14

Integral calculus 1

Learning objectives

By the end of this chapter, you should be familiar with...

- integration as antidifferentiation of functions
- calculating and applying definite integrals
- finding areas under curves (between the curve and the x-axis), and areas between curves
- antidifferentiation with a boundary condition
- solving kinematic problems involving displacement s, velocity v, acceleration a, and total distance travelled
- working with integration of polynomial functions, trigonometric functions and their inverses, and exponential functions
- integration by inspection (reverse chain rule), use of partial fractions, integration by substitution, integration by parts, and repeated integration by parts
- finding volumes of revolution about the x-axis or y-axis.

In Chapters 12 and 13 we learned about the process of differentiation. That is, finding the derivative of a given function. In this chapter we will reverse the process. That is, given a function $f(x)$ how can we find a function $F(x)$ whose derivative is $f(x)$? This process is the opposite of differentiation and is therefore called **antidifferentiation** or **integration**.

14.1 Antiderivative

An **antiderivative** of the function $f(x)$ is a function $F(x)$ such that $\frac{d}{dx}F(x) = F'(x) = f(x)$ wherever $f(x)$ is defined.

For instance, let $f(x) = x^2$. It is not difficult to discover an antiderivative of $f(x)$. Keep in mind that this is a power function. Since the power rule reduces the power of the function by 1, we examine the derivative of x^3: $\frac{d}{dx}(x^3) = 3x^2$

This derivative, however, is 3 times $f(x)$. To 'compensate' for the 'extra' 3 we have to multiply by $\frac{1}{3}$ so that the antiderivative is $\frac{1}{3}x^3$. Now $\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$, and therefore $\frac{1}{3}x^3$ is an antiderivative of x^2.

Table 14.1 shows some examples of functions, each paired with one of its antiderivatives. The diagrams show the relationship between the derivative and the integral as opposite operations.

Function $f(x)$	Antiderivative $F(x)$
1	x
x	$\frac{x^2}{2}$
$3x^2$	x^3
x^4	$\frac{x^5}{5}$
$\cos x$	$\sin x$
$\cos 2x$	$\frac{1}{2}\sin 2x$
e^x	e^x
$\sin x$	$-\cos x$

Table 14.1 Examples of functions paired to antiderivatives

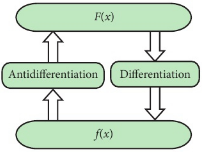


Figure 14.1 The relationship between a derivative and its integral

Example 14.1

Given the function $f(x) = 3x^2$. Find an antiderivative of $f(x)$.

Solution

$F_1(x) = x^3$ is one such antiderivative because $\frac{d}{dx}(F_1(x)) = 3x^2$

The following functions are also antiderivatives because the derivative of each one of them is also $3x^2$.

$$F_2(x) = x^3 + 27 \quad F_3(x) = x^3 - \pi \quad F_4(x) = x^3 + \sqrt{5}$$

Indeed, $F(x) = x^3 + c$ is an antiderivative of $f(x) = 3x^2$ for any constant c.

This is simply because

$$(F(x) + c)' = F'(x) + c' = F'(x) + 0 = f(x)$$

Thus we can say that any single function $f(x)$ has many antiderivatives, whereas a function has only one derivative.

If $F(x)$ is an antiderivative of $f(x)$, then so is $F(x) + c$ for any choice of the constant c. This statement is an indirect conclusion of one of the results of the mean value theorem. Two functions with the same derivative on an interval differ only by a constant on that interval.

The **mean value theorem** states that a function $H(x)$, continuous over an interval $[a, b]$ and differentiable over $]a, b[$ satisfies:
$H(b) - H(a) = (b - a)H'(c)$
for some $c \in]a, b[$

Let $F(x)$ and $G(x)$ be any antiderivatives of $f(x)$; that is, $F'(x) = G'(x)$.

