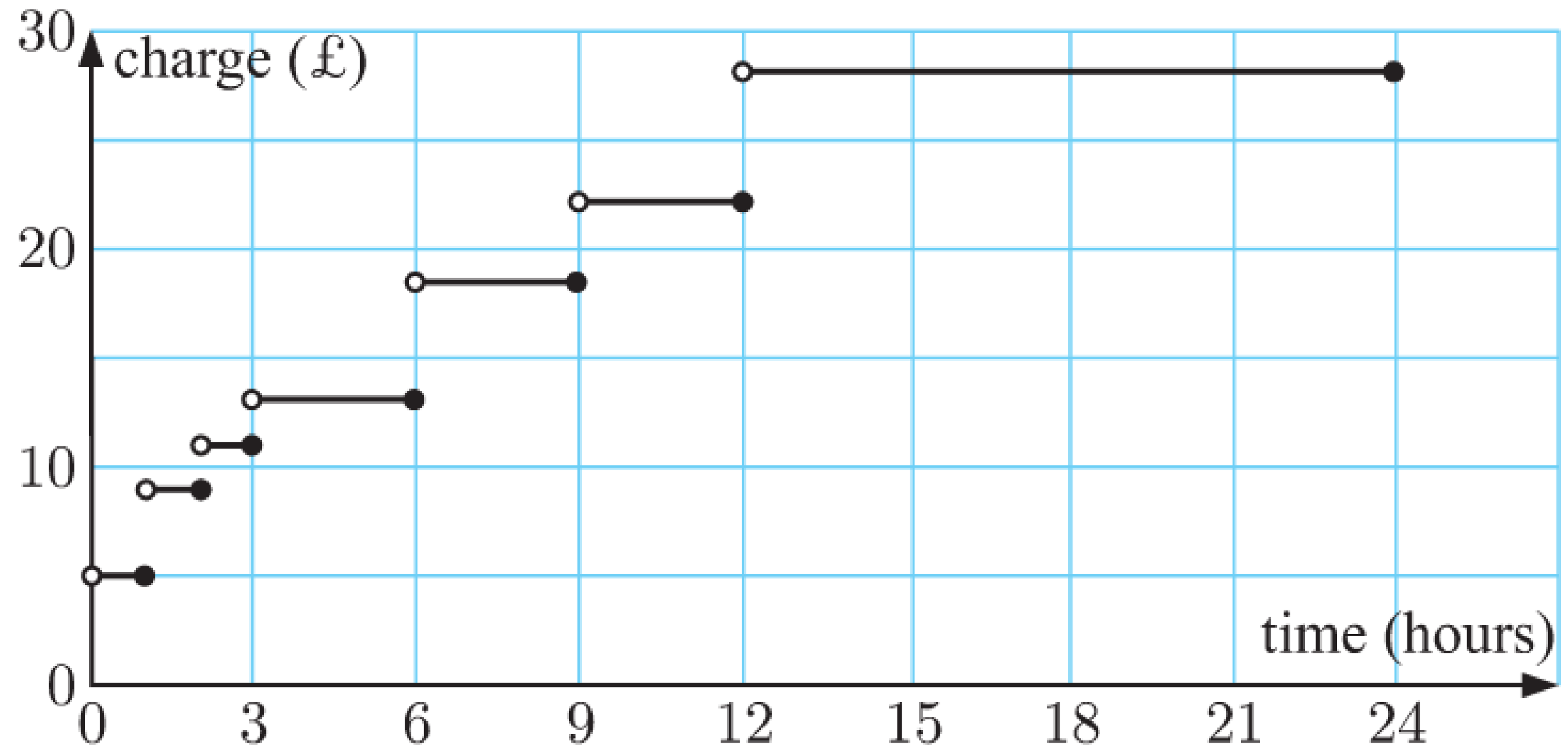

The domain and range of a relation can be described in several ways. Examples are given in the table below:

| *Set notation* | *Bracket notation* | *Number line graph* | *Meaning* |
|---|---|---|---|
| $\{x \mid x \geqslant 3\}$ | $x \in [3, \infty[$ |  | the set of all $x$ such that $x$ is greater than or equal to 3 |
| $\{x \mid x < 2\}$ | $x \in ]-\infty, 2[$ |  | the set of all $x$ such that $x$ is less than 2 |
| $\{x \mid -2 < x \leqslant 1\}$ | $x \in ]-2, 1]$ |  | the set of all $x$ such that $x$ is between $-2$ and 1, including 1 |
| $\{x \mid x \leqslant 0$ or $x > 4\}$ | $x \in ]-\infty, 0]$ or $]4, \infty[$ |  | the set of all $x$ such that $x$ is less than or equal to 0, or greater than 4 |

## DOMAIN AND RANGE OF FUNCTIONS

To find the domain and range of a function, we can observe its graph. For example:

**(1)** In the **Opening Problem**, the car park charges function is defined for all times $t$ such that $0 < t \leqslant 24$.
$\therefore$ the domain is $\{t \mid 0 < t \leqslant 24\}$.
The possible charges are £5, £9, £11, £13, £18, £22, and £28.
$\therefore$ the range is $\{5, 9, 11, 13, 18, 22, 28\}$.

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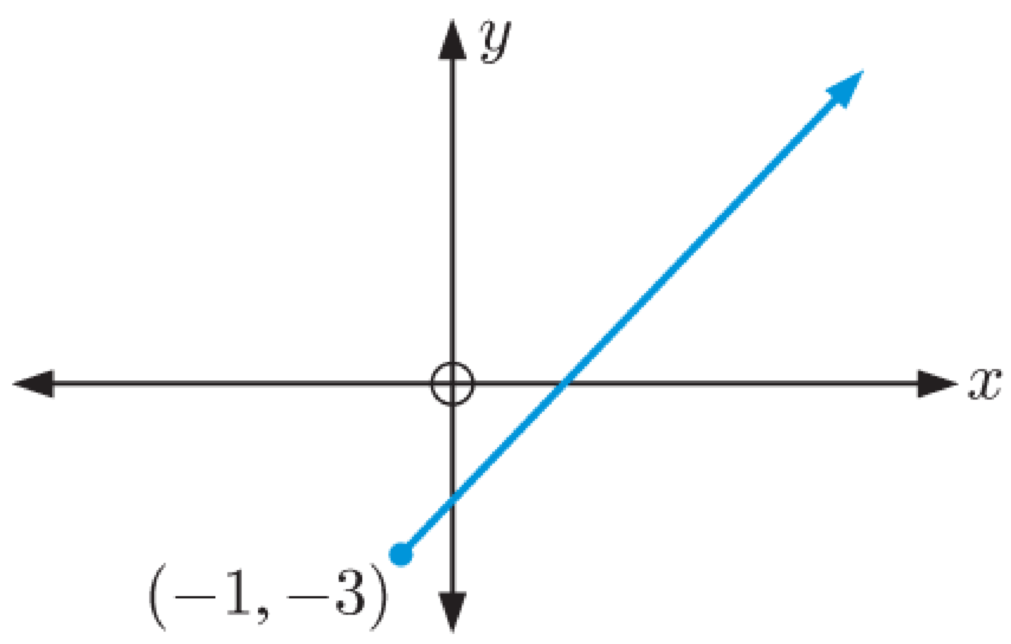
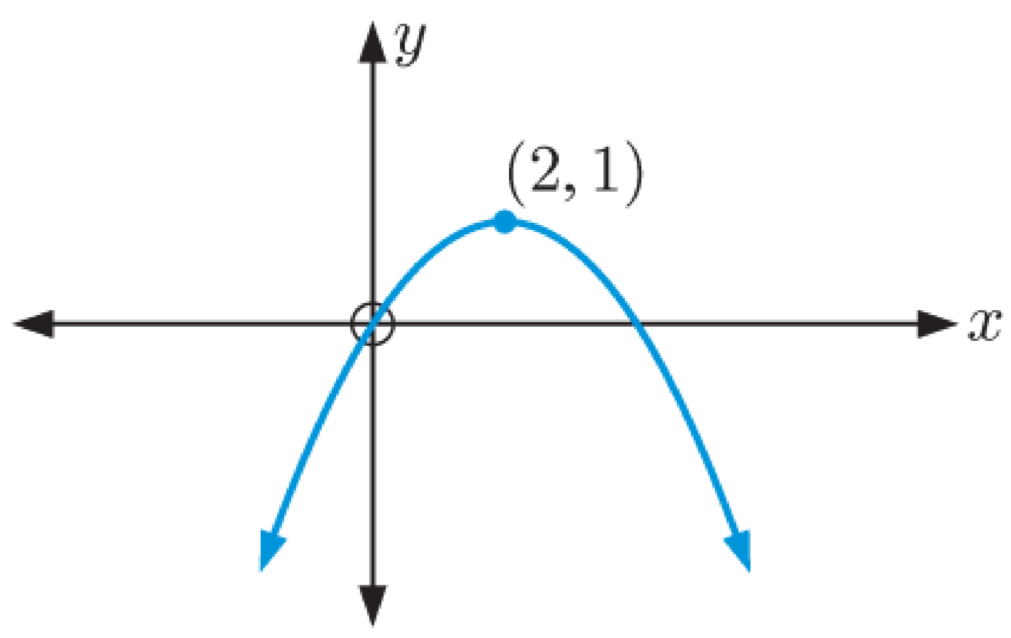
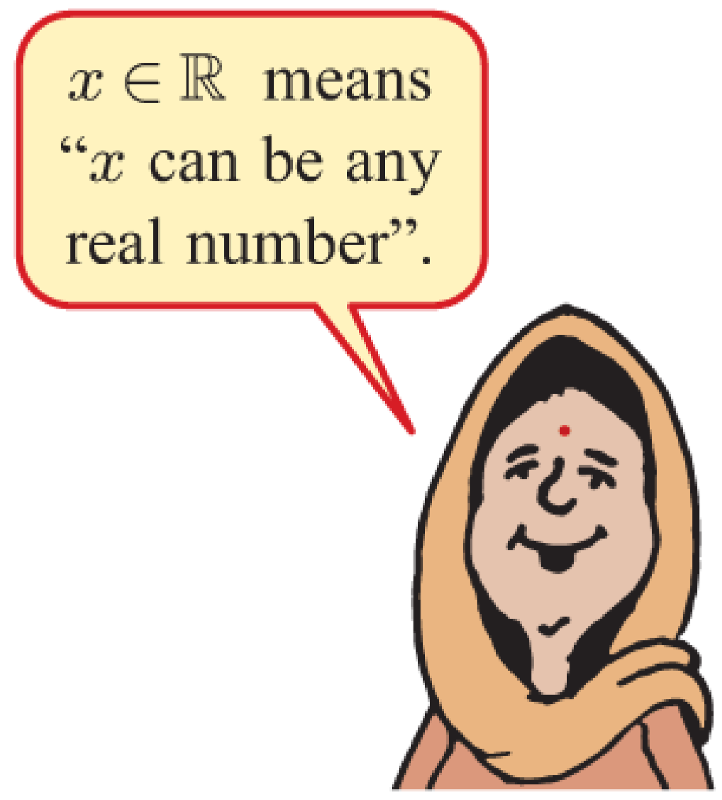



All values of $x \geqslant -1$ are included, so the domain is $\{x \mid x \geqslant -1\}$.
All values of $y \geqslant -3$ are included, so the range is $\{y \mid y \geqslant -3\}$.

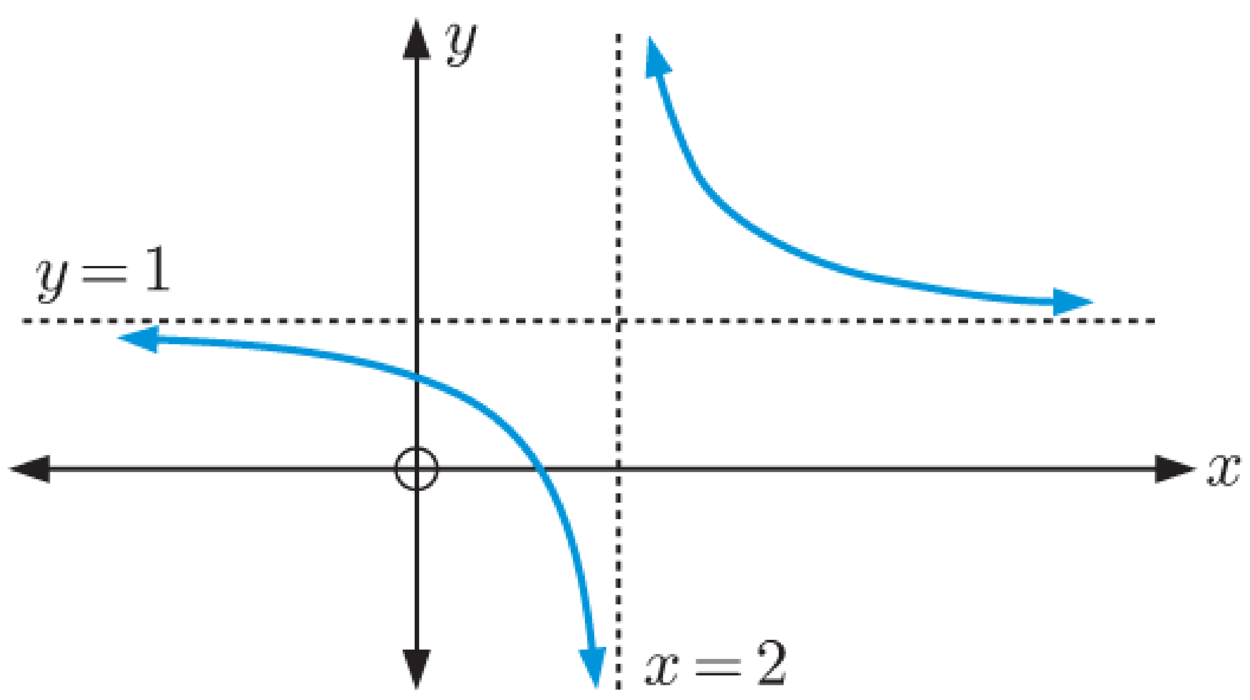
**(3)**




$x$ can take any value, so the domain is
$\{x \in \mathbb{R}\}$ or $x \in \mathbb{R}$.
$y$ cannot be $> 1$, so the range is $\{y \mid y \leqslant 1\}$.




**(4)**




$x$ can take all values except 2, so the domain is $\{x \mid x \neq 2\}$ or $x \neq 2$.
$y$ can take all values except 1, so the range is $\{y \mid y \neq 1\}$ or $y \neq 1$.

To fully describe a function, we need both a rule *and* a domain.

For example, we can specify $f(x) = x^2$ where $x \geqslant 0$.

If a domain is not specified, we use the **natural domain**, which is the largest part of $\mathbb{R}$ for which $f(x)$ is defined.

Some examples of natural domains are shown in the table opposite.

| $f(x)$ | *Natural domain* |
|---|---|
| $x^2$ | $x \in \mathbb{R}$ |
| $\sqrt{x}$ | $x \geqslant 0$ |
| $\frac{1}{x}$ | $x \neq 0$ |
| $\frac{1}{\sqrt{x}}$ | $x > 0$ |

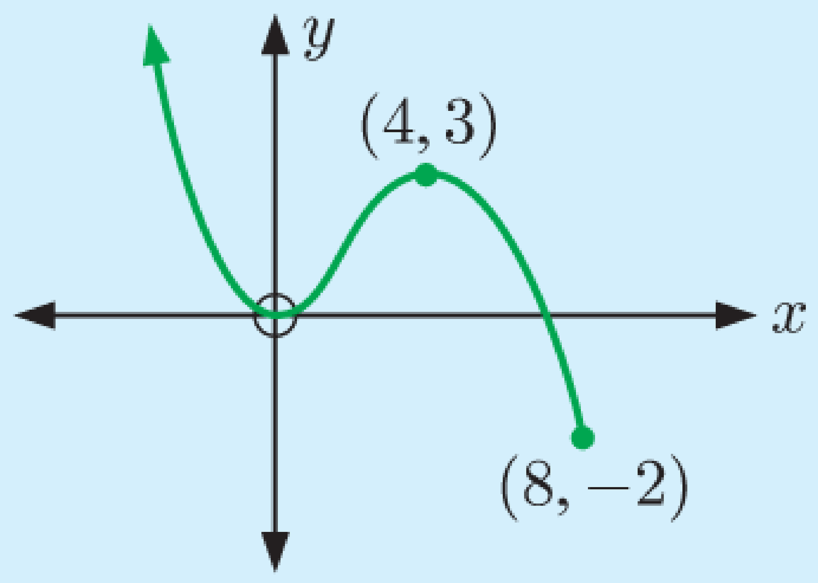
Click on the icon to obtain software for finding the natural domain and range of different functions.

DOMAIN AND RANGE

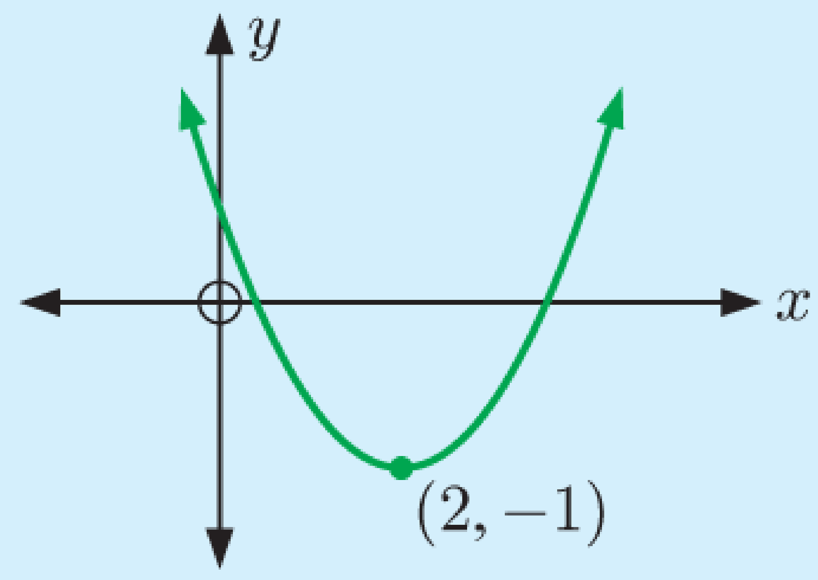
**Example 4** **Self Tutor**

For each of the following graphs, state the domain and range:

**a**




**b**




**a** Domain is $\{x \mid x \leqslant 8\}$.
Range is $\{y \mid y \geqslant -2\}$.

**b** Domain is $\{x \in \mathbb{R}\}$.
Range is $\{y \mid y \geqslant -1\}$.

## EXERCISE 15C

**1** A driver who exceeds the speed limit receives demerit points as shown in the table.

**a** Draw a graph to display this information.

**b** Is the relation a function? Explain your answer.

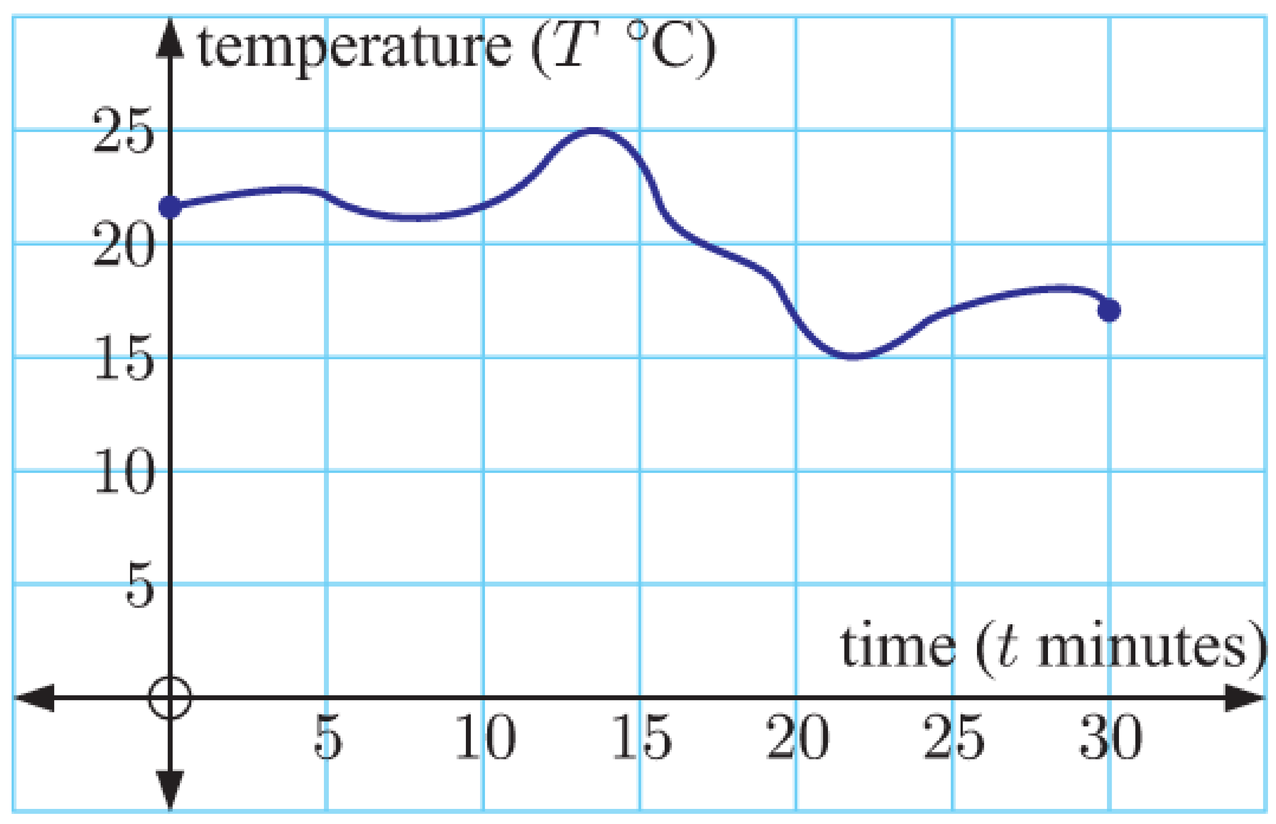
**c** Find the domain and range of the relation.

| *Amount over speed limit* ($x$ km h$^{-1}$) | *Demerit points* ($y$) |
|---|---|
| $0 < x < 10$ | 2 |
| $10 \leqslant x < 20$ | 3 |
| $20 \leqslant x < 30$ | 5 |
| $30 \leqslant x < 45$ | 7 |
| $x \geqslant 45$ | 9 |

**2** This graph shows the temperature in Barcelona over a 30 minute period as the wind shifts.

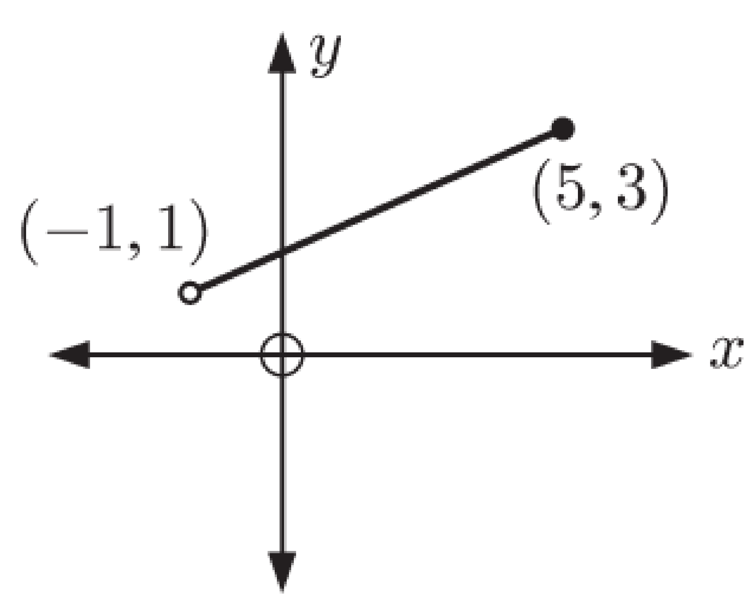
**a** Explain why a temperature graph like this must be a function.

**b** Find the domain and range of the function.

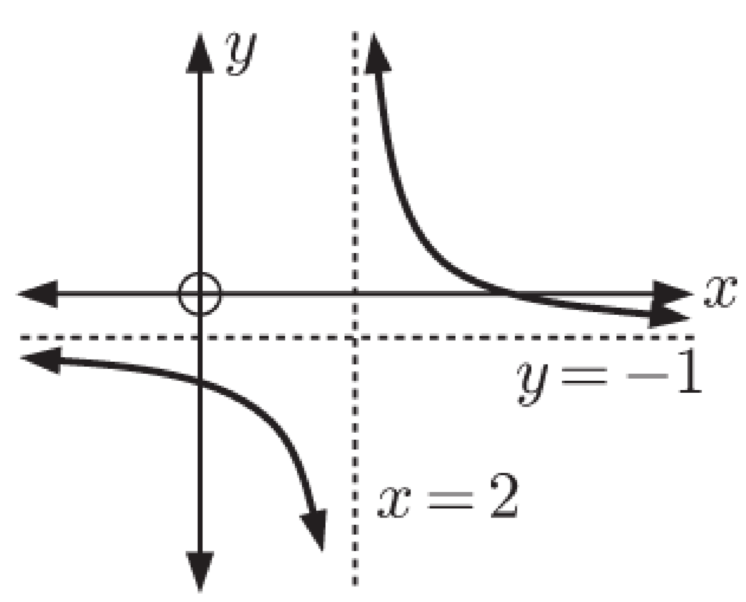



**3** For each of the following graphs, find the domain and range:

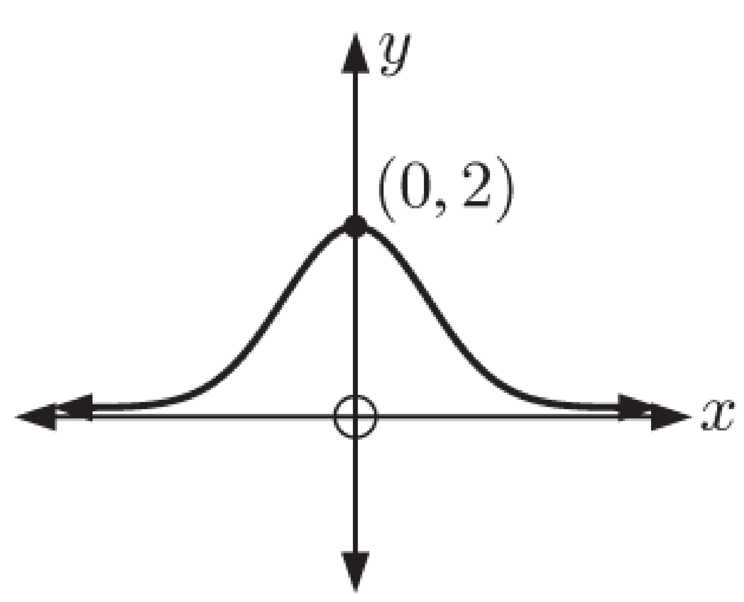
**a**



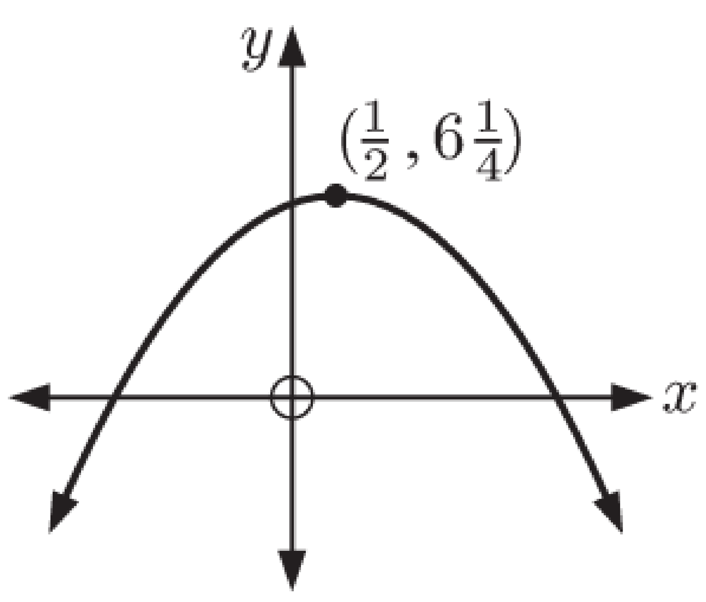

**b**



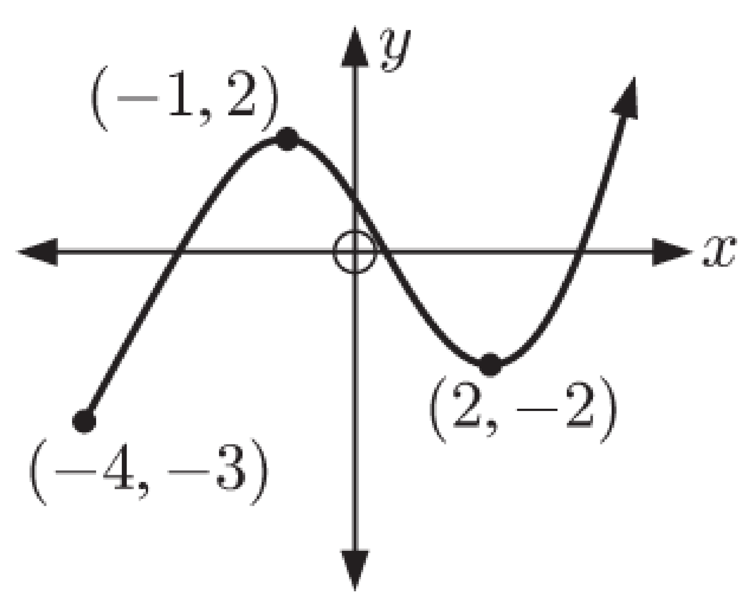

**c**



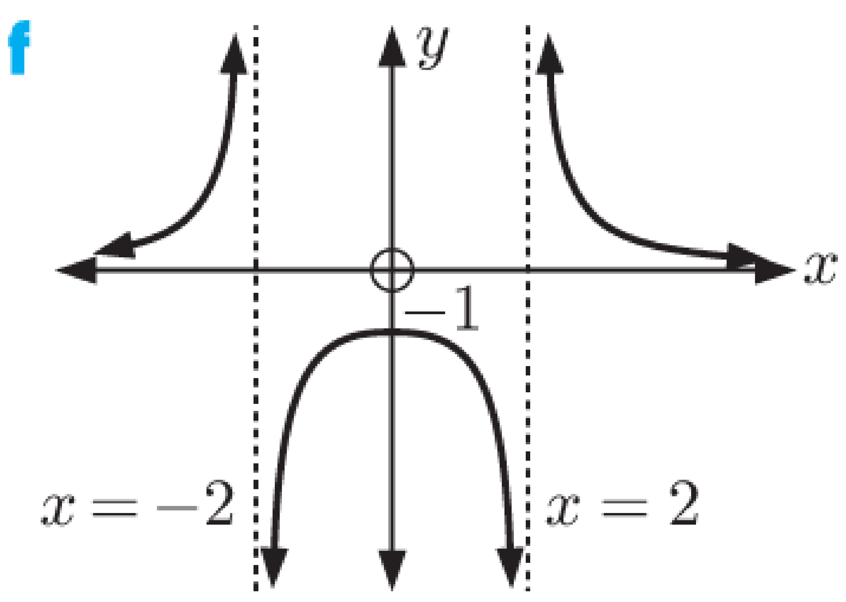

**d**




**e**




**f**



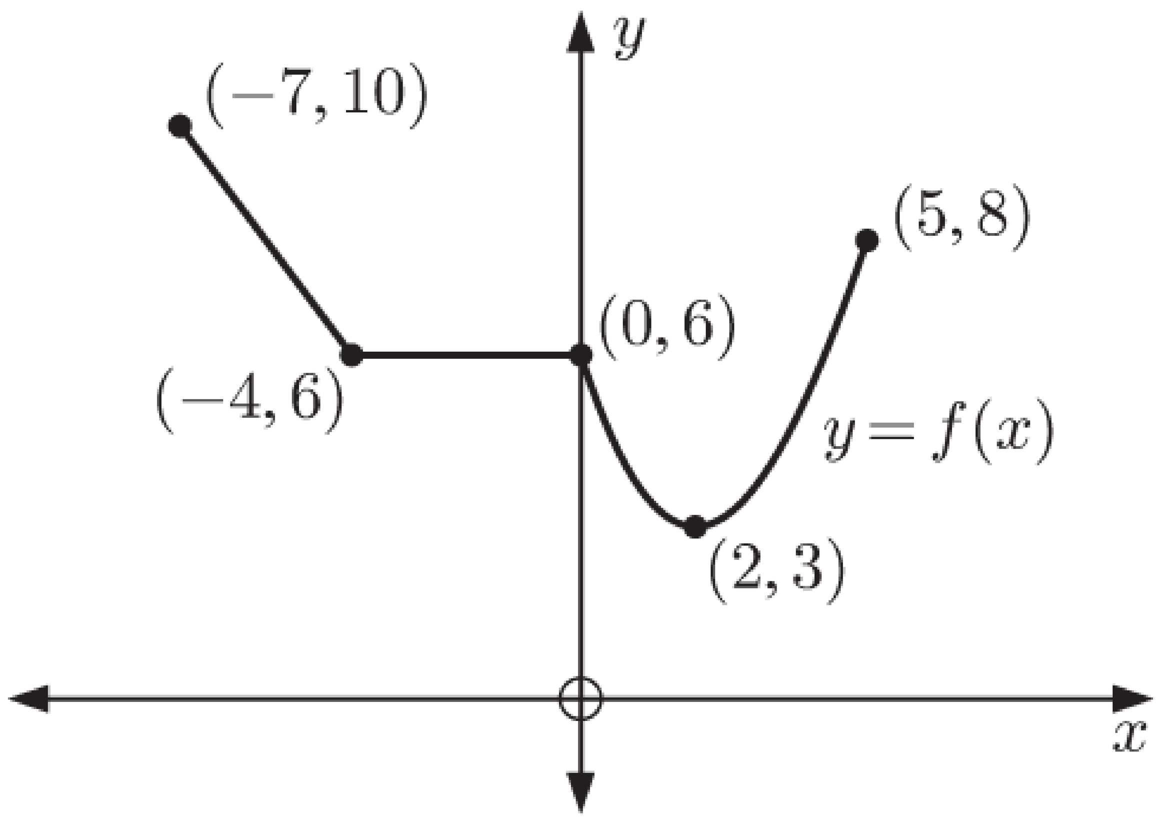
**4** Consider the graph of $y=f(x)$ alongside. Decide whether each statement is true or false:

**a** $-5$ is in the domain of $f$.

**b** $2$ is in the range of $f$.

**c** $9$ is in the range of $f$.

**d** $\sqrt{2}$ is in the domain of $f$.




**5** Use quadratic theory to find the range of each function:

**a** $y=x^2$ **b** $y=-x^2$ **c** $y=x^2+2$

**d** $y=-2(x+3)^2$ **e** $y=1-(x-2)^2$ **f** $y=(2x+1)^2+3$

**g** $y=x^2-7x+10$ **h** $y=-x^2+2x+8$ **i** $f: x \mapsto 5x-3x^2$

## Example 5

Self Tutor

State the domain and range of each of the following functions:

**a** $f(x)=\sqrt{x-5}$ **b** $f(x)=\dfrac{1}{x-5}$ **c** $f(x)=\dfrac{1}{\sqrt{x-5}}$

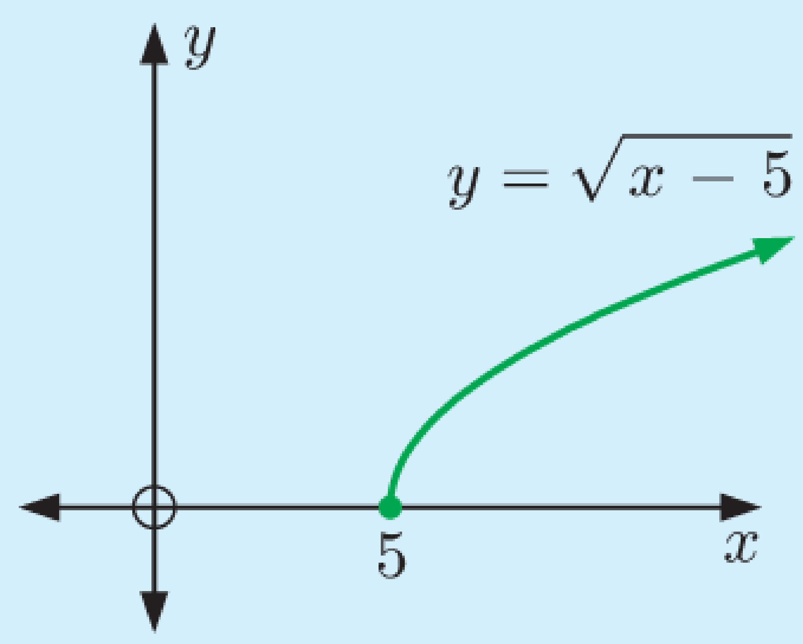
**a** $\sqrt{x-5}$ is defined when $x-5 \geqslant 0$

$\therefore\ x \geqslant 5$

$\therefore$ the domain is $\{x \mid x \geqslant 5\}$.

A square root cannot be negative.

$\therefore$ the range is $\{y \mid y \geqslant 0\}$.




**b** $\dfrac{1}{x-5}$ is defined when $x-5 \neq 0$

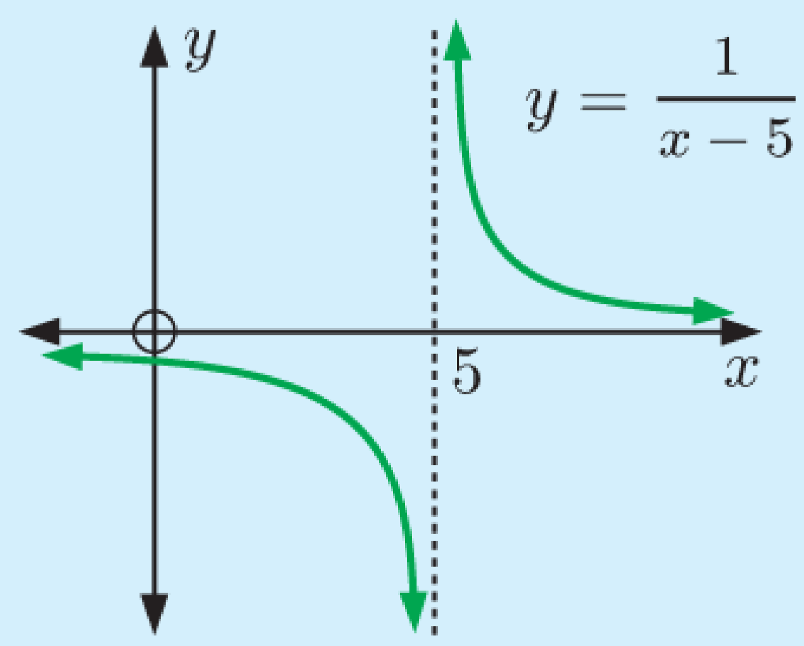
$\therefore\ x \neq 5$

$\therefore$ the domain is $\{x \mid x \neq 5\}$.

No matter how large or small $x$ is,

$y=f(x)$ is never zero.

$\therefore$ the range is $\{y \mid y \neq 0\}$.



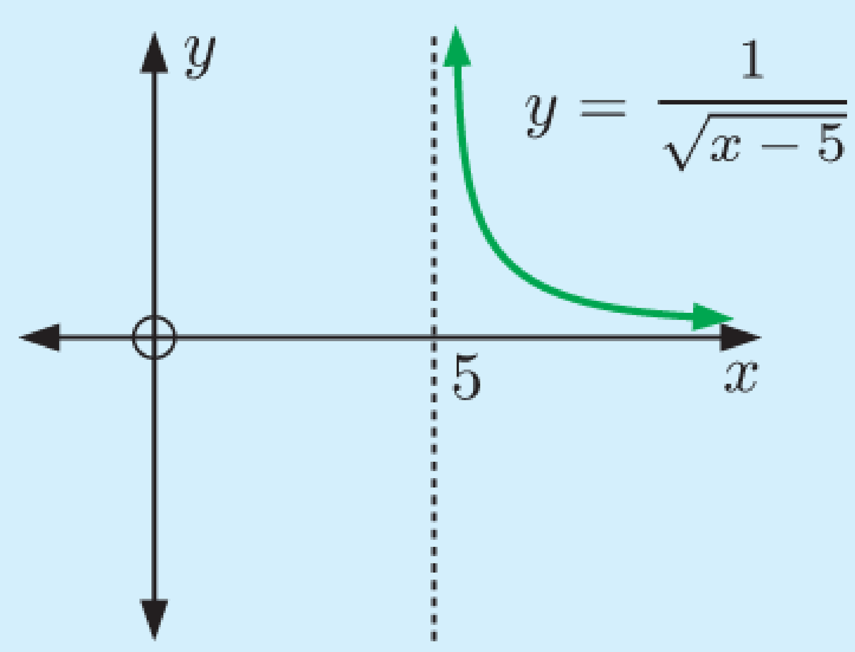

**c** $\dfrac{1}{\sqrt{x-5}}$ is defined when $x-5>0$

$\therefore\ x>5$

$\therefore$ the domain is $\{x \mid x>5\}$.

$y=f(x)$ is always positive and never zero.

$\therefore$ the range is $\{y \mid y>0\}$.



**6** Consider the function $f(x) = \sqrt{x}$.

**a** State the domain of the function.

**b** Copy and complete this table of values:

| $x$ | 0 | 1 | 4 | 9 | 16 |
|---|---|---|---|---|---|
| $f(x)$ | | | | | |

DOMAIN AND RANGE

**c** Hence sketch the graph of the function.

**d** Find the range of the function.

**7** State the domain and range of each function:

**a** $f(x) = \sqrt{x+6}$ **b** $f : x \mapsto \dfrac{1}{x^2}$ **c** $f(x) = \dfrac{1}{x+1}$

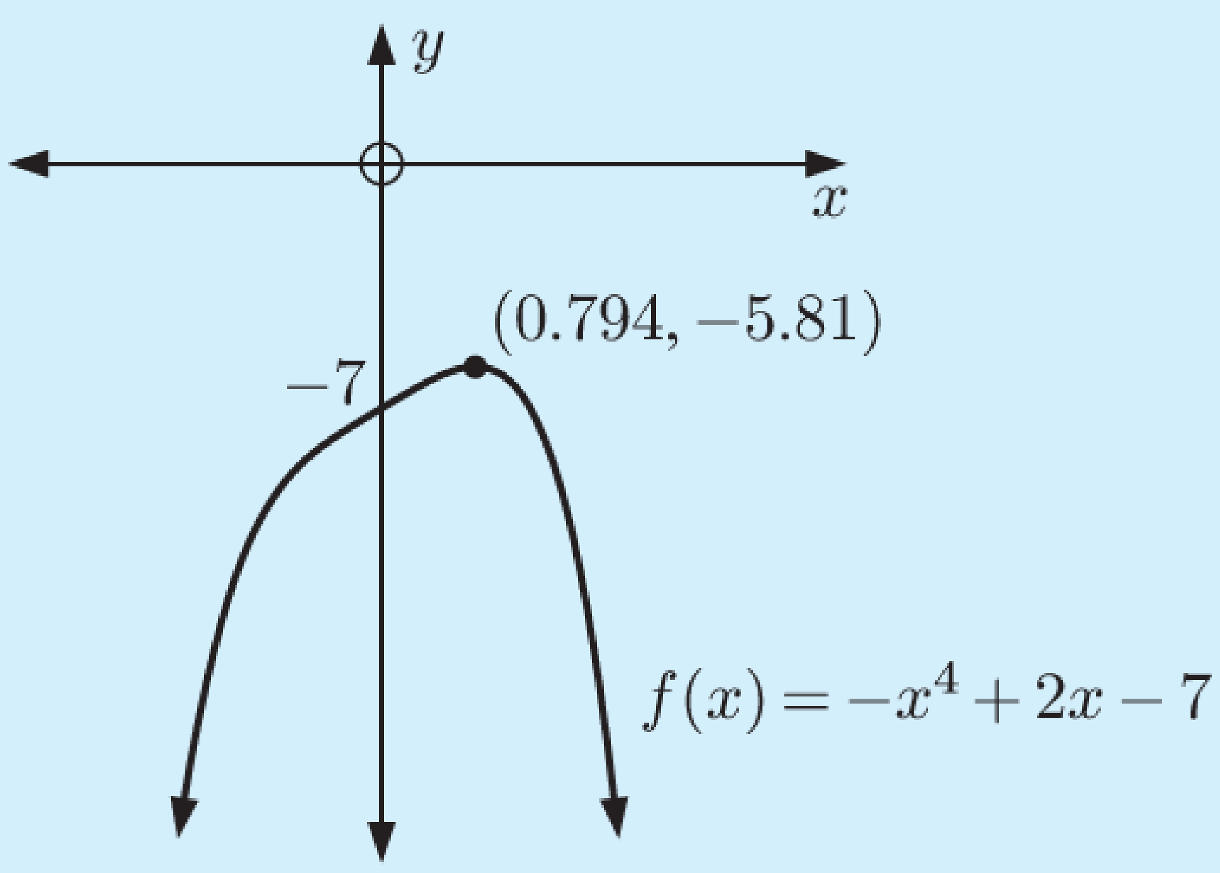
**d** $y = -\dfrac{1}{\sqrt{x}}$ **e** $f : x \mapsto \dfrac{1}{3-x}$ **f** $f : x \mapsto \sqrt{4-x}$

**Example 6** Self Tutor

Use technology to help sketch these functions. Locate any turning points. Hence state the domain and range of the function.

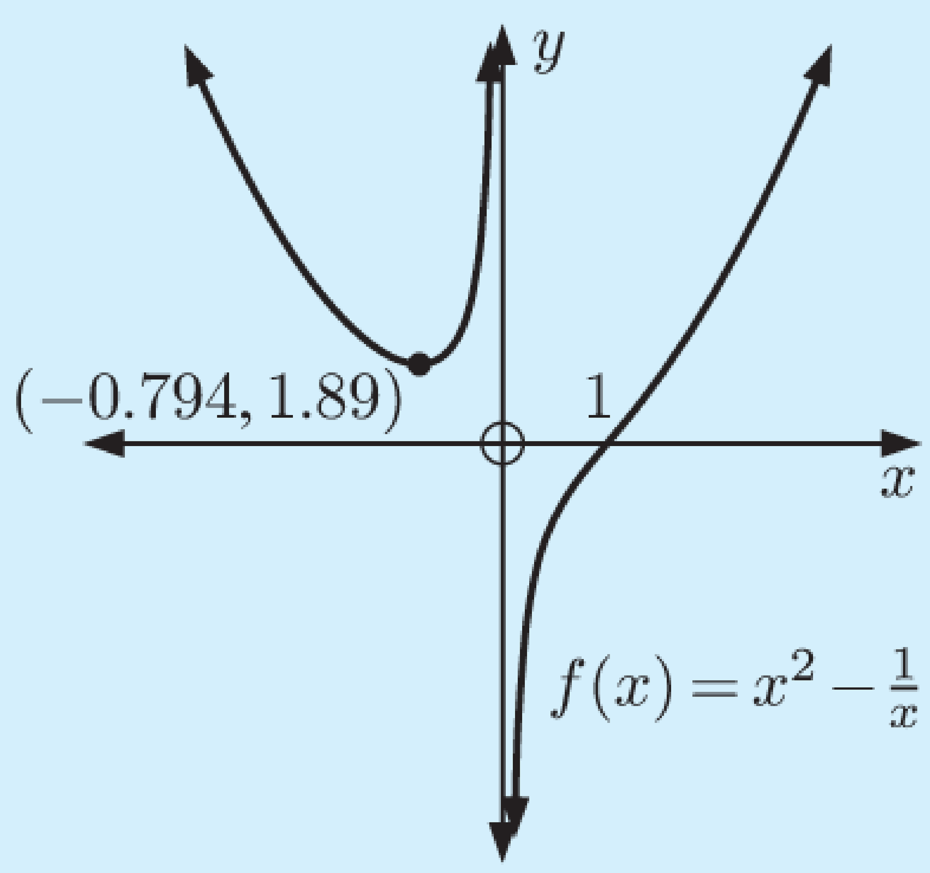
**a** $f(x) = -x^4 + 2x - 7$ **b** $f(x) = x^2 - \dfrac{1}{x}$

**a**




The domain is $\{x \mid x \in \mathbb{R}\}$.
The range is $\{y \mid y \leqslant -5.81\}$.

**b**




The domain is $\{x \mid x \neq 0\}$.
The range is $\{y \mid y \in \mathbb{R}\}$.

**8** Use technology to help sketch these functions. Locate any turning points. Hence state the domain and range of the function.