Take $H(x) = F(x) - G(x)$ and any two numbers x_1 and x_2 in the interval $[a, b]$ such that $x_1 < x_2$, then

$$H(x_2) - H(x_1) = (x_2 - x_1)H'(c) = (x_2 - x_1) \cdot (F'(c) - G'(c))$$
$$= (x_2 - x_1) \cdot 0 = 0 \Rightarrow H(x_1) = H(x_2)$$

which means $H(x)$ is a constant function. Hence $H(x) = F(x) - G(x) =$ constant. That is, any two antiderivatives of a function differ by a constant.

Notation

The notation

$$\int f(x)\mathrm{d}x = F(x) + c \qquad \textbf{(1)}$$

where c is an arbitrary constant, means that $F(x) + c$ is an antiderivative of $f(x)$.

Equivalently, $F(x)$ satisfies the condition that

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x) = F'(x) = f(x) \qquad \textbf{(2)}$$

for all x in the domain of $f(x)$.

It is important to note that that **(1)** and **(2)** are just different notations to express the same fact. For example

$$\int x^2\,\mathrm{d}x = \frac{1}{3}x^3 + c \text{ is equivalent to } \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3}x^3\right) = x^2$$

Derivative formula	Equivalent integration formula
$\frac{\mathrm{d}}{\mathrm{d}x}(x^3) = 3x^2$	$\int 3x^2\,\mathrm{d}x = x^3 + c$
$\frac{\mathrm{d}}{\mathrm{d}x}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{2\sqrt{x}}\,\mathrm{d}x = \sqrt{x} + c$
$\frac{\mathrm{d}}{\mathrm{d}t}(\tan t) = \sec^2 t$	$\int \sec^2 t\,\mathrm{d}t = \tan t + c$
$\frac{\mathrm{d}}{\mathrm{d}v}\left(v^{\frac{3}{2}}\right) = \frac{3}{2}v^{\frac{1}{2}}$	$\int \frac{3}{2}v^{\frac{1}{2}}\,\mathrm{d}v = v^{\frac{3}{2}} + c$

Table 14.2 Derivative formulae and their equivalent integration formulae

Note that if we differentiate an antiderivative of $f(x)$, we obtain $f(x)$. Thus

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int f(x)\,\mathrm{d}x\right) = f(x)$$

The expression $\int f(x)\mathrm{d}x$ is called an **indefinite integral** of $f(x)$. The function $f(x)$ is called the **integrand**, and the constant c is called the **constant of integration**.

The integral symbol $\int$ is a medieval S, used by Leibniz as an abbreviation for the Latin word *summa* ('sum').

We think of the combination $\int [\;]\mathrm{d}x$ as a single symbol; we fill in the blank with the formula of the function whose antiderivative we seek. We may regard the differential $\mathrm{d}x$ as specifying the independent variable x both in the function $f(x)$ and in its antiderivatives. This is true for any independent variable, say t, with the notation adjusted appropriately. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int f(t)\,\mathrm{d}t\right) = f(t)$$

and

$$\int f(t)\mathrm{d}t = F(t) + c$$

are equivalent statements.

Basic integration formulae

Many basic integration formulae can be obtained directly from their companion differentiation formulae. Some of the most important are given in Table 14.3.

	Derivation formula	Integration formula
1	$\frac{\mathrm{d}}{\mathrm{d}x}(x) = 1$	$\int \mathrm{d}x = x + c$
2	$\frac{\mathrm{d}}{\mathrm{d}x}(x^{n+1}) = (n+1)x^n,\ n \neq -1$	$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$
3	$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$	$\int \cos x\,\mathrm{d}x = \sin x + c$
4	$\frac{\mathrm{d}}{\mathrm{d}v}(\cos v) = -\sin v$	$\int \sin v\,\mathrm{d}v = -\cos v + c$
5	$\frac{\mathrm{d}}{\mathrm{d}t}(\tan t) = \sec^2 t$	$\int \sec^2 t\,\mathrm{d}t = \tan t + c$
6	$\frac{\mathrm{d}}{\mathrm{d}v}(\mathrm{e}^v) = \mathrm{e}^v$	$\int \mathrm{e}^v\,\mathrm{d}v = \mathrm{e}^v + c$
7	$\frac{\mathrm{d}}{\mathrm{d}x}(\ln\|x\|) = \frac{1}{x}$	$\int \frac{1}{x}\,\mathrm{d}x = \ln\|x\| + c$
8	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{a^x}{\ln a}\right) = a^x$	$\int a^x\,\mathrm{d}x = \frac{1}{\ln a}a^x + c$
9	$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + c$