GRAPHING PACKAGE

**a** $f(x) = x^3 - 3x^2 - 9x + 10$ **b** $f(x) = x^4 + 4x^3 - 16x + 3$

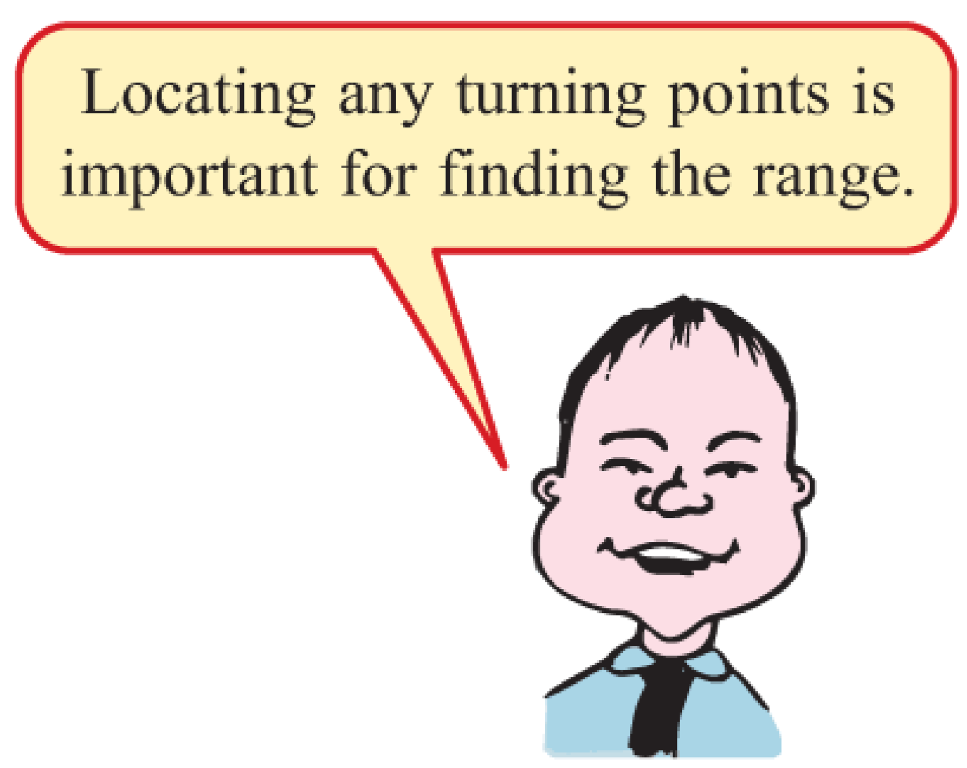
**c** $f(x) = \sqrt{x^2+4}$ **d** $f(x) = \sqrt{x^2-4}$

**e** $f(x) = \sqrt{9-x^2}$ **f** $f(x) = \dfrac{x+4}{x-2}$

**g** $f(x) = \dfrac{3x-9}{x^2-x-2}$ **h** $f(x) = x + \dfrac{1}{x}$

**i** $f(x) = x^2 + \dfrac{1}{x^2}$ **j** $f(x) = x^3 + \dfrac{1}{x^3}$

**k** $f(x) = 3^x$ **l** $f(x) = x\,2^{-x}$



**9** Use technology to sketch these functions on their given domain. Locate the points at the end(s) of the domain, as well as any turning points. Hence state the range of the function.

- **a** $y = -x^4 + 2x^3 + 5x^2 + x + 2, \quad 0 \leqslant x \leqslant 4$
- **b** $y = -2x^4 + 5x^2 + x + 2, \quad -2 \leqslant x \leqslant 2$
- **c** $y = \dfrac{1}{1 + 2^{-x}}, \quad x > 0$

**10** The function $f(x) = \sqrt{x^2 + 5x + k}$ has natural domain $x \in \mathbb{R}$.

- **a** Find the possible values of $k$.
- **b** Find the range of $f(x)$ in terms of $k$.

**11** State the domain and range of each relation:

- **a** $\{(x, y) \mid x^2 + y^2 = 4\}$
- **b** $\{(x, y) \mid x^2 + y^2 = 4, \ x \in \mathbb{Z}\}$

# D RATIONAL FUNCTIONS

Linear and quadratic functions are the first members of a family called the **polynomials**. The polynomials can all be written in the form $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$.

When a polynomial is divided by another polynomial, we call it a **rational function**.

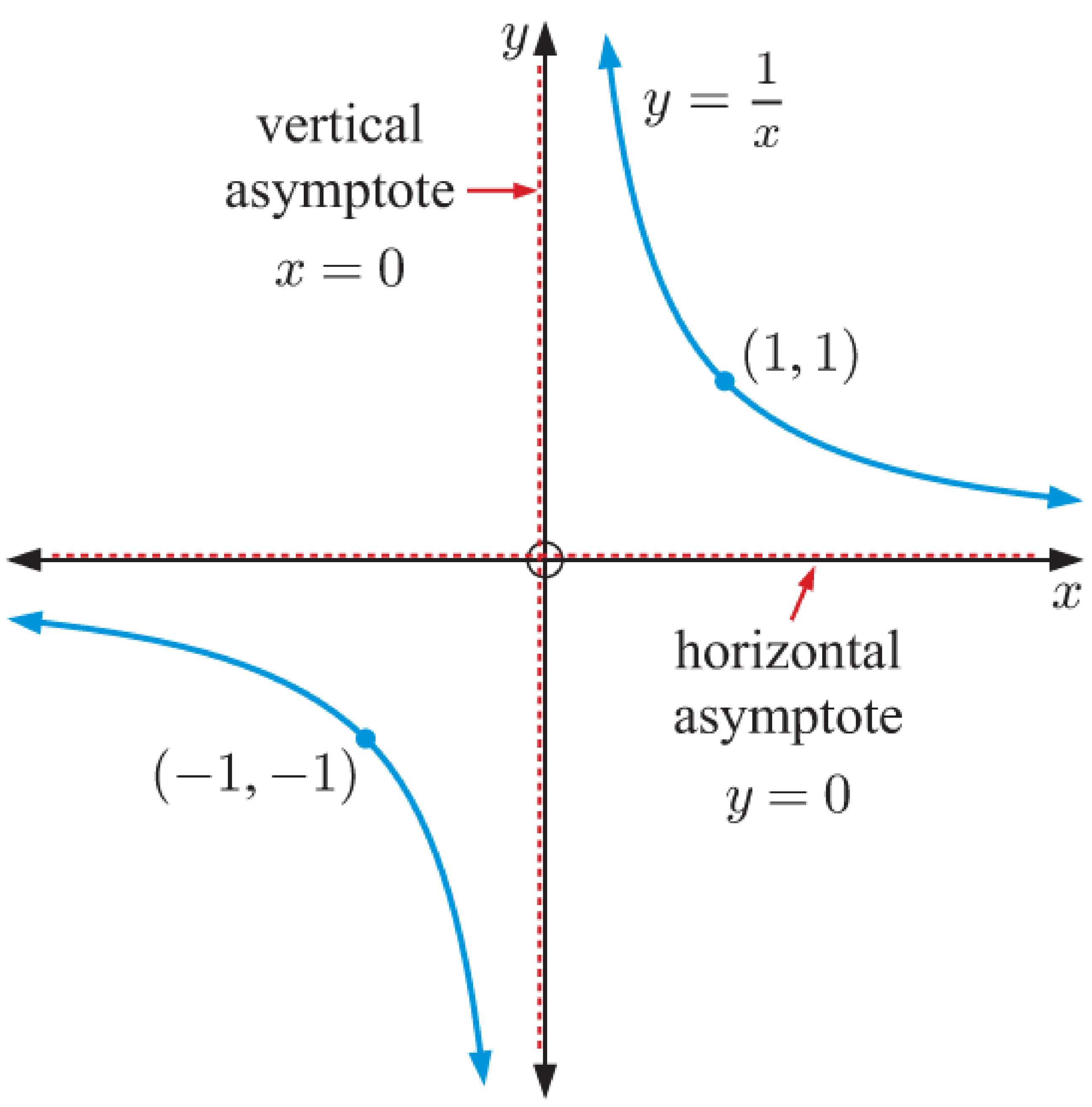
In this Section we consider only the simplest cases of a linear function divided by another linear function.

## RECIPROCAL FUNCTIONS

A **reciprocal function** is a function of the form $y = \dfrac{k}{x}, \ k \neq 0$.

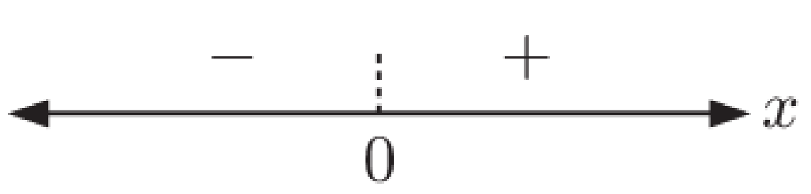
The graph of a reciprocal function is called a **rectangular hyperbola**.

The simplest example of a reciprocal function is $f(x) = \dfrac{1}{x}$. Its graph is shown below.




Notice that:

- The graph has two branches.
- $y = \dfrac{1}{x}$ is undefined when $x = 0$, so the domain is $\{x \mid x \neq 0\}$.
  On a sign diagram, we indicate this value with a dashed line.




- The graph includes two **asymptotes**, which are lines the graph approaches but never reaches.
  - $x = 0$ is a **vertical asymptote**.

    We write: as $x \to 0^-$, $\dfrac{1}{x} \to -\infty$

    as $x \to 0^+$, $\dfrac{1}{x} \to \infty$

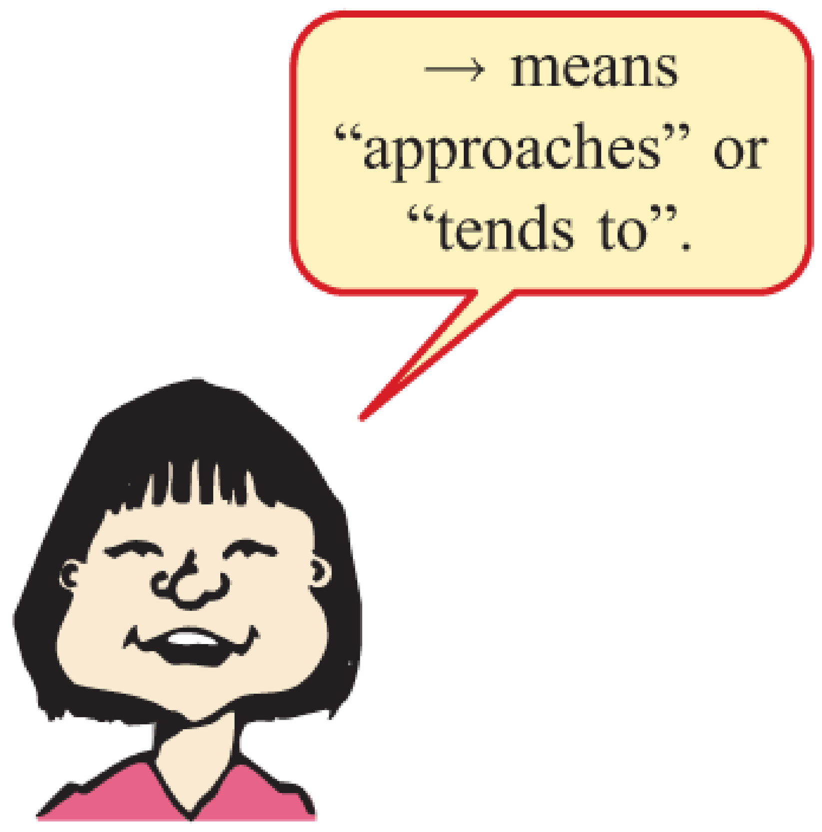
    When "as $x \to 0^+$, $\dfrac{1}{x} \to \infty$" is read out loud, we say "as $x$ tends to zero from the right, $\dfrac{1}{x}$ tends to infinity."

- $y = 0$ is a **horizontal asymptote**.

  We write: as $x \to \infty$, $\frac{1}{x} \to 0^+$

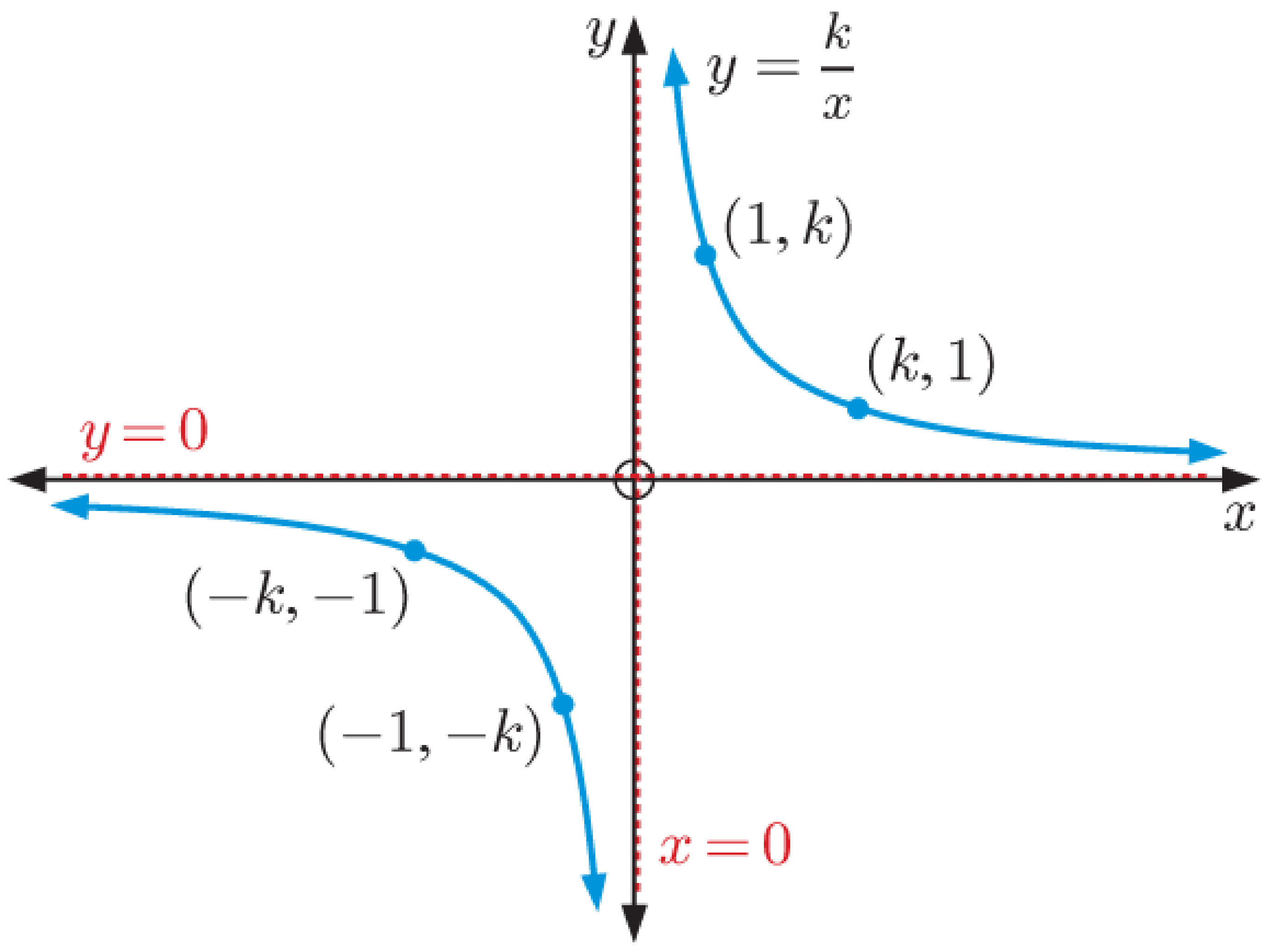
  as $x \to -\infty$, $\frac{1}{x} \to 0^-$

  When "as $x \to \infty$, $\frac{1}{x} \to 0^+$" is read out loud, we say "as $x$ tends to infinity, $\frac{1}{x}$ tends to zero from above."



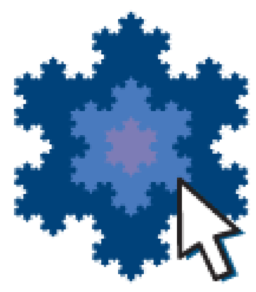

When sketching the graph of a reciprocal function, it is useful to determine some points which lie on the graph.

The reciprocal function $y = \frac{k}{x}$ passes through the points $(1, k)$, $(k, 1)$, $(-1, -k)$, and $(-k, -1)$.




## EXERCISE 15D.1

**1** **a** Sketch the graphs of $y = \frac{1}{x}$, $y = \frac{2}{x}$, and $y = \frac{4}{x}$ on the same set of axes.




**b** For the function $y = \frac{k}{x}$, $k > 0$:

  **i** Describe the effect of varying $k$.

  **ii** State the quadrants in which the graph lies.

  **iii** Draw a sign diagram for the function.

**2** **a** Sketch the graphs of $y = -\frac{1}{x}$, $y = -\frac{2}{x}$, and $y = -\frac{4}{x}$ on the same set of axes.

**b** For the function $y = \frac{k}{x}$, $k < 0$:

  **i** Describe the effect of varying $k$.

  **ii** State the quadrants in which the graph lies.

  **iii** Draw a sign diagram for the function.

**3** For the reciprocal function $y = \frac{k}{x}$, $k \neq 0$, state:

**a** the domain

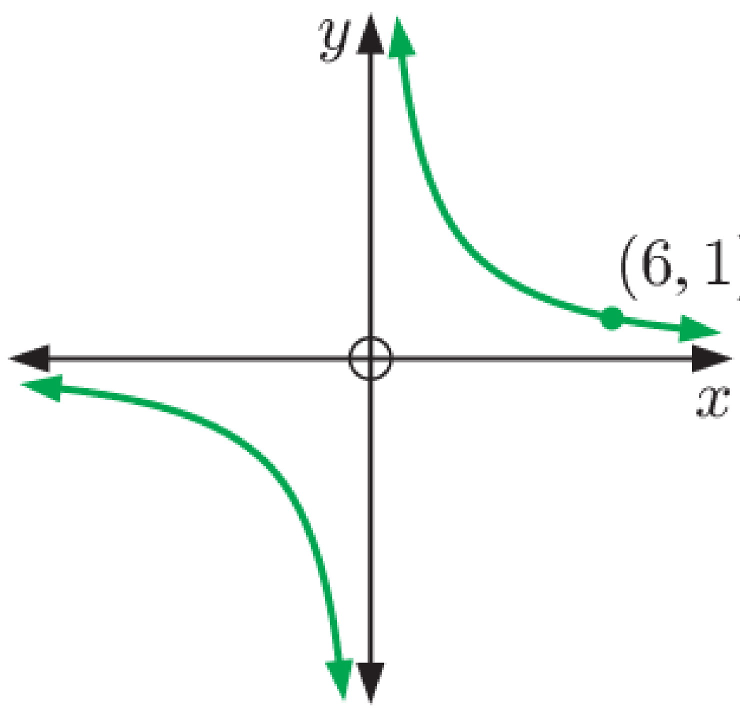
**b** the range

**c** the vertical asymptote

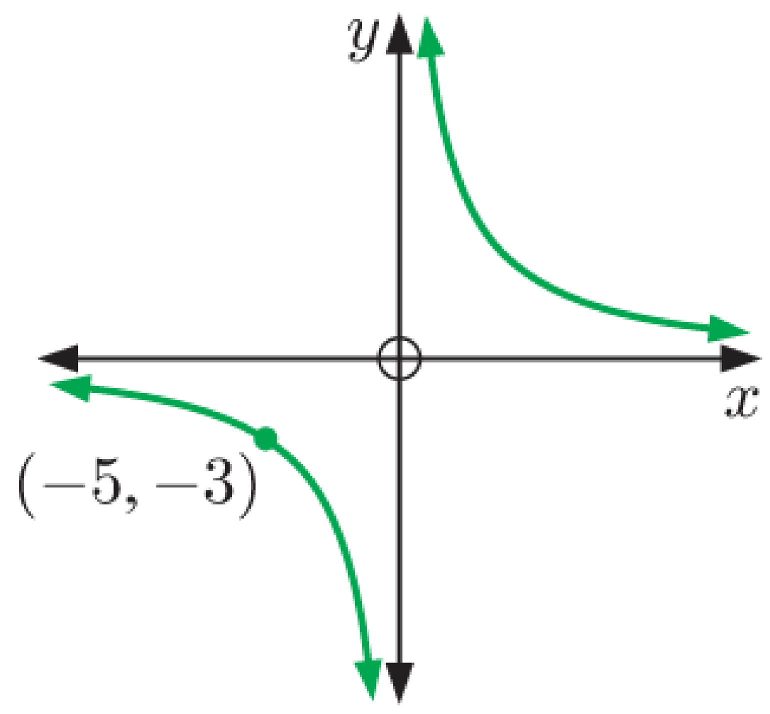
**d** the horizontal asymptote.

**4** Determine the equation of each reciprocal function:

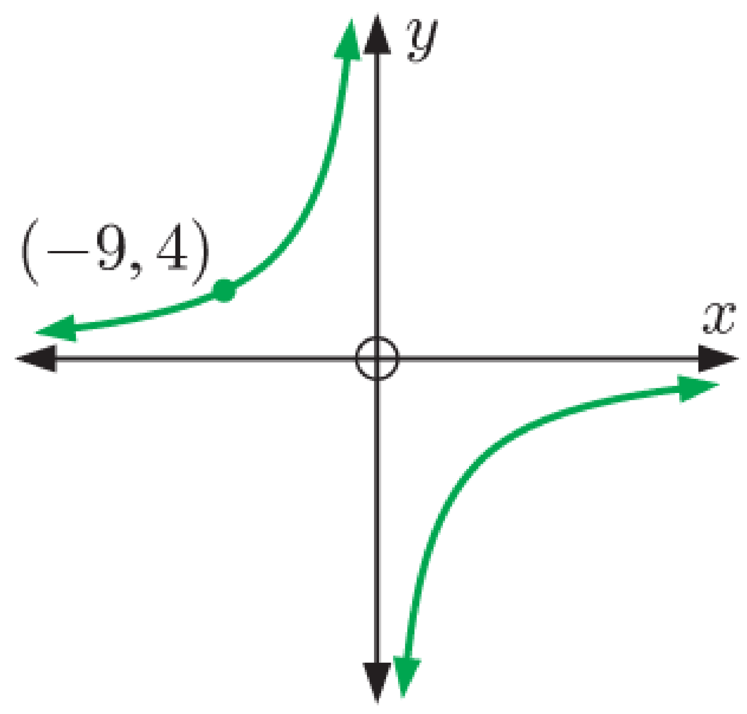
**a**




**b**




**c**




## RATIONAL FUNCTIONS OF THE FORM $y = \dfrac{ax+b}{cx+d}$, $c \neq 0$

We now consider the rational functions which result when a linear function is divided by another linear function.

The graphs of these rational functions also have horizontal and vertical asymptotes.

### INVESTIGATION — RATIONAL FUNCTIONS

**What to do:**

GRAPHING PACKAGE

**1** Use technology to examine graphs of the following functions. For each graph:

**i** State the domain. **ii** Write down the equations of the asymptotes.

**a** $y = -1 + \dfrac{3}{x-2}$ **b** $y = \dfrac{3x+1}{x+2}$ **c** $y = \dfrac{2x-9}{3-x}$

**2** Experiment with functions of the form $y = \dfrac{b}{cx+d} + a$ where $b, c \neq 0$.

For an equation of this form, state the equation of:

**a** the horizontal asymptote **b** the vertical asymptote.

**3** Experiment with functions of the form $y = \dfrac{ax+b}{cx+d}$ where $c \neq 0$.

**a** For an equation of this form, state the equation of the vertical asymptote.

**b** Can you see how to quickly write down the equation of the horizontal asymptote? Explain your answer.

- For a function written in the form $y = \dfrac{b}{cx+d} + a$ where $b, c \neq 0$:
  - the vertical asymptote is $x = -\dfrac{d}{c}$
  - the horizontal asymptote is $y = a$.
- For a function written in the form $y = \dfrac{ax+b}{cx+d}$ where $c \neq 0$:
  - the vertical asymptote is $x = -\dfrac{d}{c}$
  - the horizontal asymptote is $y = \dfrac{a}{c}$.

**Example 7** **Self Tutor**

Consider the function $y = \frac{6}{x-2} + 4$.

**a** Find the asymptotes of the function.

**b** Find the axes intercepts.

**c** Use technology to help sketch the function, including the features found in **a** and **b**.

**a** The vertical asymptote is $x = 2$.
The horizontal asymptote is $y = 4$.

**b** When $y = 0$, $\frac{6}{x-2} = -4$

$\therefore\ -4(x-2) = 6$

$\therefore\ -4x + 8 = 6$

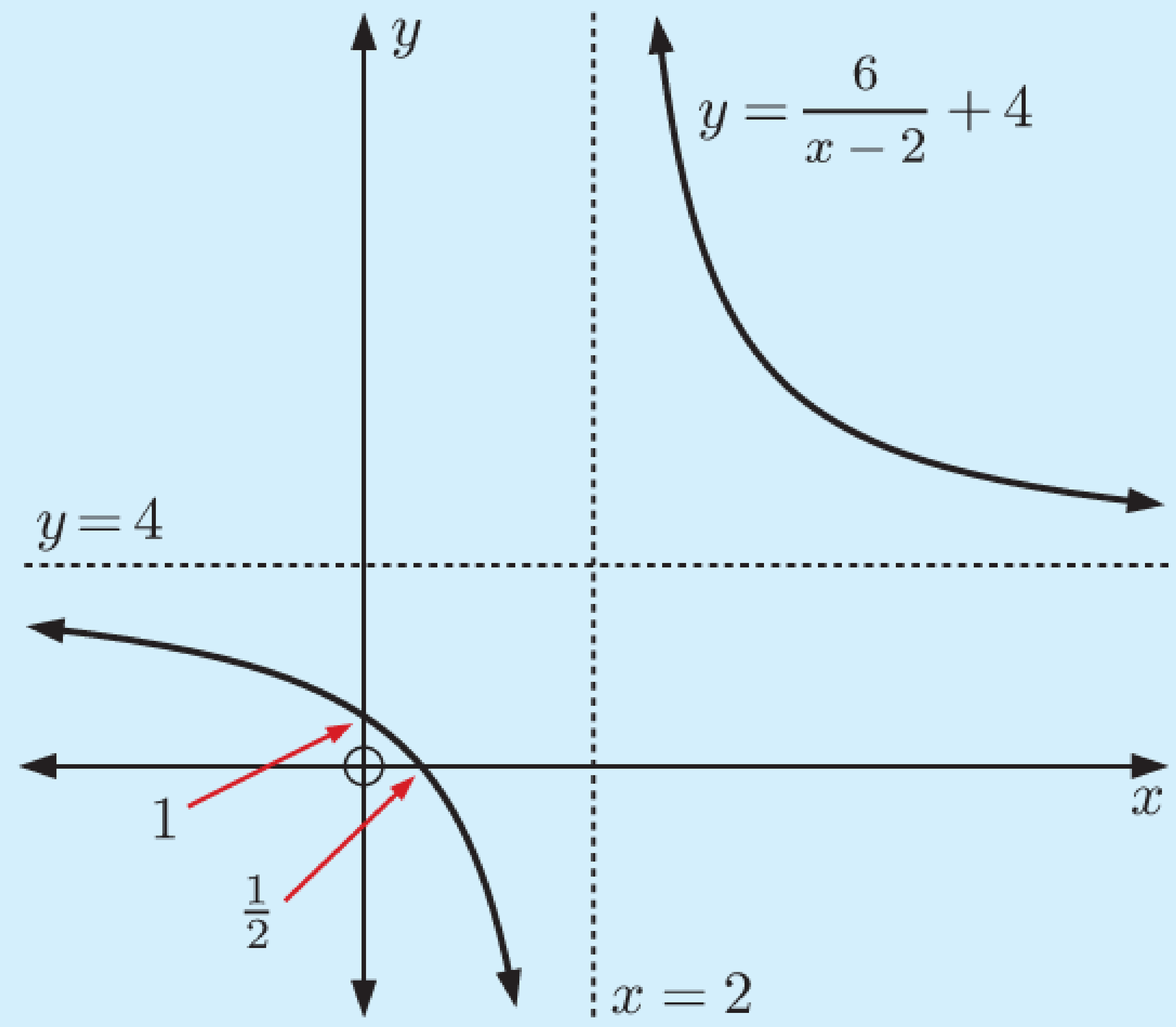
$\therefore\ -4x = -2$

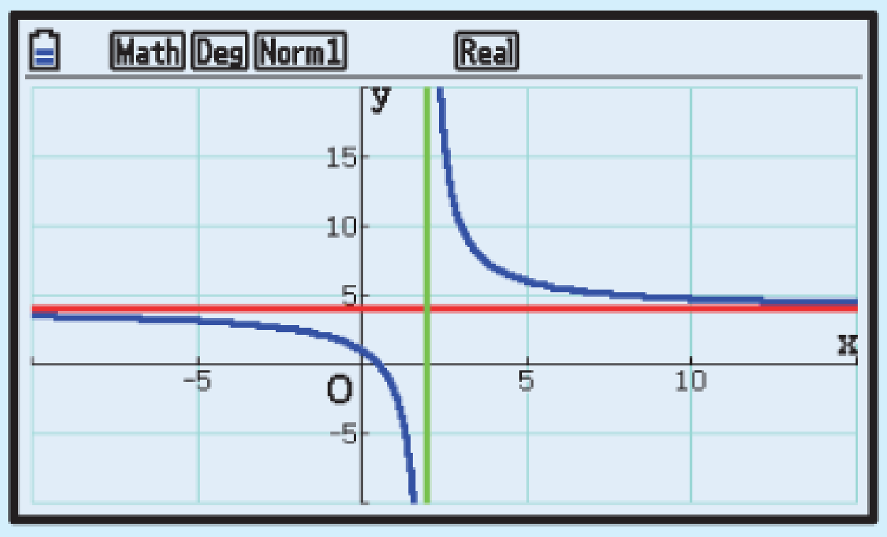
$\therefore\ x = \frac{1}{2}$

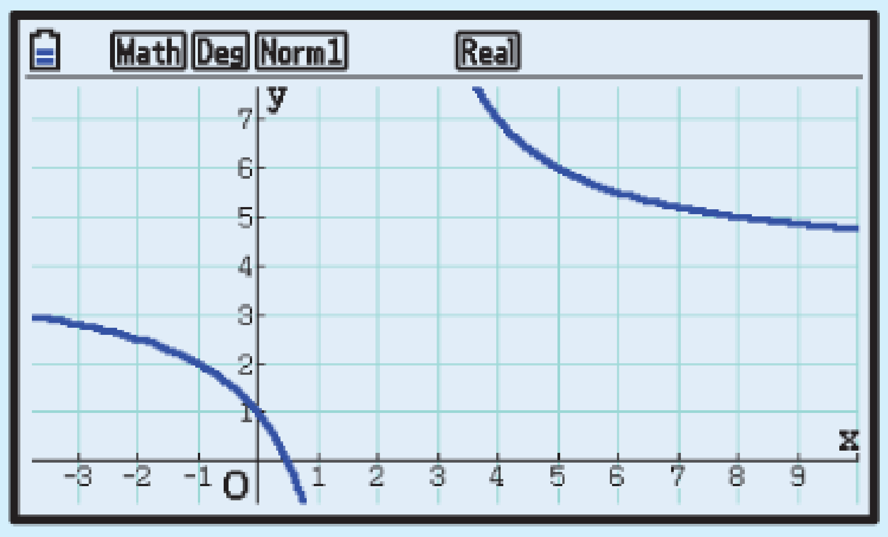
When $x = 0$, $y = \frac{6}{-2} + 4 = 1$

So, the $x$-intercept is $\frac{1}{2}$ and the $y$-intercept is 1.

**c**





## EXERCISE 15D.2

**1** For each of the following functions:

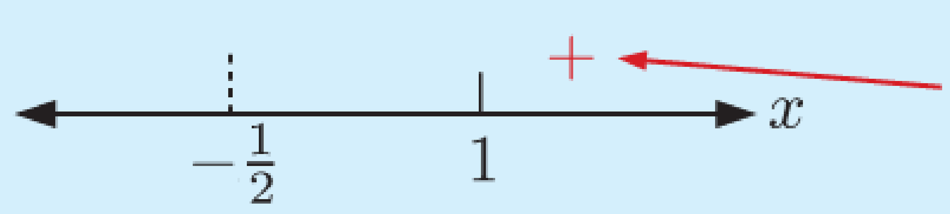
- **i** Find the equations of the asymptotes.
- **ii** State the domain and range.
- **iii** Find the axes intercepts.
- **iv** Discuss the behaviour of the function as it approaches its asymptotes.
- **v** Sketch the graph of the function.

**a** $f(x) = \frac{3}{x-2}$ **b** $f : x \mapsto 2 + \frac{1}{x-3}$ **c** $f(x) = 2 - \frac{3}{x+1}$

**Example 8** **Self Tutor**

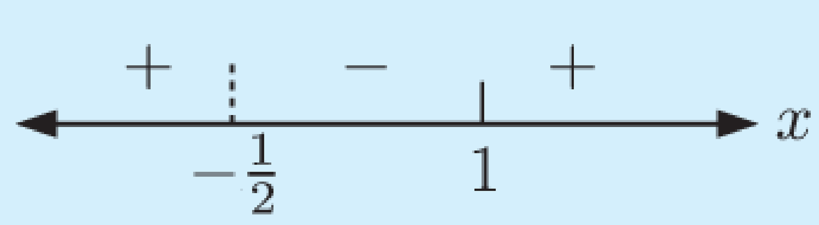
Draw a sign diagram for $\frac{x-1}{2x+1}$.

$\frac{x-1}{2x+1}$ is zero when $x = 1$ and undefined when $x = -\frac{1}{2}$.



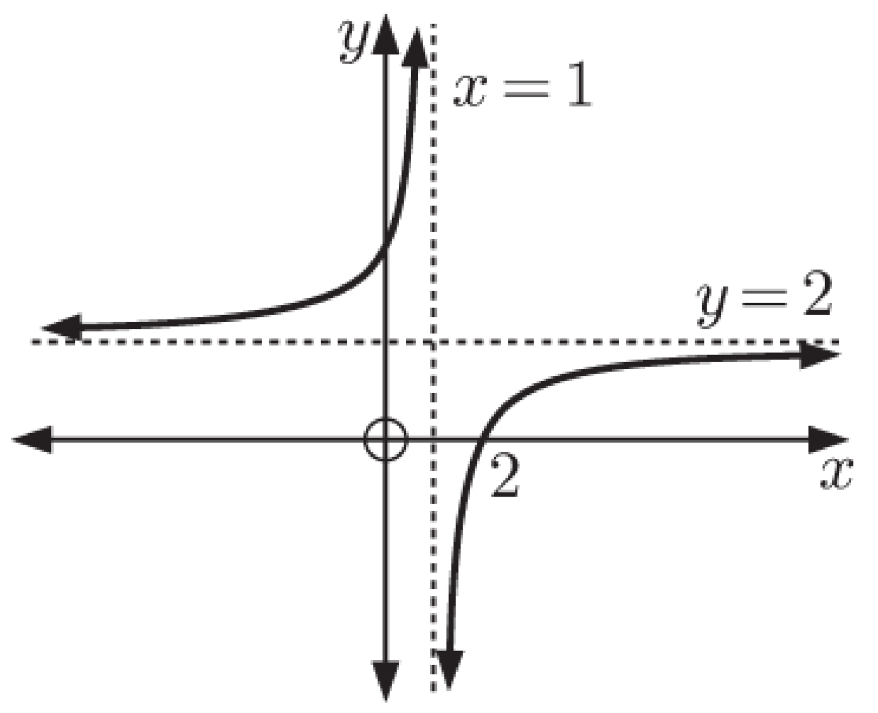

When $x = 10$, $\frac{x-1}{2x+1} = \frac{9}{21} > 0$

Since $(x-1)$ and $(2x+1)$ are single factors, the signs alternate.

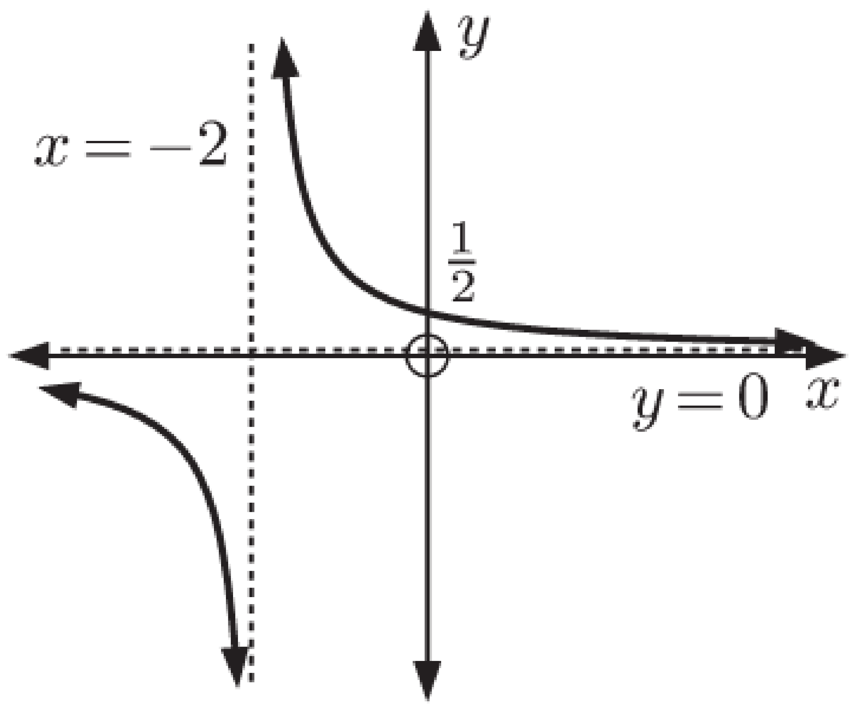


**2** Draw the sign diagram for:

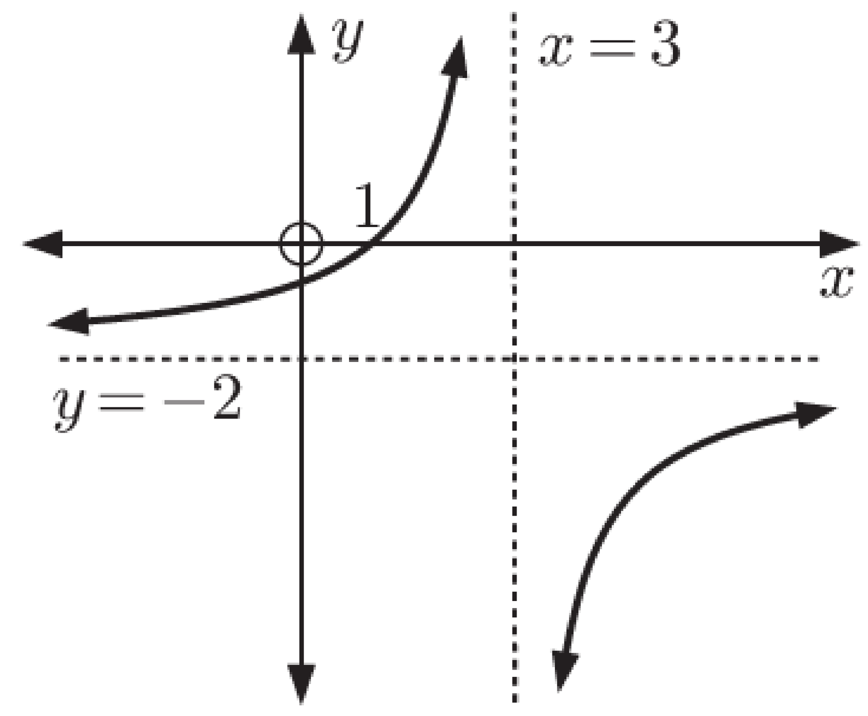
**a**




**b**




**c**




**3** Draw a sign diagram for:

**a** $\dfrac{x+2}{x-1}$ **b** $\dfrac{x}{x+3}$ **c** $\dfrac{x+1}{x+5}$ **d** $\dfrac{x-2}{2x+1}$

**e** $\dfrac{2x+3}{4-x}$ **f** $\dfrac{4x-1}{2-x}$ **g** $\dfrac{3x}{x-2}$ **h** $\dfrac{-8x}{3-x}$

**Example 9** **Self Tutor**

Consider the function $f(x) = \dfrac{2x+1}{x-1}$.

**a** Find the vertical asymptote of the function. **b** Find the axes intercepts.

**c** Rearrange the function to find the horizontal asymptote.

**d** Draw a sign diagram of the function.

**e** Hence discuss the behaviour of the function near the asymptotes.

**f** Sketch the function, showing the features you have found.

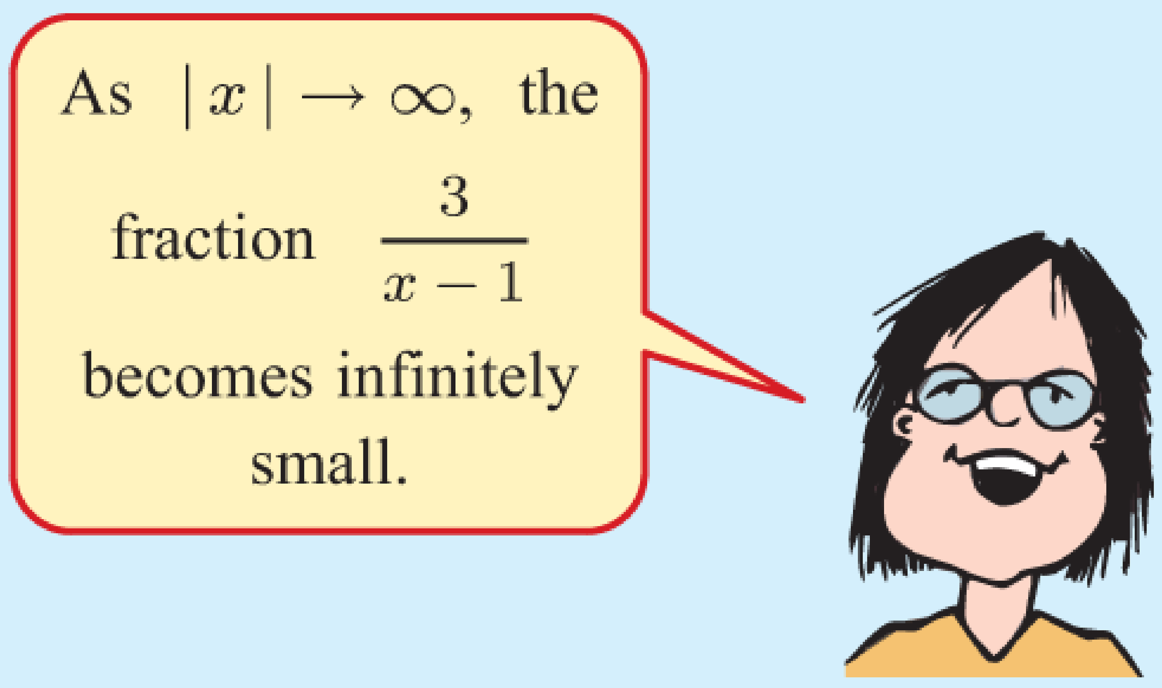
**a** The vertical asymptote is $x = 1$.

**b** $f(0) = \dfrac{1}{-1} = -1$, so the $y$-intercept is $-1$.

$f(x) = 0$ when $2x + 1 = 0$

$\therefore\ x = -\frac{1}{2}$

$\therefore$ the $x$-intercept is $-\frac{1}{2}$.



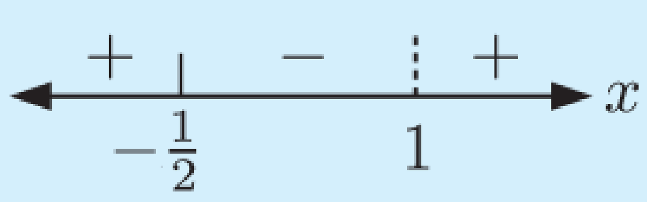

**c** $f(x) = \dfrac{2x+1}{x-1}$

$= \dfrac{2(x-1)+3}{x-1}$

$= 2 + \dfrac{3}{x-1}$

$\therefore$ the horizontal asymptote is $y = 2$.

**d**



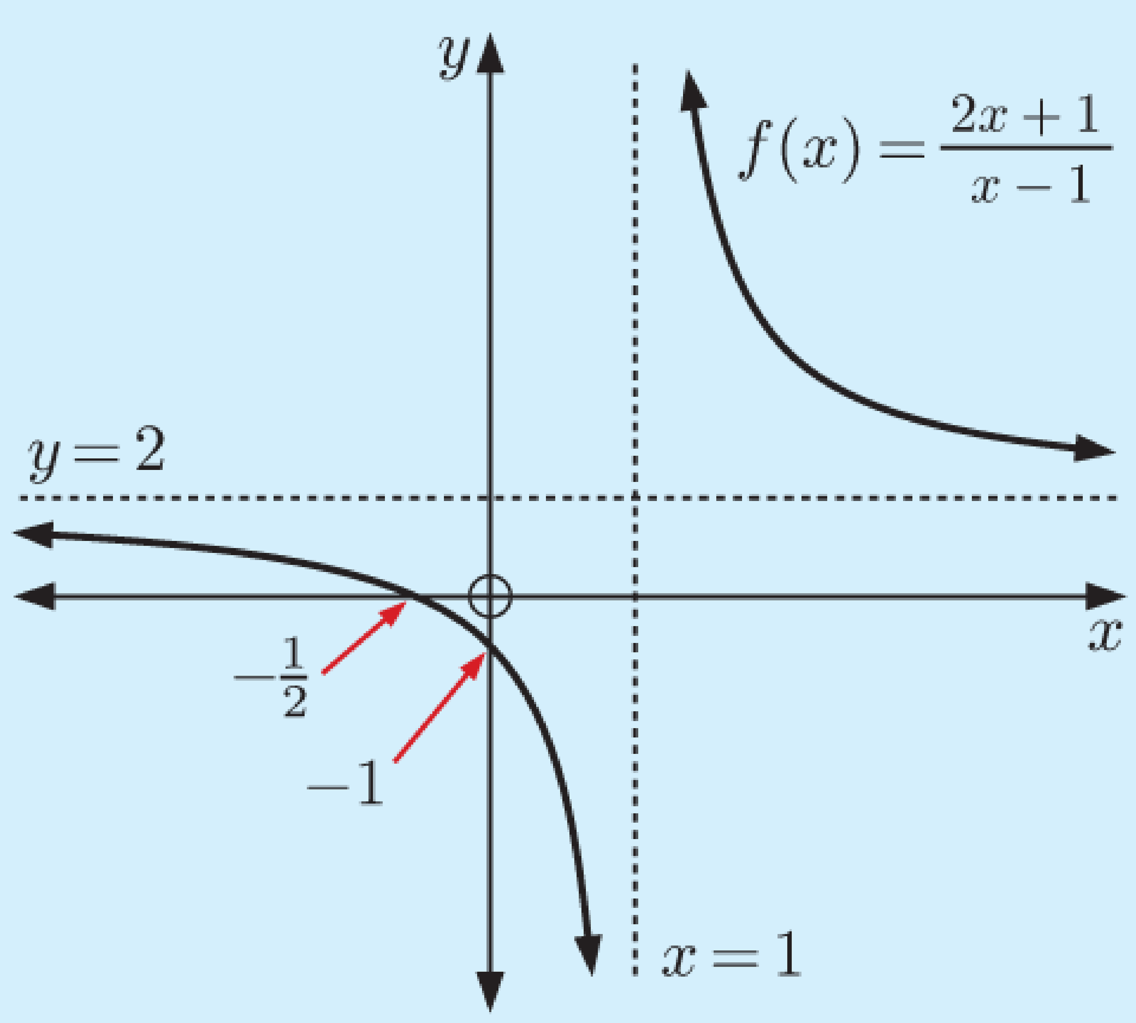

**e** As $x \to 1^-$, $f(x) \to -\infty$

As $x \to 1^+$, $f(x) \to \infty$

As $x \to -\infty$, $f(x) \to 2^-$

As $x \to \infty$, $f(x) \to 2^+$

**f**



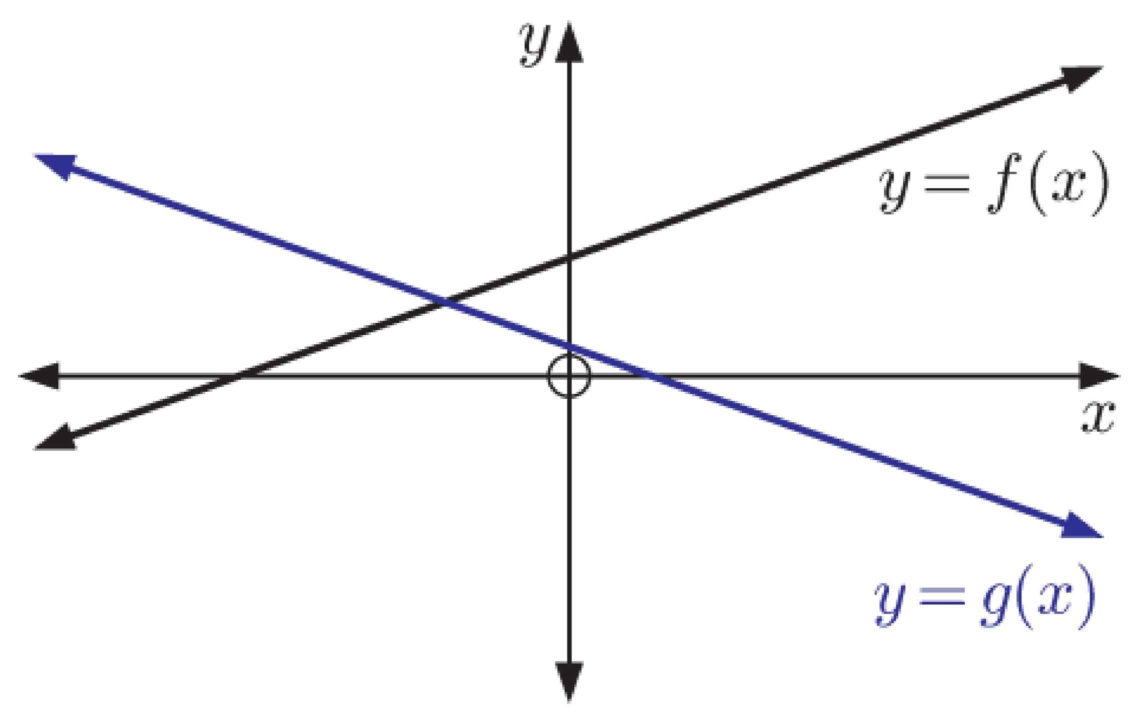
**4** For each of the following functions:

**i** Find the equation of the vertical asymptote.
**ii** Find the axes intercepts.
**iii** Rearrange the function to find the horizontal asymptote.
**iv** Draw a sign diagram of the function.
**v** Hence discuss the behaviour of the function near its asymptotes.
**vi** Sketch the graph of the function.

**a** $f(x) = \dfrac{x}{x-1}$ **b** $f : x \mapsto \dfrac{x+3}{x-2}$ **c** $f(x) = \dfrac{3x-1}{x+2}$

**d** $f(x) = -\dfrac{2x+1}{x-3}$ **e** $f : x \mapsto \dfrac{2x+4}{3-x}$ **f** $f(x) = \dfrac{x+3}{2x-1}$

**5** The graph alongside shows the linear functions $f(x)$ and $g(x)$.
Copy the graph, and on the same set of axes, graph $y = \dfrac{f(x)}{g(x)}$. Indicate clearly where any $x$-intercepts and asymptotes occur.