The integral sign and differential serve as delimiters, adjoining the integrand on the left and right, respectively. In particular we do not write $\int \mathrm{d}x\, f(x)$ when we mean $\int f(x)\mathrm{d}x$

10	$\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-dx}{\sqrt{1-x^2}} = \arccos x + c = -\arcsin x + c$
11	$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$	$\int \frac{dx}{1+x^2} = \arctan x + c$
12	$\frac{d}{dt}\sec t = \tan t \sec t$	$\int \tan t \sec t\, dt = \sec t + c$
13	$\frac{d}{dx}\cot x = -\mathrm{cosec}^2 x$	$\int \mathrm{cosec}^2 t\, dt = -\cot t + c$
14	$\frac{d}{dt}\mathrm{cosec}\, t = -\cot t\, \mathrm{cosec}\, t$	$\int \cot t\, \mathrm{cosec}\, t\, dt = -\mathrm{cosec}\, t + c$

Table 14.3 Many basic integration formulae can be obtained directly from their companion differentiation formulae

Formula 7 is a special case of the 'power' rule shown in formula 2, but needs some modification.

If we are asked to integrate $\frac{1}{x}$, we may attempt to do it using the power rule:

$\int \frac{1}{x}dx = \int x^{-1}dx = \frac{1}{(-1)+1}x^{(-1)+1} + c = \frac{1}{0}x^0 + c$, which is undefined.

However, the solution is found by observing what we learned in Chapter 13:

$\frac{d}{dx}(\ln x) = \frac{1}{x}, x > 0$, implies $\int \frac{1}{x}\, dx = \ln x + c, x > 0$.

The function $\frac{1}{x}$ is differentiable for $x < 0$ too. So, we must be able to find its integral.

The solution lies in the chain rule.

If $x < 0$, then we can write $x = -u$ where $u > 0$. Then $dx = -du$, and

$\int \frac{1}{x}\, dx = \int \frac{1}{-u}(-du) = \int \frac{1}{u}\, du = \ln u + c, u > 0$

But $u = -x$, therefore when $x < 0$:

$\int \frac{1}{x}\, dx = \ln u + c = \ln(-x) + c$, and combining the two results, we have

$\int \frac{1}{x}\, dx = \ln|x| + c, x \neq 0$

Suppose that $f(x)$ and $g(x)$ are differentiable functions and k is a constant. Then:

A constant factor can be moved through an integral sign; that is,

$\int kf(x)\, dx = k\int f(x)\, dx$

An antiderivative of a sum (difference) is the sum (difference) of the antiderivatives; i.e.,

$\int (f(x) \pm g(x))dx$
$= \int f(x)\, dx \pm \int g(x)\, dx$

Example 14.2

Evaluate

(a) $\int 3\cos x\, dx$ (b) $\int (x^3 + x^2)dx$

Solution

(a) $\int 3\cos x\, dx = 3\int \cos x\, dx = 3\sin x + c$

(b) $\int (x^3 + x^2)dx = \int x^3\, dx + \int x^2\, dx = \frac{x^4}{4} + \frac{x^3}{3} + c$

Sometimes it is useful to rewrite the integrand in a different form before performing the integration.