**6** Consider the function $y = \dfrac{ax+b}{cx+d}$, where $a, b, c, d$ are constants and $c \neq 0$.

**a** State the domain of the function.
**b** State the equation of the vertical asymptote.
**c** Find the axes intercepts.
**d** Show that for $c \neq 0$, $\dfrac{ax+b}{cx+d} = \dfrac{a}{c} + \dfrac{b - \frac{ad}{c}}{cx+d}$.

Hence explain why the horizontal asymptote is $y = \dfrac{a}{c}$.

## ACTIVITY

Click on the icon to run a card game for rational functions.

GAME

# E COMPOSITE FUNCTIONS

Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the **composite function** of $f$ and $g$ will convert $x$ into $f(g(x))$.

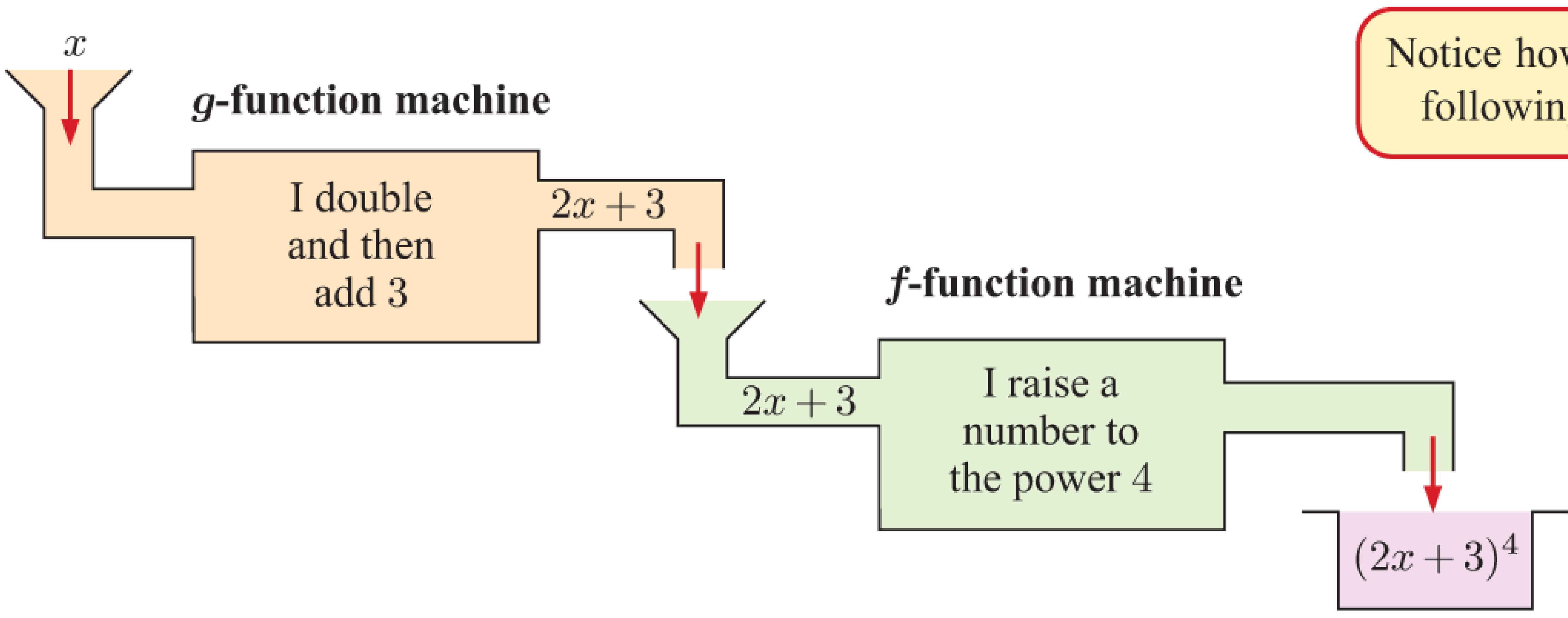
$f \circ g$ is used to represent the composite function of $f$ and $g$. It means "$f$ following $g$".

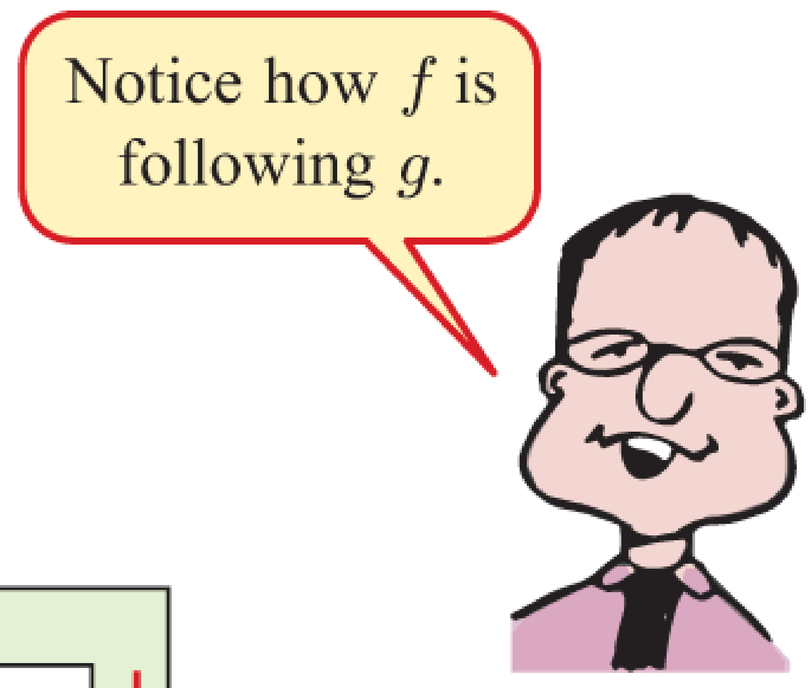
$$(f \circ g)(x) = f(g(x)) \quad \text{or} \quad f \circ g : x \mapsto f(g(x)).$$

Consider $f : x \mapsto x^4$ and $g : x \mapsto 2x + 3$.

$f \circ g$ means that $g$ converts $x$ to $2x + 3$ and then $f$ converts $(2x + 3)$ to $(2x + 3)^4$.

This is illustrated by the two function machines below.




Algebraically, if $f(x) = x^4$ and $g(x) = 2x + 3$ then

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f(2x + 3) && \{g \text{ operates on } x \text{ first}\} \\
&= (2x + 3)^4 && \{f \text{ operates on } g(x) \text{ next}\}
\end{aligned}
$$

and

$$
\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g(x^4) && \{f \text{ operates on } x \text{ first}\} \\
&= 2(x^4) + 3 && \{g \text{ operates on } f(x) \text{ next}\} \\
&= 2x^4 + 3
\end{aligned}
$$

So, $f(g(x)) \neq g(f(x))$.

> In general, $(f \circ g)(x) \neq (g \circ f)(x)$.

**Example 10** — Self Tutor

Given $f : x \mapsto 2x + 1$ and $g : x \mapsto 3 - 4x$, find in simplest form:

**a** $(f \circ g)(x)$ **b** $(g \circ f)(x)$

$f(x) = 2x + 1$ and $g(x) = 3 - 4x$

**a**
$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f(3 - 4x) \\
&= 2(3 - 4x) + 1 \\
&= 6 - 8x + 1 \\
&= 7 - 8x
\end{aligned}
$$

**b**
$$
\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g(2x + 1) \\
&= 3 - 4(2x + 1) \\
&= 3 - 8x - 4 \\
&= -8x - 1
\end{aligned}
$$

In the previous Example you should have observed how we can substitute an expression into a function.

If $f(x) = 2x + 1$ then $f(\Delta) = 2(\Delta) + 1$

and so $f(3 - 4x) = 2(3 - 4x) + 1$.

**Example 11** — Self Tutor

Given $f(x) = 6x - 5$ and $g(x) = x^2 + x$, find:

**a** $(g \circ f)(-1)$ **b** $(f \circ f)(0)$

**a** $(g \circ f)(-1) = g(f(-1))$

Now $f(-1) = 6(-1) - 5$
$= -11$

$\therefore$ $(g \circ f)(-1) = g(-11)$
$= (-11)^2 + (-11)$
$= 110$

**b** $(f \circ f)(0) = f(f(0))$

Now $f(0) = 6(0) - 5$
$= -5$

$\therefore$ $(f \circ f)(0) = f(-5)$
$= 6(-5) - 5$
$= -35$

You should be aware that the domain of the composite of two functions depends on the domains of the original functions.

For example, consider $f(x) = x^2$ with domain $x \in \mathbb{R}$ and $g(x) = \sqrt{x}$ with domain $x \geqslant 0$.

$(f \circ g)(x) = f(g(x))$
$= (\sqrt{x})^2$
$= x$

The domain of $(f \circ g)(x)$ is $x \geqslant 0$, not $\mathbb{R}$, since $(f \circ g)(x)$ is defined using function $g(x)$.

## EXERCISE 15E

**1** Given $f : x \mapsto -2x$ and $g : x \mapsto 1 + x^2$, find in simplest form:

**a** $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** $(f \circ g)(2)$ **d** $(f \circ f)(-1)$

**2** Given $f(x) = 3 - x^2$ and $g(x) = 2x + 4$, find in simplest form:

**a** $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** $(g \circ g)(\frac{1}{2})$ **d** $(f \circ f)(-\frac{1}{2})$

**3** Given $f(x) = \sqrt{6 - x}$ and $g(x) = 5x - 7$, find:

**a** $(g \circ g)(x)$ **b** $(f \circ g)(1)$ **c** $(g \circ f)(6)$ **d** $(f \circ f)(2)$

**4** Suppose $f : x \mapsto x^2 + 1$ and $g : x \mapsto 3 - x$.

**a** Find in simplest form: **i** $(f \circ g)(x)$ **ii** $(g \circ f)(x)$

**b** Find the value(s) of $x$ such that $(g \circ f)(x) = f(x)$.

**5** Suppose $f(x) = 9 - \sqrt{x}$ and $g(x) = x^2 + 4$.

**a** Find $(f \circ g)(x)$ and state its domain and range. **b** Find $(g \circ f)(4)$.

**c** Find $(f \circ f)(x)$ and state its domain and range.

**6** Suppose $f(x) = 1 - 2x$ and $g(x) = 3x + 5$.

**a** Find $f(g(x))$. **b** Hence solve $(f \circ g)(x) = f(x + 3)$.

**7** Suppose $f : x \mapsto 2x - x^2$ and $g : x \mapsto 1 + 3x$.

**a** Find in simplest form: **i** $(f \circ g)(x)$ **ii** $(g \circ f)(x)$

**b** Find the value(s) of $x$ such that $(f \circ g)(x) = 3(g \circ f)(x)$.

**8** For each pair of functions, find $(f \circ g)(x)$ and state its domain and range:

**a** $f(x) = \dfrac{1}{x}$ and $g(x) = x - 3$ **b** $f(x) = -\dfrac{1}{x}$ and $g(x) = x^2 + 3x + 2$

**9** Functions $f$ and $g$ are defined by $f = \{(0, 2), (1, 5), (2, 7), (3, 9)\}$ and $g = \{(2, 2), (5, 0), (7, 1), (9, 3)\}$. Find: **a** $f \circ g$ **b** $g \circ f$.

**10** Given $f(x) = \dfrac{x+3}{x+2}$ and $g(x) = \dfrac{x+1}{x-1}$, find in simplest form:

**a** $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** $(g \circ g)(x)$

In each case, find the domain of the composite function.

**11** **a** If $ax + b = cx + d$ for all values of $x$, show that $a = c$ and $b = d$.
**Hint:** If it is true for all $x$, it is true for $x = 0$ and $x = 1$.

**b** Given $f(x) = 2x + 3$ and $g(x) = ax + b$ and that $(f \circ g)(x) = x$ for all values of $x$, deduce that $a = \frac{1}{2}$ and $b = -\frac{3}{2}$.

**c** Is the result in **b** true if $(g \circ f)(x) = x$ for all $x$?

**12** Suppose $f(x) = \sqrt{1-x}$ and $g(x) = x^2$. Find:

**a** $(f \circ g)(x)$ **b** the domain and range of $(f \circ g)(x)$

**c** $(g \circ f)(x)$ **d** the domain and range of $(g \circ f)(x)$.

**13** Suppose $f(x)$ and $g(x)$ are functions. $f(x)$ has domain $D_f$ and range $R_f$. $g(x)$ has domain $D_g$ and range $R_g$.

**a** Under what circumstance will $(f \circ g)(x)$ be defined?

**b** Assuming $(f \circ g)(x)$ is defined, find its domain.

**Example 12** Self Tutor

The outside temperature at an altitude $A$ km above ground level in a particular city is $T(A) = 25 - 6A$ °C. The altitude of an aeroplane $t$ minutes after taking off is $A(t) = 12 - \dfrac{60}{t+5}$ km.

Find the composite function $T \circ A$, and explain what it means.

$$\begin{aligned} T \circ A &= T(A(t)) \\ &= T\left(12 - \frac{60}{t+5}\right) \\ &= 25 - 6\left(12 - \frac{60}{t+5}\right) \\ &= \frac{360}{t+5} - 47 \end{aligned}$$

The function $T \circ A$ gives the temperature outside the aeroplane $t$ minutes after taking off.

**14** The value of Mila's car after it has been driven $D$ thousand kilometres is $V(D) = 10\,000 - 40D$ dollars. The distance Mila's car has been driven $t$ years after Mila purchased it is $D(t) = 80 + 10t$ thousand kilometres.

**a** Find the composite function $V \circ D$, and explain what it means.

**b** Find and interpret $(V \circ D)(6)$.

**15** Diego sells sculptures online. When a sculpture with marked price €$x$ is bought, Diego adds a €50 shipping fee and 10% tax to determine the total cost.

Let the functions $S(x) = x + 50$ and $T(x) = 1.1x$ represent the shipping and tax respectively.

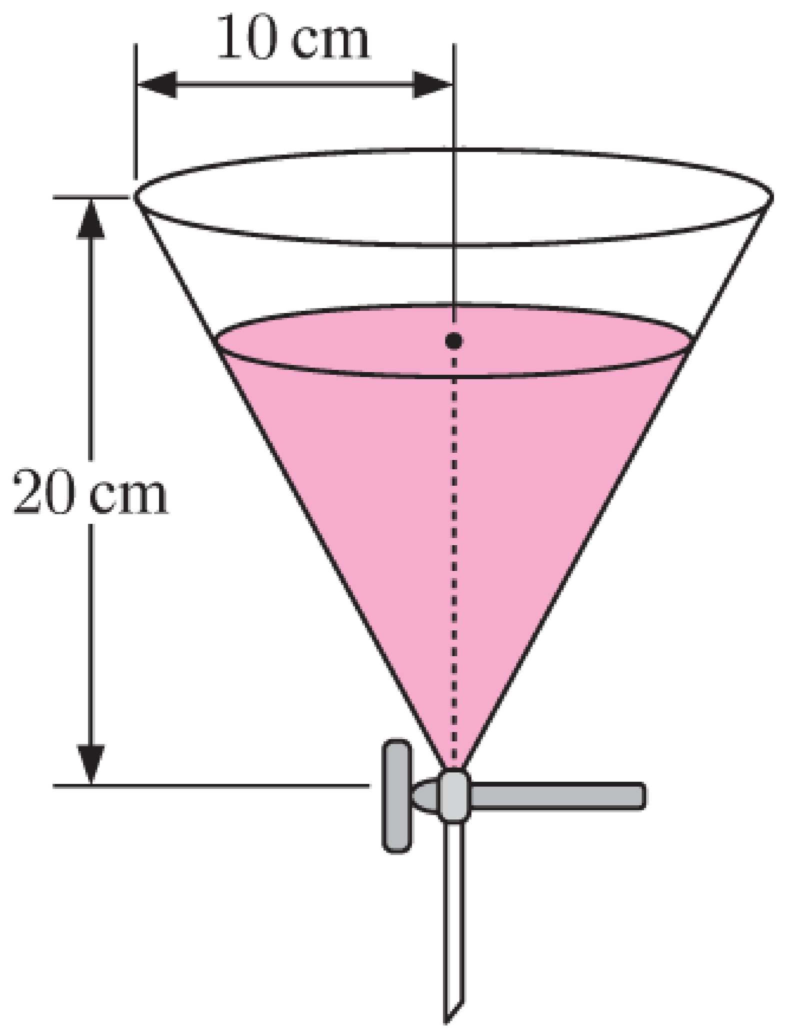
**a** What composite function should be used to determine the total cost if the tax must be paid on:

**i** both the marked price and the shipping fee

**ii** the marked price but not the shipping fee?

**b** Suppose the tax must be paid on both the marked price and the shipping fee. Use the correct composite function to find the total cost of a sculpture with marked price €600.

**16** In a laboratory, a conical funnel has radius 10 cm and height 20 cm. The funnel initially contains 2 L of solution. The tap is turned on, allowing the solution to flow from the funnel at 20 mL per minute.




**a** Find the function for the volume $V$ mL of solution remaining in the funnel after $t$ minutes.

**b** Show that if there is $V$ mL of solution in the funnel, the height of the solution is $H(V) = \sqrt[3]{\dfrac{12V}{\pi}}$ cm.

**c** Find $H \circ V$, and explain what it means.

**d** Find and interpret $(H \circ V)(30)$.

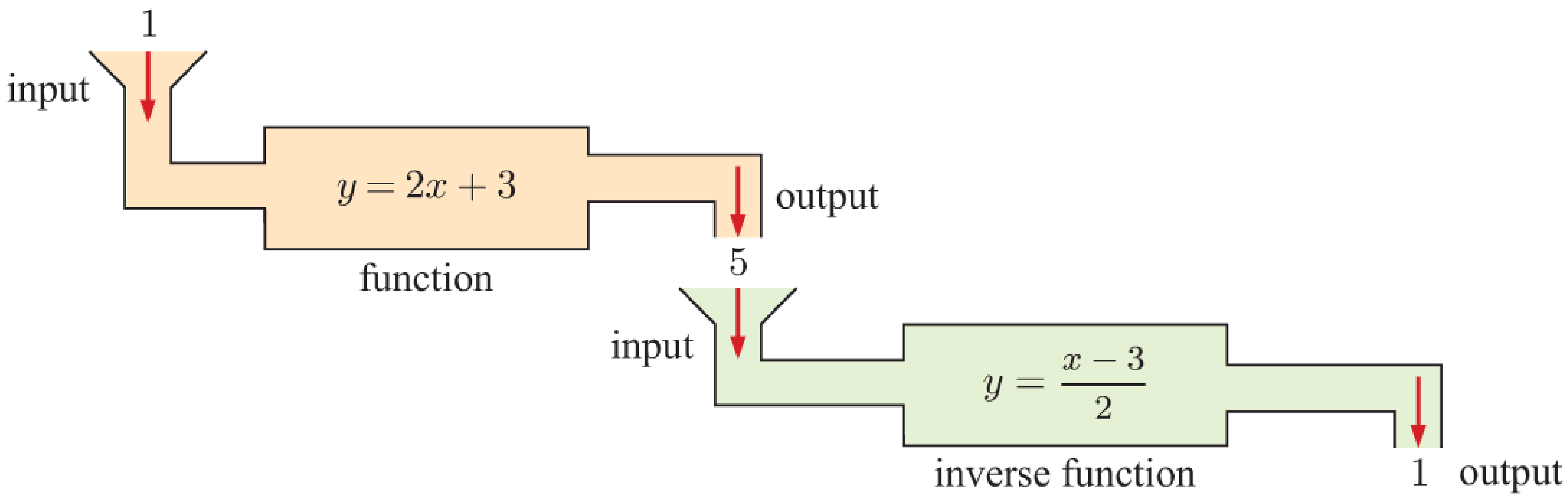
## F INVERSE FUNCTIONS

The operations of $+$ and $-$, $\times$ and $\div$, are **inverse operations** as one "undoes" what the other does.

The function $y = 2x + 3$ can be "undone" by its *inverse* function $y = \dfrac{x-3}{2}$.

We can think of this as two machines. If the machines are inverses then the second machine *undoes* what the first machine does.

No matter what value of $x$ enters the first machine, it is returned as the output from the second machine.




A function $y = f(x)$ *may or may not* have an inverse function. To understand which functions do have inverses, we need some more terminology.

## ONE-TO-ONE AND MANY-TO-ONE FUNCTIONS

A **one-to-one** function is any function where:

- for each $x$ there is only one value of $y$ and
- for each $y$ there is only one value of $x$.

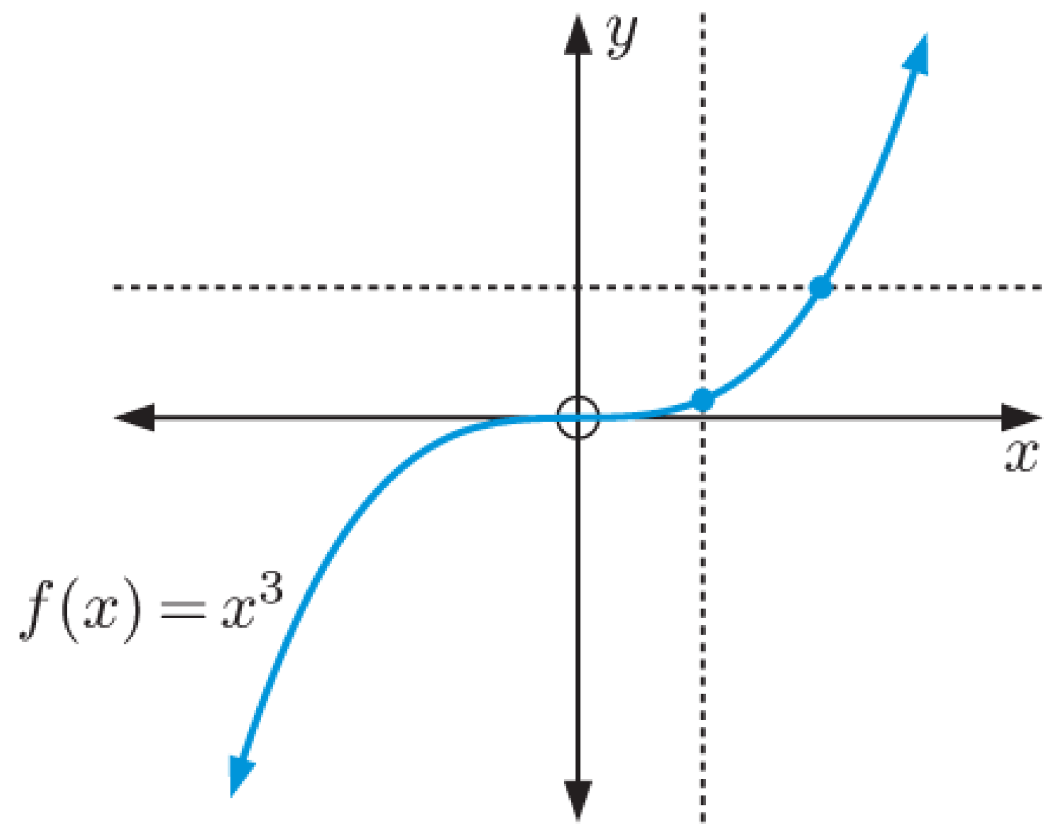
Equivalently, a function is one-to-one if $f(a) = f(b)$ only when $a = b$.

**One-to-one** functions satisfy both the **vertical line test** and the **horizontal line test**.

This means that:

- no vertical line can meet the graph more than once
- no horizontal line can meet the graph more than once.

For example, $f(x) = x^3$ is one-to-one since it passes both the vertical line and horizontal line tests.

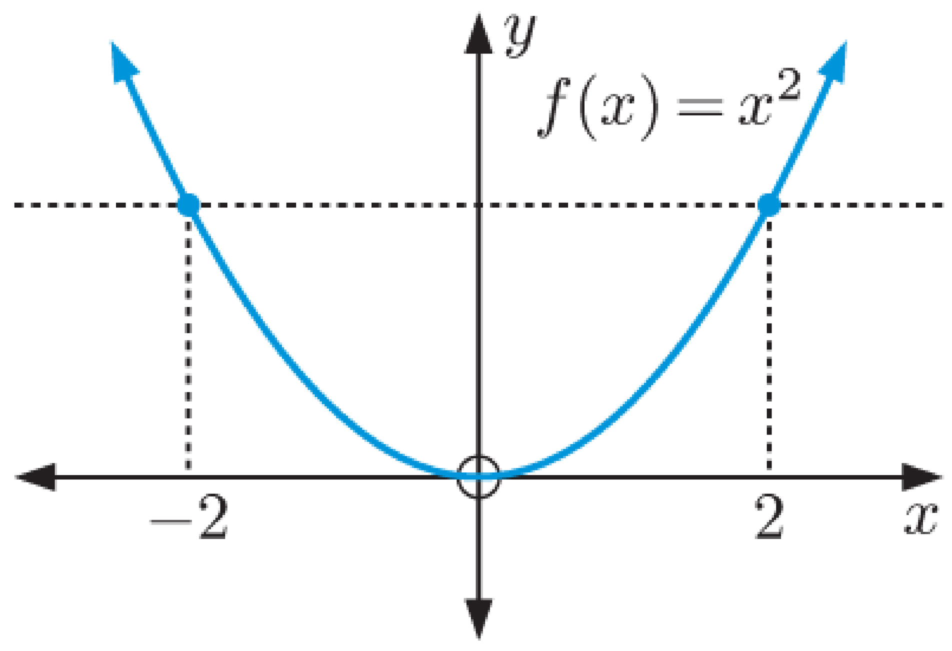



If the function $f(x)$ is **one-to-one**, it will have an inverse function which we denote $f^{-1}(x)$.

Functions that are not one-to-one are called **many-to-one**. While these functions satisfy the vertical line test, they *do not* satisfy the horizontal line test. At least one $y$-value has more than one corresponding $x$-value.

For example, $f(x) = x^2$ fails the horizontal line test, since if $f(x) = 4$ then $x = -2$ or 2.

$f(x) = x^2$ is therefore many-to-one.




If a function $f(x)$ is **many-to-one**, it *does not* have an inverse function.

However, for a many-to-one function we can often define a new function using the same formula but with a **restricted domain** to make it a one-to-one function. This new function will have an inverse function.

For example, if we restrict $f(x) = x^2$ to the domain $x \geqslant 0$ or the domain $x \leqslant 0$, then the restricted function is one-to-one, and it has an inverse function.

## PROPERTIES OF THE INVERSE FUNCTION

If $f(x)$ has an **inverse function**, this new function:

- is denoted $f^{-1}(x)$
- must satisfy the vertical line test
- has a graph which is the reflection of $y = f(x)$ in the line $y = x$
- satisfies $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.

The function $y = x$ is called the **identity function** because it is its own inverse, and when its inverse is found, $(x, y)$ maps onto itself.

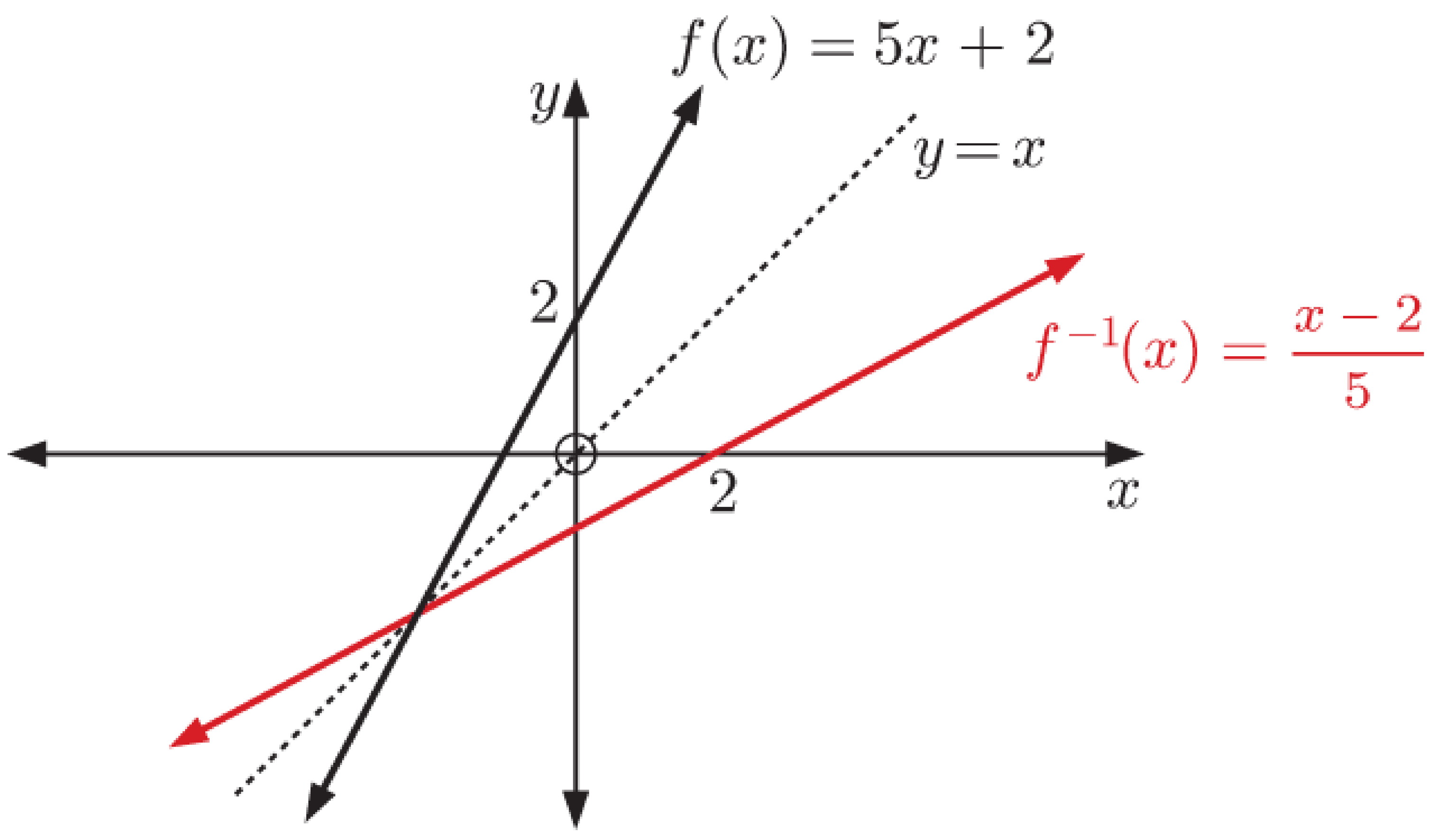
If $(x, y)$ lies on $f$, then $(y, x)$ must lie on $f^{-1}$.

Geometrically, this is achieved by *reflecting* the graph of $y = f(x)$ in the line $y = x$.

Algebraically, we find the formula for an inverse function by exchanging $x$ and $y$.

For example, $f : y = 5x + 2$ becomes $f^{-1} : x = 5y + 2$,

which we rearrange to obtain $f^{-1} : y = \dfrac{x-2}{5}$.

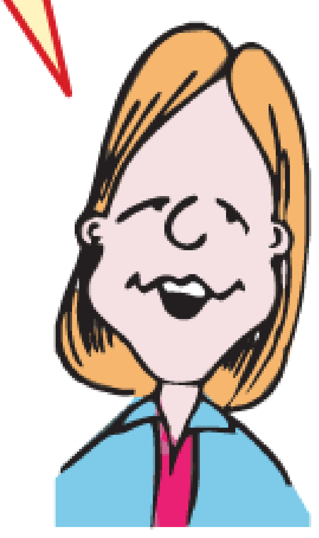



$y = f^{-1}(x)$ is the inverse of $y = f(x)$ as:

- it is also a function
- it is the reflection of $y = f(x)$ in the line $y = x$.

$f^{-1}(x)$ is the **inverse** of $f$, *not* its reciprocal.

In general, $f^{-1}(x) \neq \dfrac{1}{f(x)}$.



If $f(x)$ has an inverse function $f^{-1}(x)$, then:

The domain of $f^{-1}$ is equal to the range of $f$.

The range of $f^{-1}$ is equal to the domain of $f$.

**Example 13** — Self Tutor

Consider $f : x \mapsto 2x + 3$.

**a** On the same axes, graph $f$ and its inverse function $f^{-1}$.

**b** Find $f^{-1}(x)$ using:

  **i** coordinate geometry and the gradient of $y = f^{-1}(x)$ from **a**

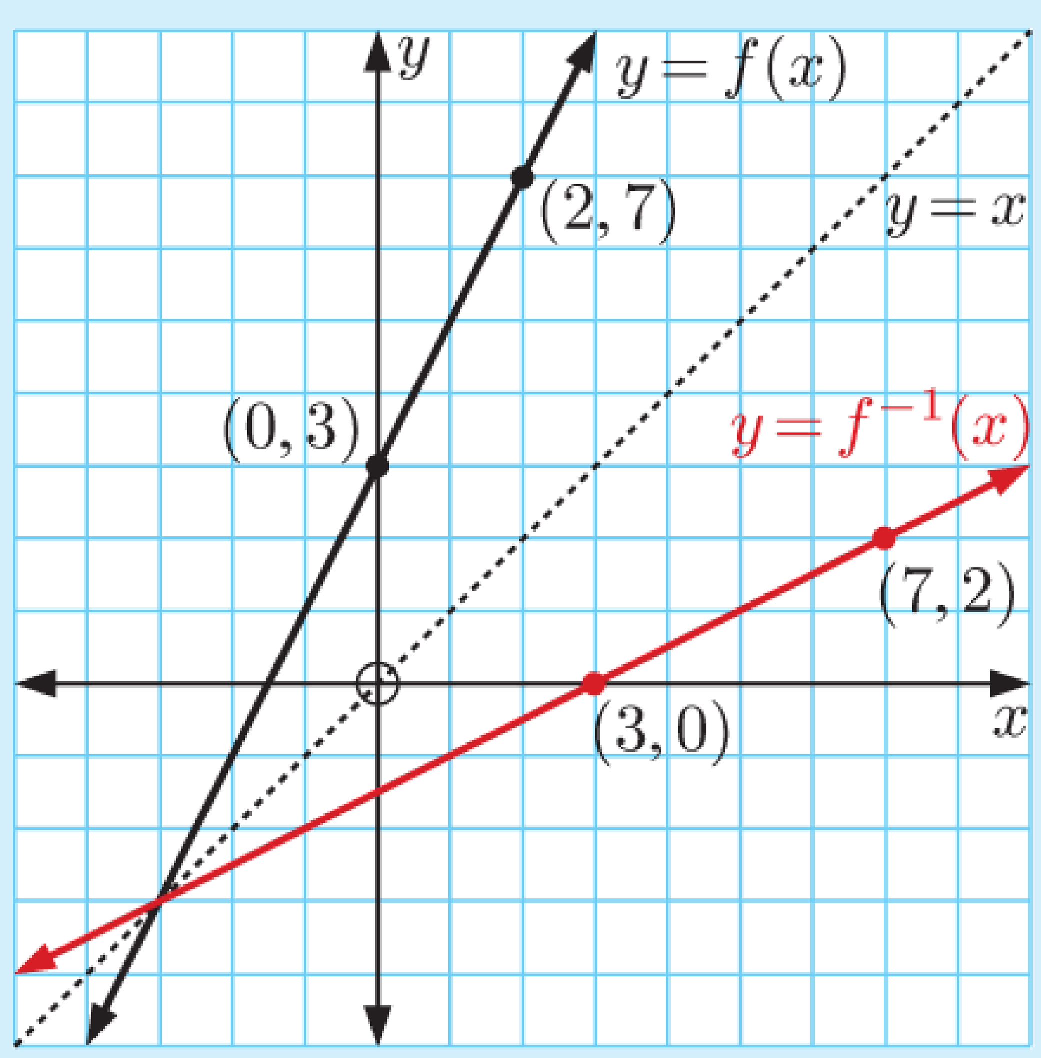
  **ii** variable interchange.

**c** Check that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$

**a** $f(x) = 2x + 3$ passes through $(0, 3)$ and $(2, 7)$.

$\therefore$ $f^{-1}(x)$ passes through $(3, 0)$ and $(7, 2)$.

If $f$ includes point $(a, b)$ then $f^{-1}$ includes point $(b, a)$.



**b** **i** $y = f^{-1}(x)$ has gradient $\dfrac{2-0}{7-3} = \dfrac{1}{2}$

Its equation is $\dfrac{y-0}{x-3} = \dfrac{1}{2}$

$\therefore\ y = \dfrac{x-3}{2}$

$\therefore\ f^{-1}(x) = \dfrac{x-3}{2}$

**ii** $f$ is $y = 2x+3$,

$\therefore\ f^{-1}$ is $x = 2y+3$

$\therefore\ x-3 = 2y$

$\therefore\ \dfrac{x-3}{2} = y$

$\therefore\ f^{-1}(x) = \dfrac{x-3}{2}$

**c** $(f \circ f^{-1})(x) = f(f^{-1}(x))$

$= f\left(\dfrac{x-3}{2}\right)$

$= 2\left(\dfrac{x-3}{2}\right) + 3$

$= x$

and $(f^{-1} \circ f)(x) = f^{-1}(f(x))$

$= f^{-1}(2x+3)$

$= \dfrac{(2x+3)-3}{2}$

$= \dfrac{2x}{2}$

$= x$

**Example 14** **Self Tutor**

Consider $f : x \mapsto x^2$.

**a** Explain why this function does not have an inverse function.

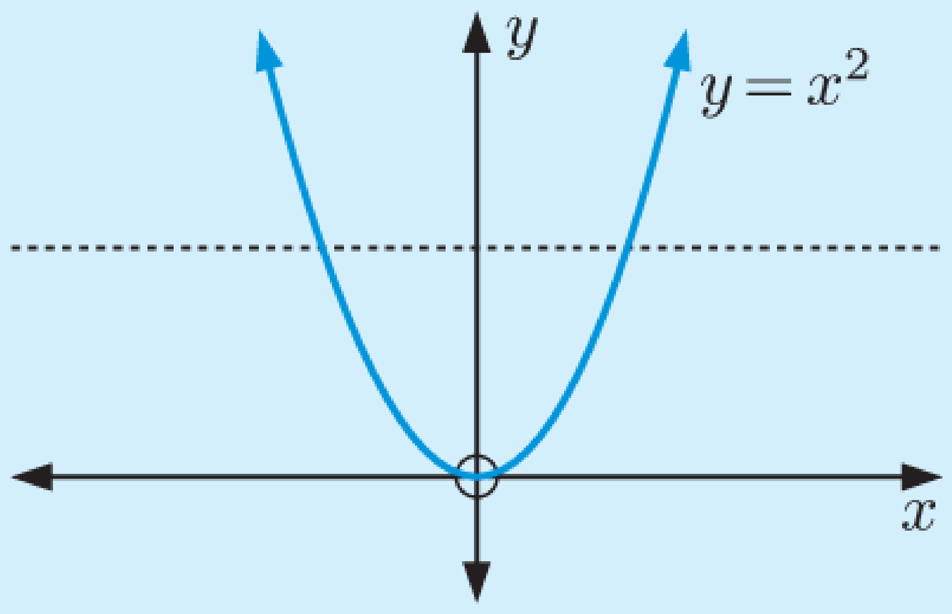
**b** Does $f : x \mapsto x^2$ where $x \geqslant 0$ have an inverse function?

**c** Find $f^{-1}(x)$ for $f : x \mapsto x^2,\ x \geqslant 0$.

**d** Sketch $y = f(x)$, $y = x$, and $y = f^{-1}(x)$ for $f$ in **b** and $f^{-1}$ in **c**.

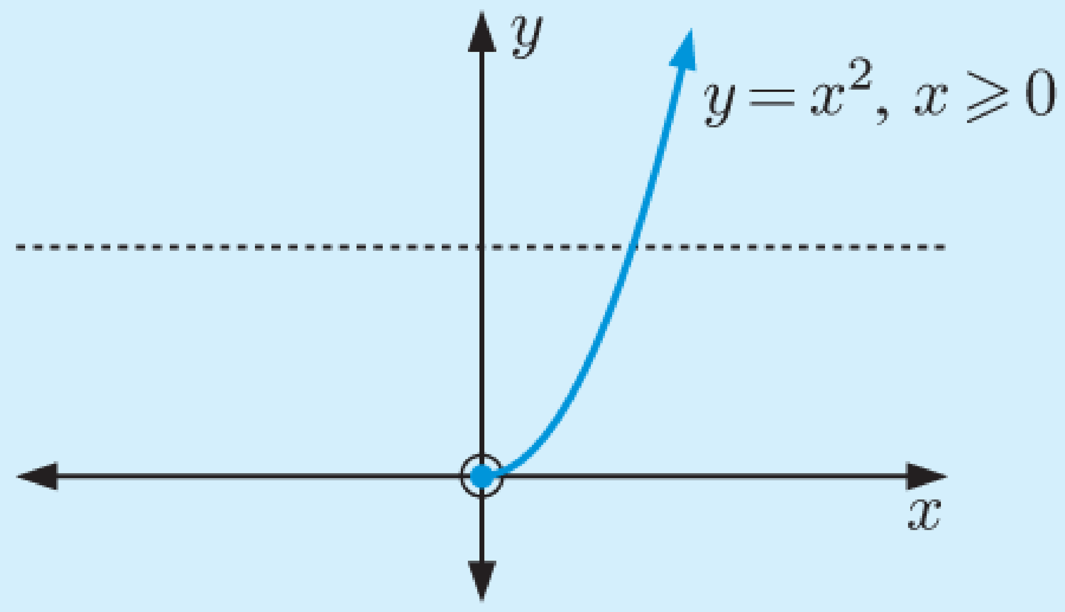
**a** From the graph, we can see that $f : x \mapsto x^2$ does not pass the horizontal line test.

$\therefore$ it is many-to-one and does not have an inverse function.




**b** If we restrict the domain to $x \geqslant 0$ or $x \in [0, \infty[$, the function now satisfies the horizontal line test.

$\therefore$ it is one-to-one and has an inverse function.




**c** $f$ is defined by $y = x^2,\quad x \geqslant 0$

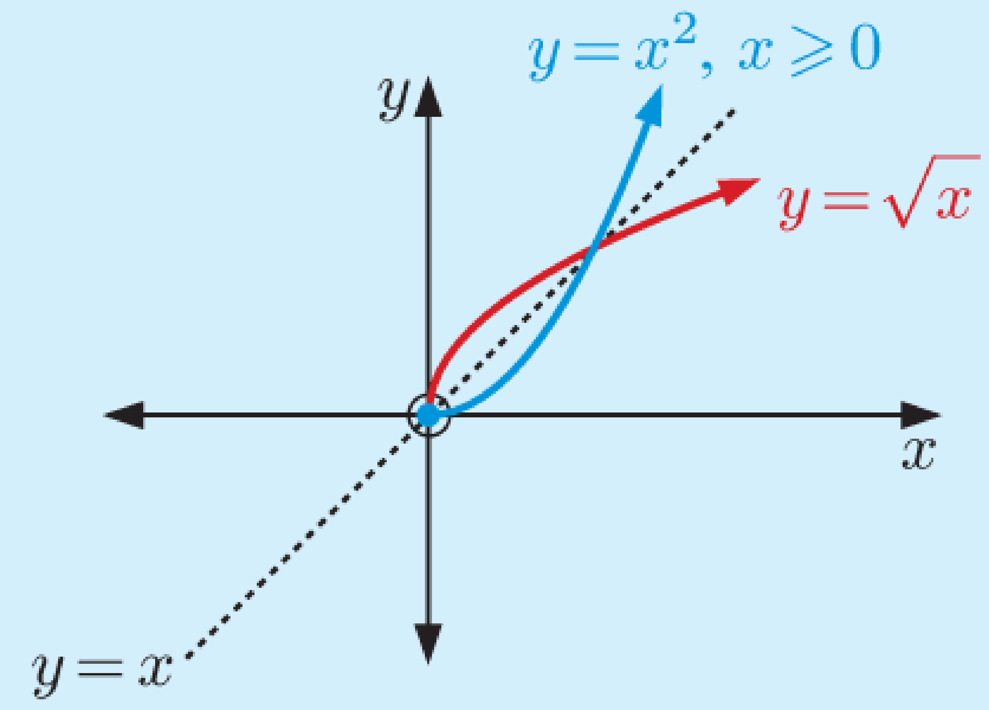
$\therefore\ f^{-1}$ is defined by $x = y^2,\quad y \geqslant 0$

$\therefore\ y = \pm\sqrt{x},\quad y \geqslant 0$

$\therefore\ y = \sqrt{x}\quad$ {as $-\sqrt{x}$ is $\leqslant 0$}

So, $f^{-1}(x) = \sqrt{x}$

**d**



## SELF-INVERSE FUNCTIONS

Any function which has an inverse, and whose graph is symmetrical about the line $y = x$, is a **self-inverse function**.

For example:

- The function $f(x) = x$ is the **identity function**, and is also a self-inverse function.
- The function $f(x) = \frac{1}{x}$, $x \neq 0$, is also a self-inverse function, as $f = f^{-1}$.

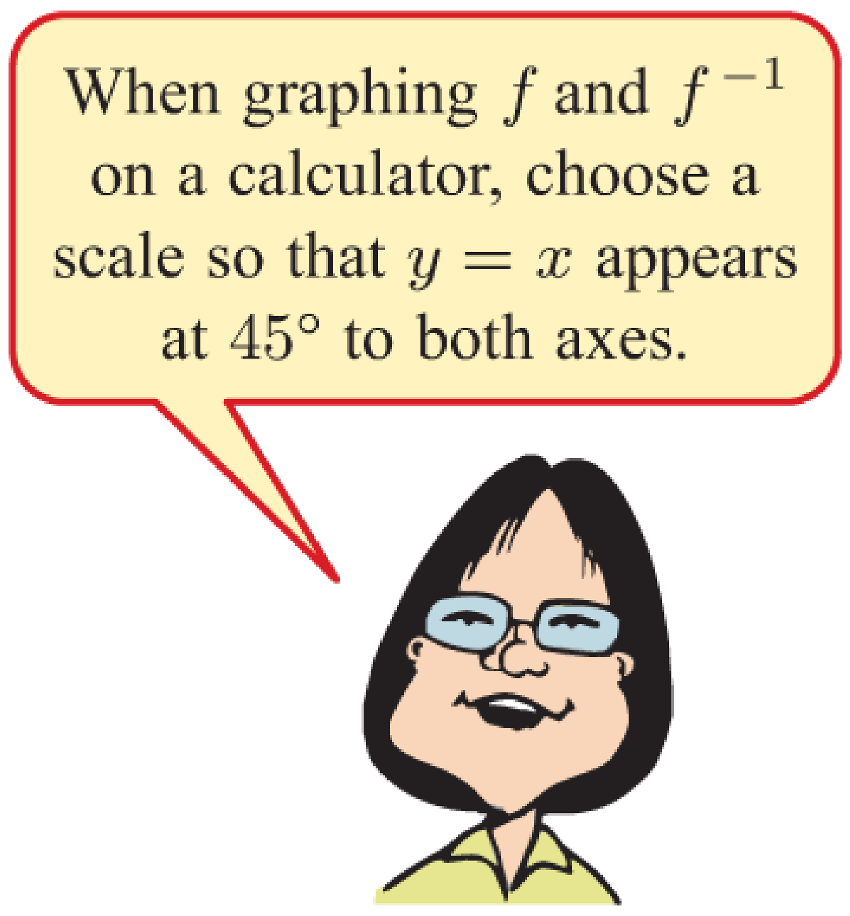
## EXERCISE 15F

**1** For each of the following functions $f$:

**i** On the same set of axes, graph $y = x$, $y = f(x)$, and $y = f^{-1}(x)$.

**ii** Find $f^{-1}(x)$ using coordinate geometry and the gradient of $y = f^{-1}(x)$ from **i**.

**iii** Find $f^{-1}(x)$ using variable interchange.




**a** $f : x \mapsto 3x + 1$ **b** $f : x \mapsto \frac{x+2}{4}$

**2** For each of the following functions $f$:

**i** Find $f^{-1}(x)$.

**ii** Sketch $y = f(x)$, $y = f^{-1}(x)$, and $y = x$ on the same set of axes.

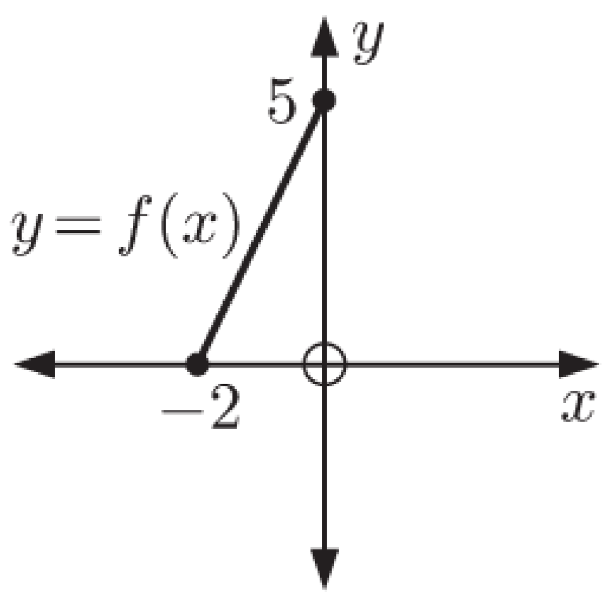
**iii** Show that $(f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$, the identity function.

**a** $f : x \mapsto 2x + 5$ **b** $f : x \mapsto \frac{3 - 2x}{4}$ **c** $f : x \mapsto x + 3$

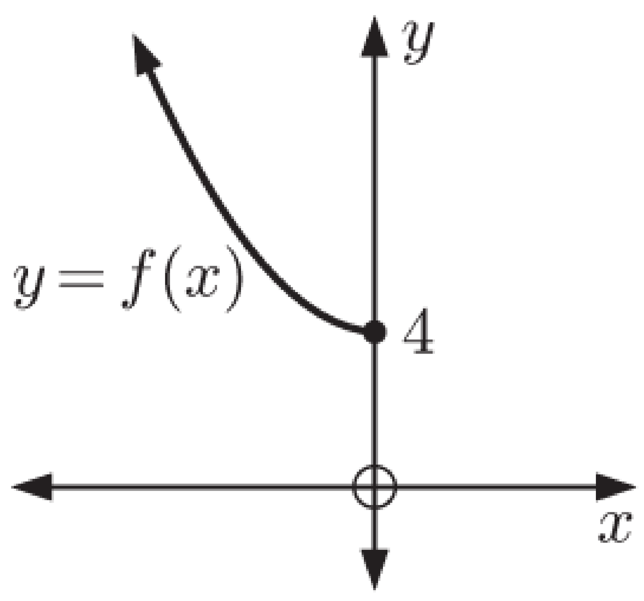
**3** Copy the graphs of the following functions and draw the graphs of $y = x$ and $y = f^{-1}(x)$ on the same set of axes. In each case, state the domain and range of both $f$ and $f^{-1}$.

PRINTABLE GRAPHS

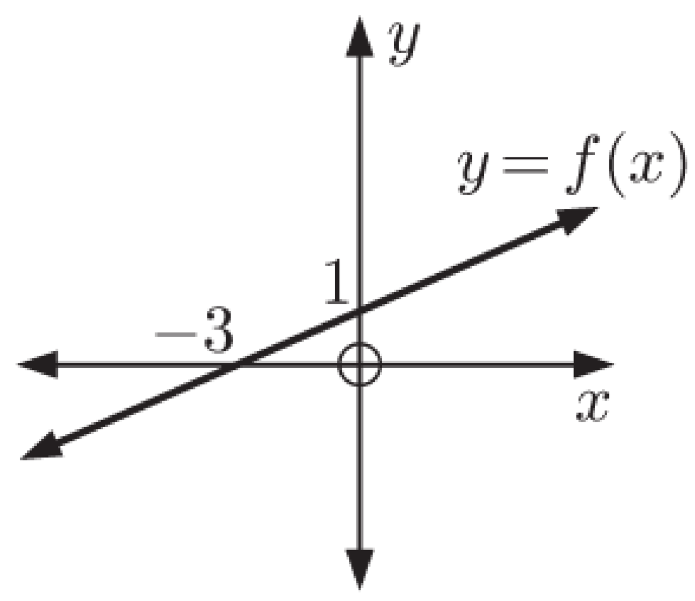
**a**



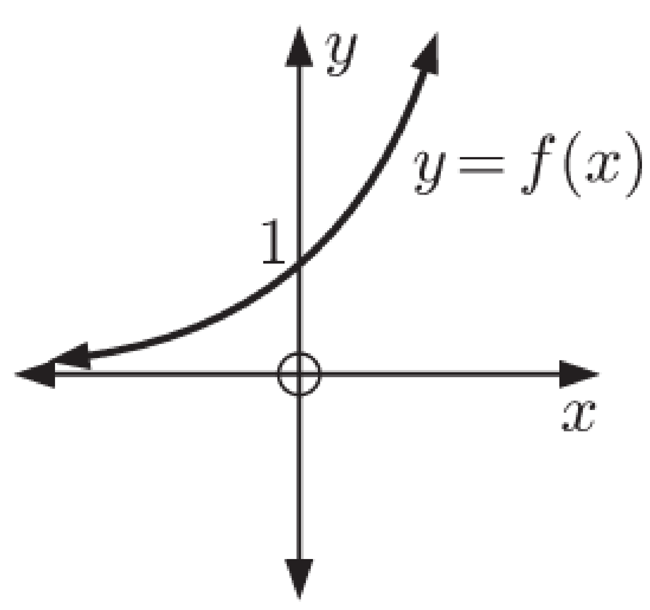

**b**



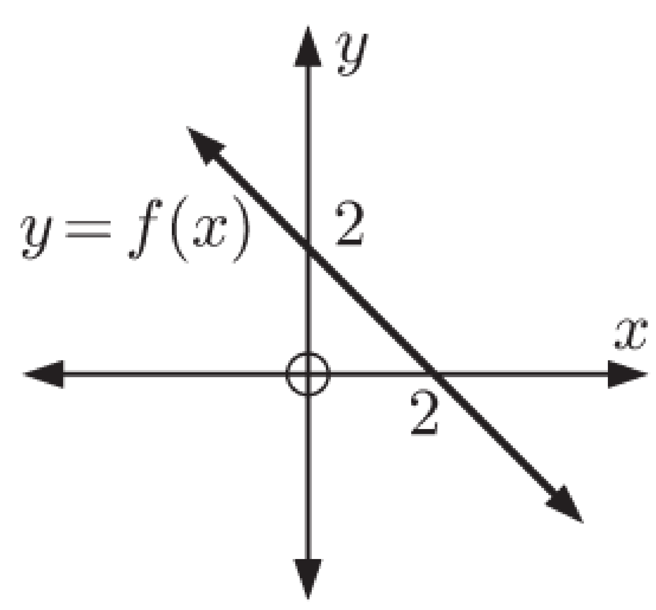

**c**



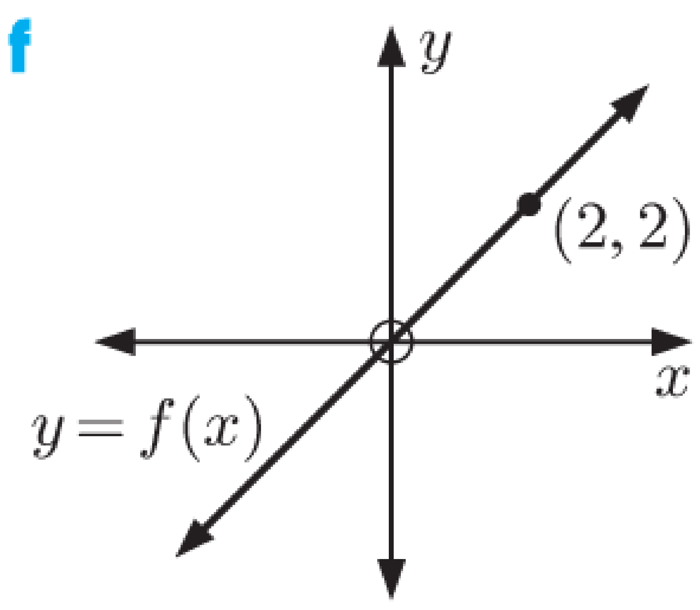

**d**




**e**




**f**




**4** Given $f(x) = 2x - 5$, find $(f^{-1})^{-1}(x)$. What do you notice?

**5** Which of the following functions have inverses? Where an inverse exists, write down the inverse function.

**a** $\{(1, 2), (2, 4), (3, 5)\}$ **b** $\{(-1, 3), (0, 2), (1, 3)\}$

**c** $\{(2, 1), (-1, 0), (0, 2), (1, 3)\}$ **d** $\{(-1, -1), (0, 0), (1, 1)\}$

**6** Find *all* linear functions which are self-inverse.

**7** If the one-to-one function $H(x)$ has domain $\{x \mid -2 \leqslant x < 3\}$, find the range of its inverse $H^{-1}(x)$.

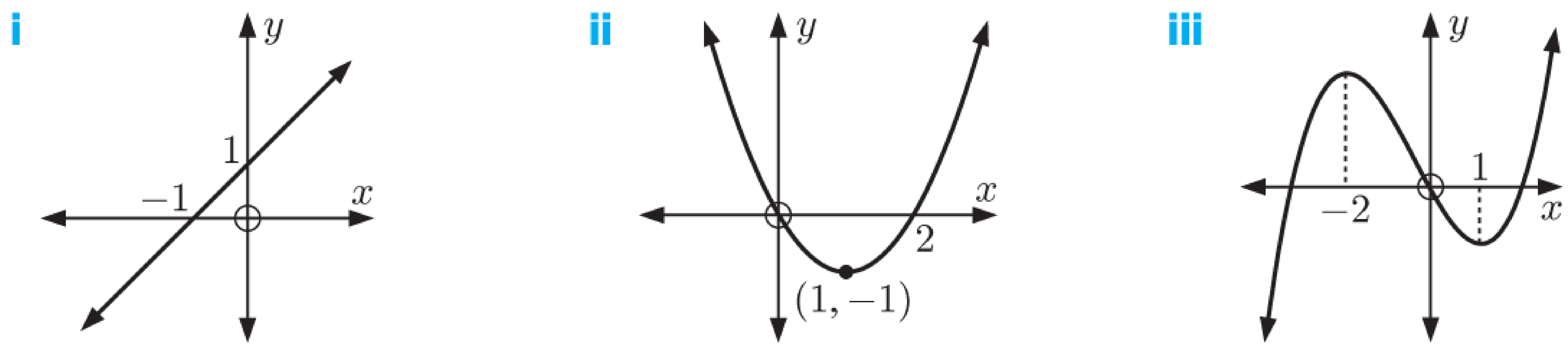
**8** **a** Sketch the graph of $f : x \mapsto x^2 - 4$ and reflect it in the line $y = x$.
**b** Does $f$ have an inverse function?
**c** Does $f$ with restricted domain $x \geqslant 0$ have an inverse function?

**9** Sketch the graph of $f : x \mapsto x^3$ and its inverse function $f^{-1}(x)$.

**10** Given $f(x) = \frac{1}{x}$, $x \neq 0$, show that $f$ is self-inverse.

**11** The **horizontal line test** says: *For a function to have an inverse function, no horizontal line can cut its graph more than once.*

**a** Explain why this is a valid test for the existence of an inverse function.
**b** Which of the following functions have an inverse function?




**c** For the functions in **b** which do not have an inverse, specify restricted domains as wide as possible such that the resulting function does have an inverse.

**12** Consider $f : x \mapsto x^2$, $x \leqslant 0$.
**a** Find $f^{-1}(x)$.
**b** Sketch $y = f(x)$, $y = x$, and $y = f^{-1}(x)$ on the same set of axes.

**13** **a** Explain why $f(x) = x^2 - 4x + 3$ is a function but does not have an inverse function.
**b** Explain why $g(x) = x^2 - 4x + 3$, $x \geqslant 2$, has the inverse function $g^{-1}(x) = 2 + \sqrt{1 + x}$.
**c** State the domain and range of $g$ and $g^{-1}$.
**d** Show that $(g \circ g^{-1})(x) = (g^{-1} \circ g)(x) = x$, the identity function.

**14** Consider $f : x \mapsto (x + 1)^2 + 3$, $x \geqslant -1$.
**a** Find $f^{-1}(x)$.
**b** Use technology to help sketch the graphs of $y = f(x)$, $y = x$, and $y = f^{-1}(x)$.
**c** State the domain and range of $f$ and $f^{-1}$.

**15** Consider $f(x) = 4 + 6x - x^2$, $x \leqslant 3$.
**a** Find $f^{-1}(x)$.
**b** State the domain and range of $f$ and $f^{-1}$.

**16** Let $f(x) = 2x^2 - 10x + 6$, $x \leqslant k$.
**a** Find the largest value of $k$ such that $f^{-1}(x)$ exists.
**b** For this value of $k$:
**i** Find $f^{-1}(x)$.
**ii** State the domain and range of $f^{-1}(x)$.

**17** Consider the functions $f: x \mapsto 2x+5$ and $g: x \mapsto \frac{8-x}{2}$.

**a** Find $g^{-1}(x)$.

**b** Hence solve $g(x) = -1$.

**c** Show that $f^{-1}(-3) - g^{-1}(6) = 0$.

**d** Find $x$ such that $(f \circ g^{-1})(x) = 9$.

**18** Consider $f: x \mapsto 2x$ and $g: x \mapsto 4x-3$.

**a** Find $(f \circ g)(x)$.

**b** Given $(f \circ g)^{-1}(k) = 2$, find $k$.

**c** Show that $(f^{-1} \circ g^{-1})(x) = (g \circ f)^{-1}(x)$.

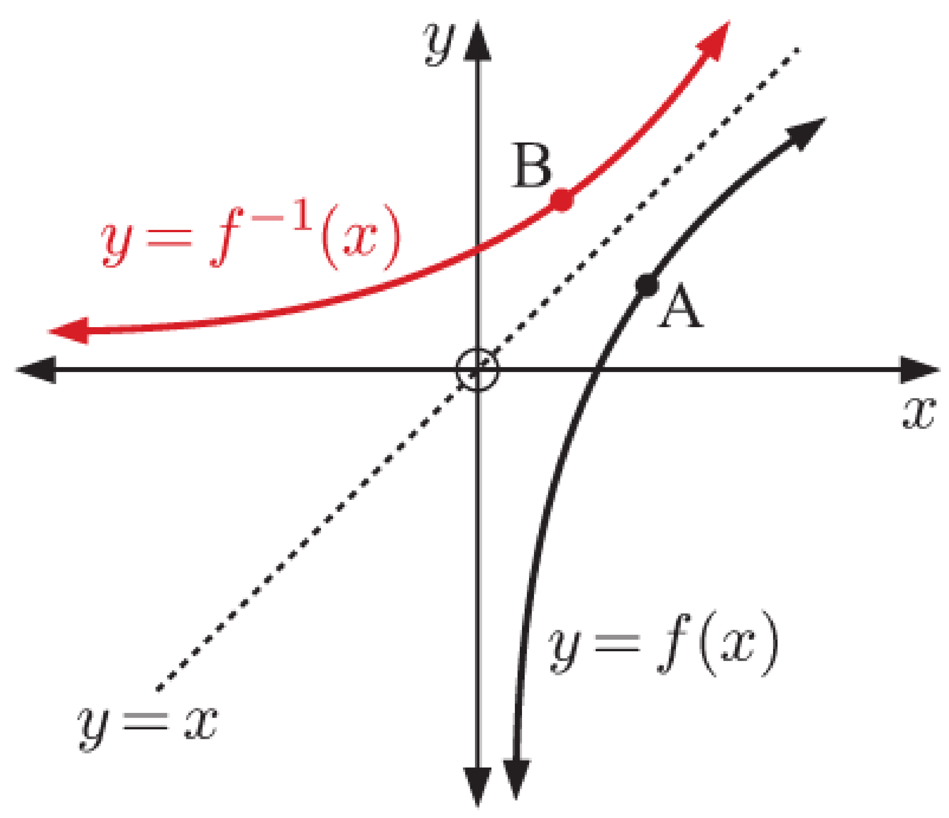
**19** Show that $f: x \mapsto \frac{3x-8}{x-3}$, $x \neq 3$ is self-inverse by:

**a** referring to its graph

**b** using algebra.

**20** Under what conditions is the rational function $y = \frac{ax+b}{cx+d}$, $c \neq 0$, a self-inverse function?

**21**

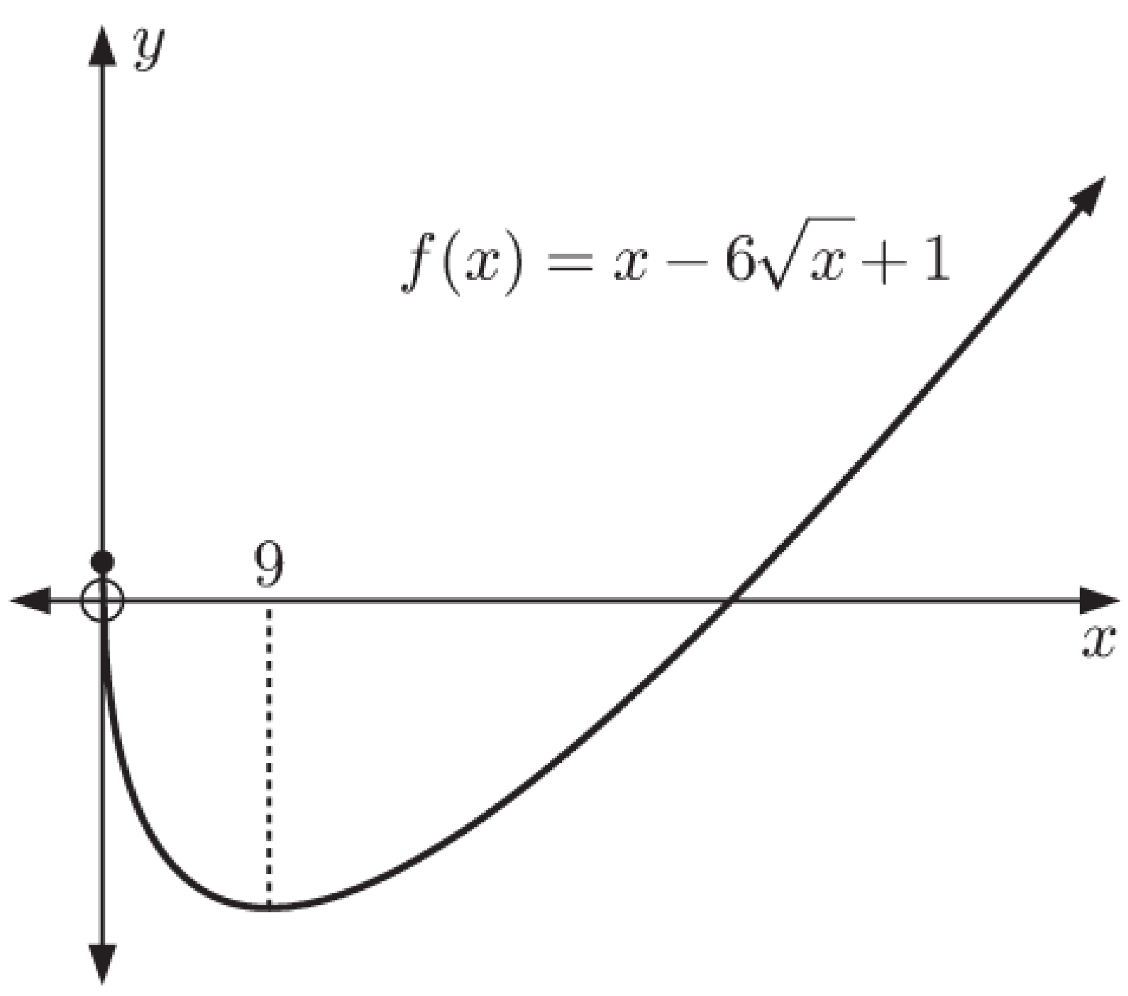



**a** B is the image of A under a reflection in the line $y = x$. If A is $(x, f(x))$, find the coordinates of B.

**b** By substituting your result from **a** into $y = f^{-1}(x)$, show that $f^{-1}(f(x)) = x$.

**c** Using a similar method, show that $f(f^{-1}(x)) = x$.

**22** The graph of $f(x) = x - 6\sqrt{x} + 1$ is shown alongside.




**a** State the natural domain of $f(x)$.

**b** Does $f(x)$ have an inverse function? Explain your answer.

**c** Let $g(x) = x - 6\sqrt{x} + 1$, $0 \leqslant x \leqslant 9$.

**i** Find $g^{-1}(x)$, and verify that $(g \circ g^{-1})(4) = 4$.

**ii** State the domain and range of $g$ and $g^{-1}$.

**d** Let $h(x) = x - 6\sqrt{x} + 1$, $x \geqslant 9$.

**i** Find $h^{-1}(x)$ and verify that $(h \circ h^{-1})(16) = 16$.

**ii** State the domain and range of $h$ and $h^{-1}$.

**iii** Find the value of $x$ such that $g^{-1}(x) = h^{-1}(x)$.

## THEORY OF KNOWLEDGE

The notation and terminology of mathematics has rules which tell us how to construct expressions or mathematical "sentences". This allows us to communicate mathematical ideas in a written form.

For example, the expression "$1+1=2$" tells us that "one added to one is equal to two".

**1** What does it mean for something to be a "language"?

**2** Does mathematics have a "grammar" or **syntax** in the same sense as the English language?

In computer science, **Backus-Naur form** (BNF) is commonly used to define the syntax of programming languages. BNF can also be used to describe the rules of non-programming related languages.

**3** Research how BNF works and use it to define the syntax of mathematical function notation.

**4** Mathematical expressions can also be represented with diagrams such as **abstract syntax trees** and **syntax (railroad) diagrams**. Which form is more *efficient* in conveying its information? Which form is more useful?

The fact that something is *grammatically* correct does not make it *logically* true.

For example, consider the grammatically correct but illogical English sentence: "The sun is cold."

**5** The **syntax** of a language refers to its structure and rules. The **semantics** of a language is all about meaning.

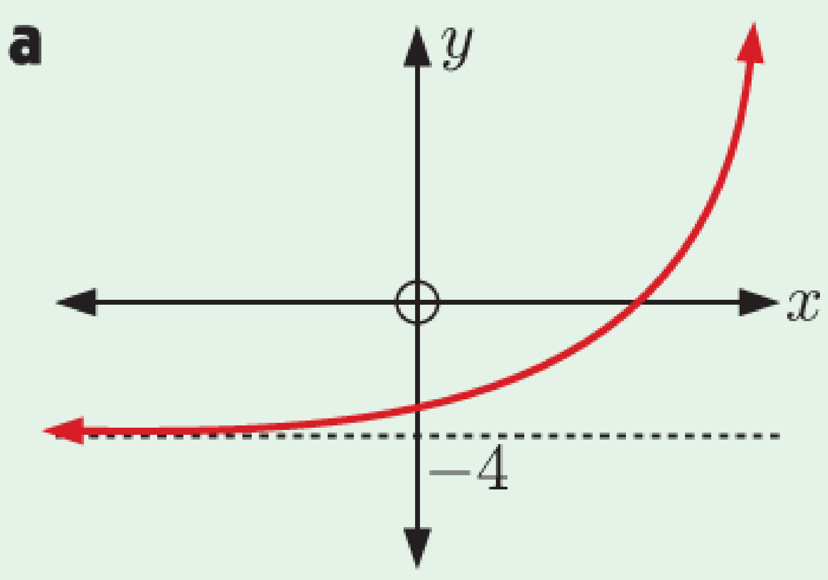
- **a** Why is it important to distinguish between these two concepts?
- **b** In mathematics, is one more important than the other?

## REVIEW SET 15A

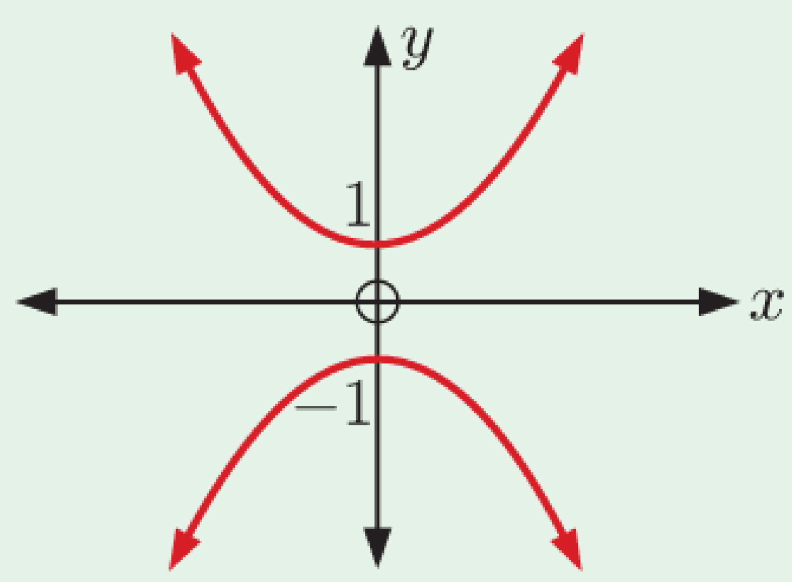
**1** For each graph, state:

**i** the domain **ii** the range **iii** whether the graph shows a function.

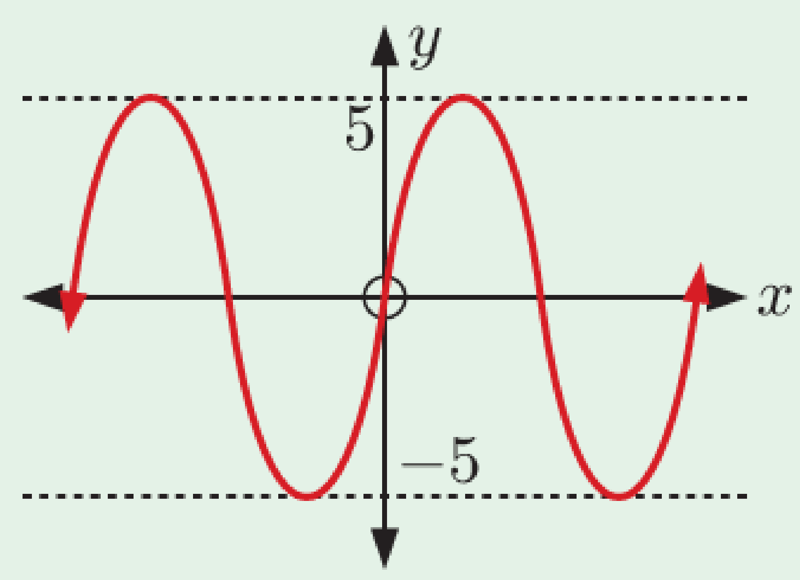
**a**




**b**




**c**




**2** If $f(x) = 2x - x^2$, find:

- **a** $f(2)$
- **b** $f(-3)$
- **c** $f(-\frac{1}{2})$

**3** Suppose $f(x) = ax + b$ where $a$ and $b$ are constants. If $f(1) = 7$ and $f(3) = -5$, find $a$ and $b$.

**4** Consider $f(x) = \dfrac{-2}{x^2}$.

- **a** For what value of $x$ is $f(x)$ undefined?
- **b** Sketch the function using technology.
- **c** State the domain and range of the function.

**5** Consider $f(x) = x^2$ and $g(x) = 1 - 6x$.

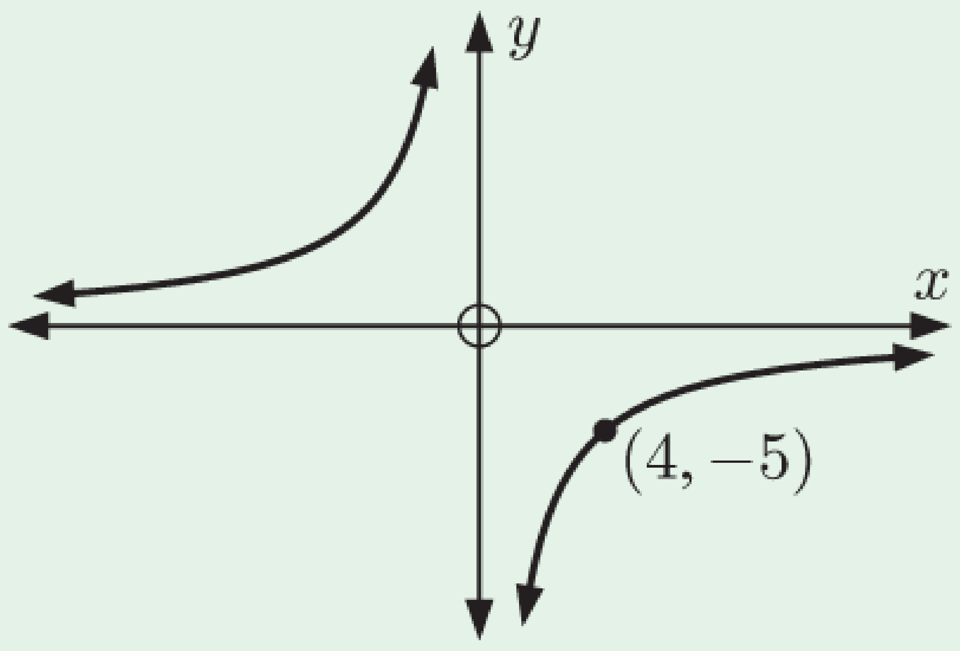
- **a** Show that $f(-3) = g(-\frac{4}{3})$.
- **b** Find $x$ such that $g(x) = f(5)$.

**6** Find the domain and range of:

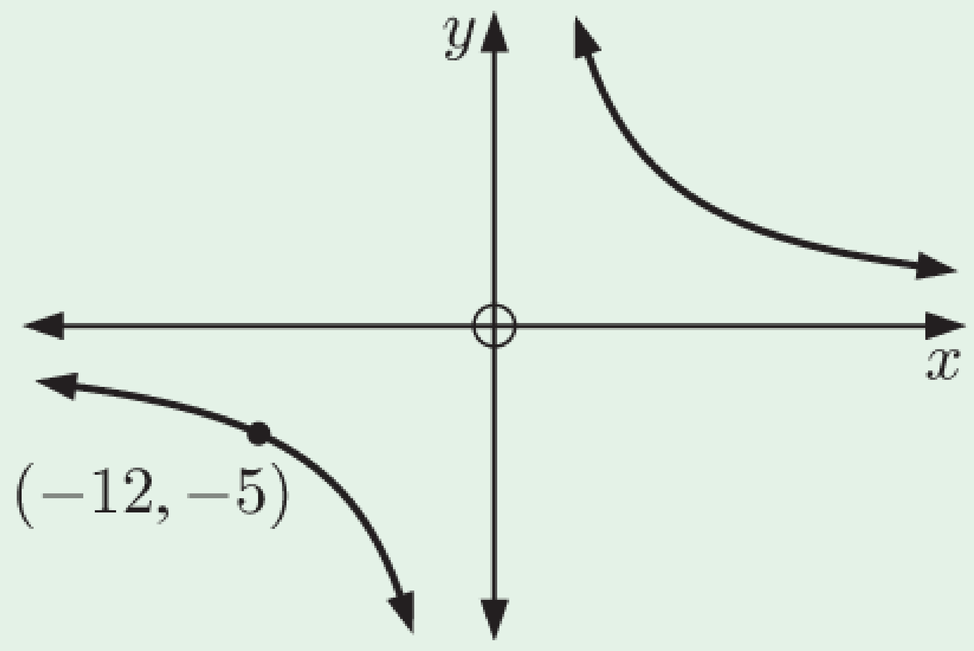
- **a** $y = \sqrt{x+4}$
- **b** $y = -(1-x)^2 + 1$
- **c** $y = 2x^2 - 3x + 1$

**7** Determine the equation of the reciprocal functions:

**a**




**b**




**8** Consider the function $f : x \mapsto \dfrac{4x+1}{2-x}$.

**a** Find the equations of the asymptotes.

**b** State the domain and range of the function.

**c** Draw a sign diagram of the function. Hence discuss the behaviour of the function as it approaches its asymptotes.

**d** Find the axes intercepts.

**e** Sketch the function.

**9** Suppose $f(x) = 2x - 5$ and $g(x) = 3x + 1$.

**a** Find $(f \circ g)(x)$.

**b** Solve $(f \circ g)(x) = f(x+3)$.

**10** If $f(x) = 1 - 2x$ and $g(x) = \sqrt{x}$, find in simplest form:

**a** $(f \circ g)(x)$

**b** $(g \circ f)(x)$

**c** $(g \circ g)(81)$

**11** Suppose $f(x) = \sqrt{x+2}$ and $g(x) = x^2 - 3$.

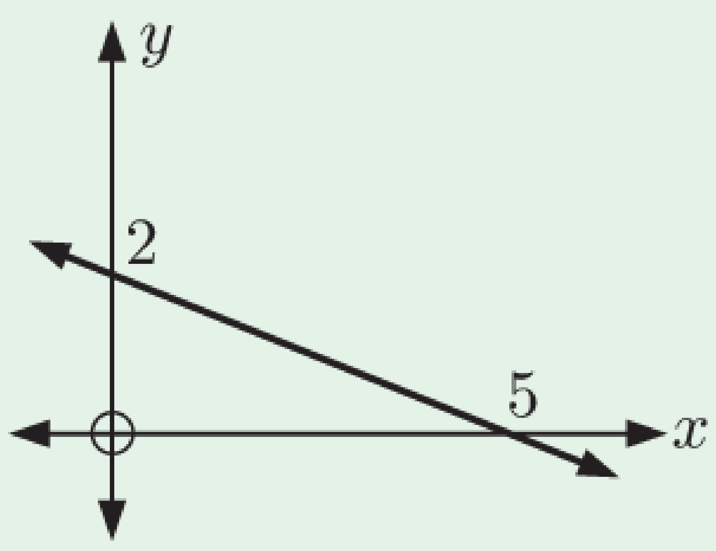
**a** Find $(f \circ g)(x)$, and state its domain and range.

**b** Find $(g \circ f)(x)$, and state its domain and range.

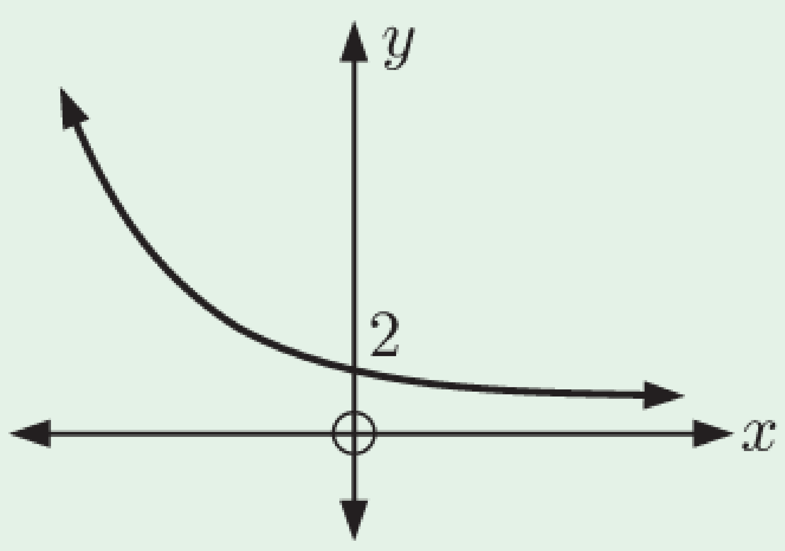
**12** If $f(x) = ax + b$, $f(2) = 1$, and $f^{-1}(3) = 4$, find $a$ and $b$.

**13** Copy the following graphs and draw the inverse function on the same set of axes:

**a**




**b**

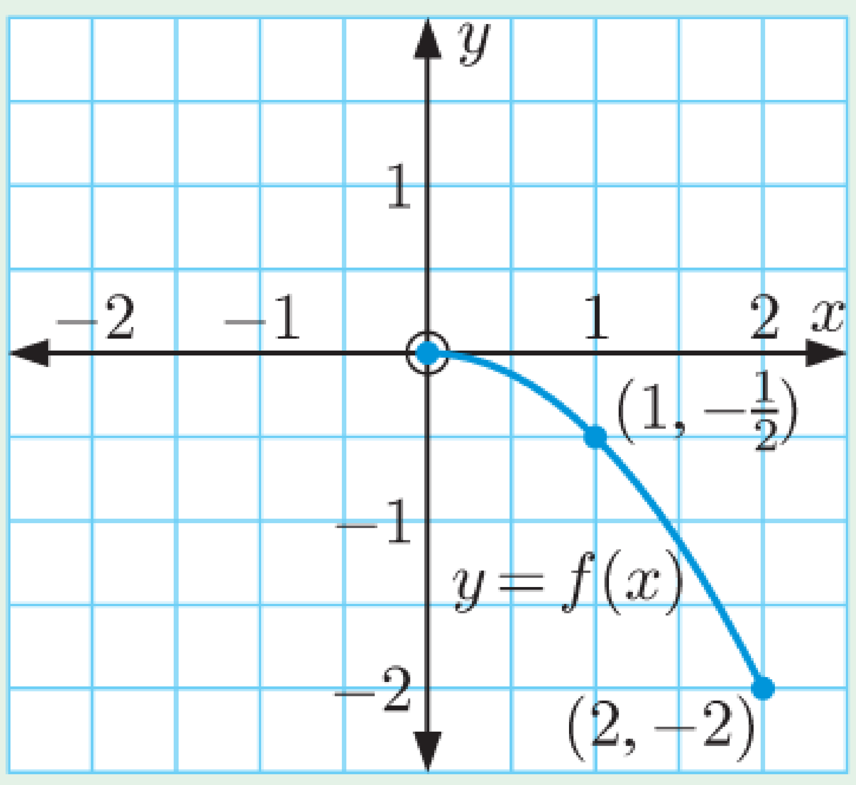



**14** Find $f^{-1}(x)$ given that $f(x)$ is:

**a** $4x + 2$

**b** $\dfrac{3-5x}{4}$

**15** The graph of the function $f(x) = -\frac{1}{2}x^2$, $0 \leqslant x \leqslant 2$ is shown alongside.




**a** Sketch the graph of $y = f^{-1}(x)$.

**b** State the range of $f^{-1}$.

**c** Solve:

**i** $f(x) = -\frac{3}{2}$

**ii** $f^{-1}(x) = 1$

**16** Given $f : x \mapsto 3x + 6$ and $h : x \mapsto \dfrac{x}{3}$, show that $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$.

**17** Show that $f : x \mapsto \dfrac{5x-1}{x-5}$, $x \neq 5$ is self-inverse by:

**a** referring to its graph **b** using algebra.

**18** If $f : x \mapsto \sqrt{x}$ and $g : x \mapsto 3 + x$, find:

**a** $f^{-1}(2) \times g^{-1}(2)$ **b** $(f \circ g)^{-1}(2)$.

**19** **a** Sketch the graph of $g : x \mapsto x^2 + 6x + 7$ for $x \in \,]-\infty, -3]$.

**b** Explain why $g$ has an inverse function $g^{-1}$.

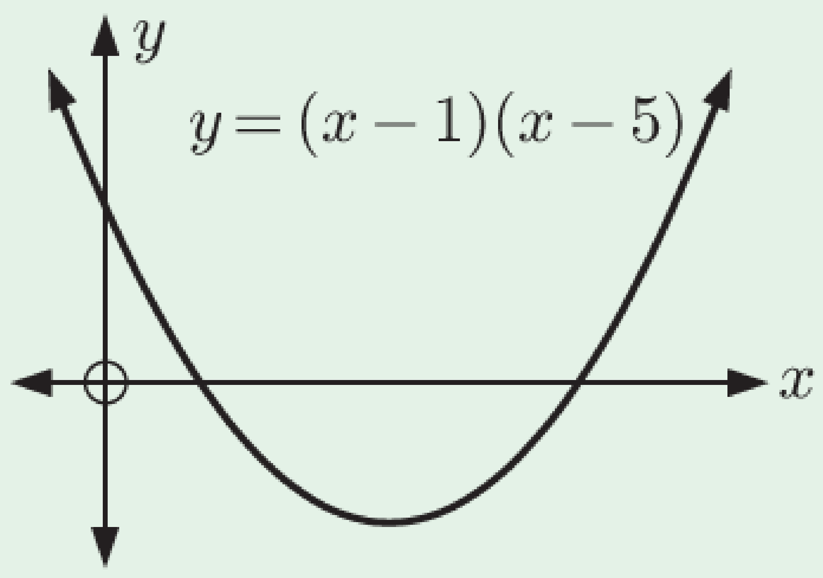
**c** Find algebraically, a formula for $g^{-1}$. **d** Sketch the graph of $y = g^{-1}(x)$.

**e** Find the range of $g$. **f** Find the domain and range of $g^{-1}$.

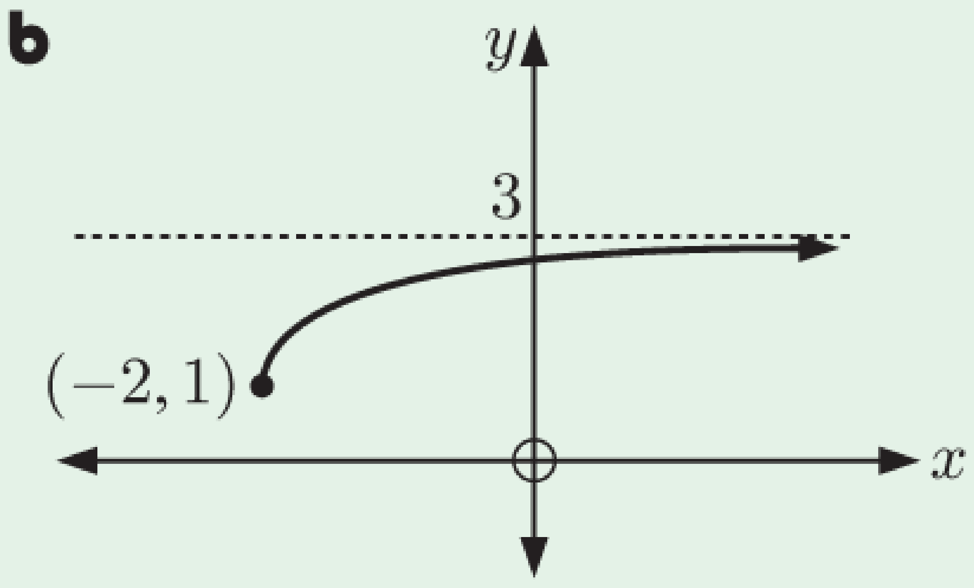
## REVIEW SET 15B

**1** State the domain and range of each function:

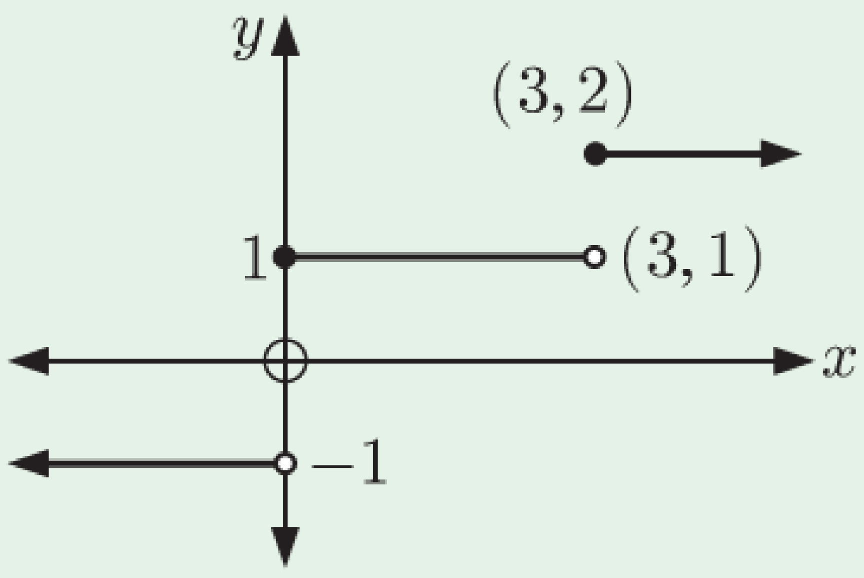
**a**




**b**




**c**




**2** If $g(x) = x^2 - 3x$, find in simplest form:

**a** $g(x+1)$ **b** $g(4x)$

**3** Use algebraic methods to determine whether these relations are functions:

**a** $x + 2y = 10$ **b** $x + y^2 = 10$

**4** State the domain and range of:

**a** $f(x) = 10 + \dfrac{3}{2x-1}$ **b** $f(x) = \sqrt{x+7}$

**5** **a** Use technology to help sketch the graph of the relation $y = \sqrt{9-x}$.

**b** Determine whether the relation is a function.

**c** Find the domain and range of the relation.

**6** Find $a$ such that the range of $y = ax^2 + 12x - 13$ is $\{y \mid y \leqslant 5\}$.

**7** Suppose $f(x) = ax^2 + bx + c$. Find $a$, $b$, and $c$ if $f(0) = 5$, $f(-2) = 21$, and $f(3) = -4$.

**8** For the function $f(x) = -1 + \dfrac{3}{x+2}$:

**a** Find the equations of the asymptotes. **b** State the domain and range.

**c** Find the axes intercepts.

**d** Discuss the behaviour of the function as it approaches its asymptotes.

**e** Sketch the graph of the function.

**16** Given $f : x \mapsto 3x + 6$ and $h : x \mapsto \frac{x}{3}$, show that $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$.

**17** Show that $f : x \mapsto \frac{5x - 1}{x - 5}$, $x \neq 5$ is self-inverse by:

**a** referring to its graph **b** using algebra.

**18** If $f : x \mapsto \sqrt{x}$ and $g : x \mapsto 3 + x$, find:

**a** $f^{-1}(2) \times g^{-1}(2)$ **b** $(f \circ g)^{-1}(2)$.

**19** **a** Sketch the graph of $g : x \mapsto x^2 + 6x + 7$ for $x \in\ ]-\infty, -3]$.

**b** Explain why $g$ has an inverse function $g^{-1}$.

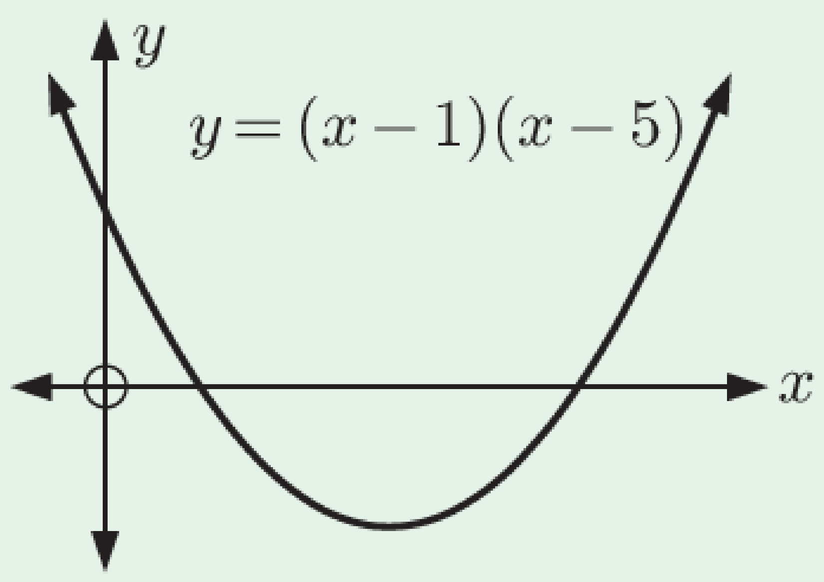
**c** Find algebraically, a formula for $g^{-1}$. **d** Sketch the graph of $y = g^{-1}(x)$.

**e** Find the range of $g$. **f** Find the domain and range of $g^{-1}$.

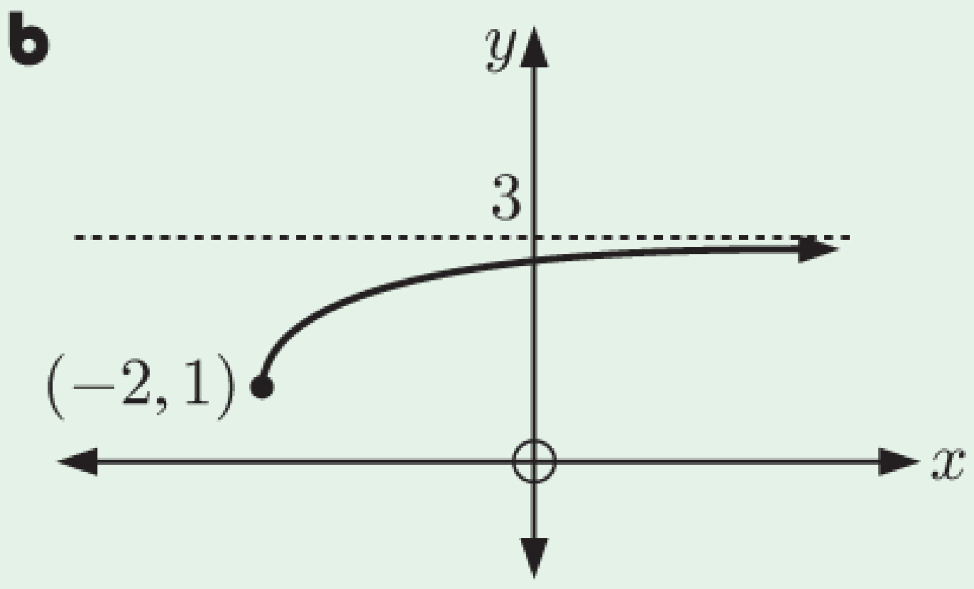
## REVIEW SET 15B

**1** State the domain and range of each function:

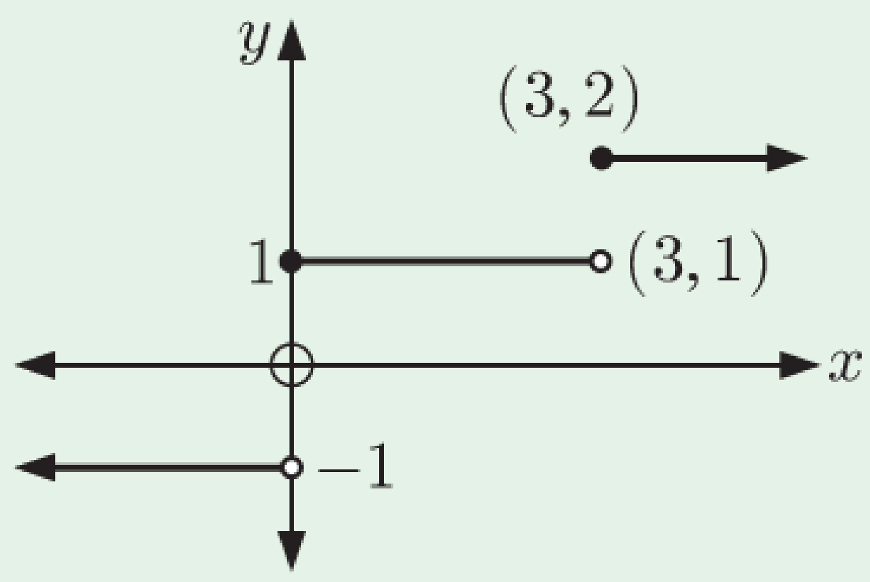
**a**




**b**




**c**




**2** If $g(x) = x^2 - 3x$, find in simplest form:

**a** $g(x + 1)$ **b** $g(4x)$

**3** Use algebraic methods to determine whether these relations are functions:

**a** $x + 2y = 10$ **b** $x + y^2 = 10$

**4** State the domain and range of:

**a** $f(x) = 10 + \frac{3}{2x - 1}$ **b** $f(x) = \sqrt{x + 7}$

**5** **a** Use technology to help sketch the graph of the relation $y = \sqrt{9 - x}$.

**b** Determine whether the relation is a function.

**c** Find the domain and range of the relation.

**6** Find $a$ such that the range of $y = ax^2 + 12x - 13$ is $\{y \mid y \leqslant 5\}$.

**7** Suppose $f(x) = ax^2 + bx + c$. Find $a$, $b$, and $c$ if $f(0) = 5$, $f(-2) = 21$, and $f(3) = -4$.

**8** For the function $f(x) = -1 + \frac{3}{x + 2}$:

**a** Find the equations of the asymptotes. **b** State the domain and range.

**c** Find the axes intercepts.

**d** Discuss the behaviour of the function as it approaches its asymptotes.

**e** Sketch the graph of the function.