Example 14.3

Evaluate

(a) $\int \frac{t^3 - 3t^5}{t^5} \mathrm{d}t$ (b) $\int \frac{x + 5x^4}{x^2} \mathrm{d}x$

Solution

(a) $\int \frac{t^3 - 3t^5}{t^5} \mathrm{d}t = \int \frac{t^3}{t^5} \mathrm{d}t - \int \frac{3t^5}{t^5} \mathrm{d}t = \int t^{-2} \mathrm{d}t - \int 3\mathrm{d}t = \frac{t^{-1}}{-1} - 3t + c$

$= -\frac{1}{t} - 3t + c$

(b) $\int \frac{x + 5x^4}{x^2} \mathrm{d}x = \int \frac{x}{x^2} \mathrm{d}x + \int \frac{5x^4}{x^2} \mathrm{d}x = \int \frac{1}{x} \mathrm{d}x + \int 5x^2 \mathrm{d}x$

$= \ln|x| + \frac{5x^3}{3} + c$

Integration by simple substitution – change of variables

In this section we will study substitution, a technique that can often be used to transform complex integration problems into simpler ones.

The method of substitution depends on our understanding of the chain rule as well as the use of variables in integration. Two facts to recall:

When we find an antiderivative, we can use any other variable.

That is, $\int f(u)\, \mathrm{d}u = F(u) + c$, where u is a dummy variable in the sense that it can be replaced by any other variable.

Using the chain rule $\frac{\mathrm{d}}{\mathrm{d}x}(F(u(x))) = F'(u(x)) \cdot u'(x)$

Which can be written in integral form as $\int F'(u(x)) \cdot u'(x)\, \mathrm{d}x = F(u(x)) + c$

Or equivalently, since $F(x)$ is an antiderivative of $f(x)$,

$\int f(u(x)) \cdot u'(x)\, \mathrm{d}x = F(u(x)) + c$

For our purposes it will be useful and simpler to let $u(x) = u$ and to write $\frac{\mathrm{d}u}{\mathrm{d}x} = u'(x)$ in its differential form as $\mathrm{d}u = u'(x)\, \mathrm{d}x$ or simply $\mathrm{d}u = u'\, \mathrm{d}x$.

We can now write the integral as

$$\int f(u(x)) \cdot u'(x)\, \mathrm{d}x = \int f(u) \mathrm{d}u = F(u(x)) + c$$

Example 14.4 demonstrates how the method works.

Example 14.4

Evaluate

(a) $\int (x^3 + 2)^{10} \cdot 3x^2 \, dx$ (b) $\int \tan x \, dx$ (c) $\int \cos 5x \, dx$

(d) $\int \cos x^2 \cdot x \, dx$ (e) $\int e^{3x+1} \, dx$

Solution

(a) To integrate this function, it is simplest to make the substitution $u = x^3 + 2$, and so $du = 3x^2 dx$. Now we can write the integral as

$$\int (x^3 + 2)^{10} \cdot 3x^2 \, dx = \int u^{10} \, du = \frac{u^{11}}{11} + c = \frac{(x^3 + 2)^{11}}{11} + c$$

(b) This integrand has to be rewritten first and then we make the substitution:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \int \frac{1}{\cos x} \cdot \sin x \, dx$$

We now let $u = \cos x \Rightarrow du = -\sin x \, dx$, and

$$\int \tan x \, dx = \int \frac{1}{\cos x} \cdot \sin x \, dx = \int \frac{1}{u} \cdot (-du) = -\int \frac{1}{u} du = -\ln|u| + c$$

This last result can be then expressed in two ways:

$$\int \tan x \, dx = -\ln|\cos x| + c, \text{ or}$$

$$\int \tan x \, dx = -\ln|\cos x| + c = \ln|(\cos x)^{-1}| + c = \ln\left|\frac{1}{(\cos x)}\right| + c$$

$$= \ln|\sec x| + c$$

(c) We let $u = 5x$, then $du = 5dx \Rightarrow dx = \frac{1}{5} du$, and so

$$\int \cos 5x \, dx = \int \cos u \cdot \frac{1}{5} du = \frac{1}{5} \int \cos u \, du = \frac{1}{5} \sin u + c$$

$$= \frac{1}{5} \sin 5x + c$$

Another method can be applied here:

The substitution $u = 5x$ requires $du = 5dx$. As there is no factor of 5 in the integrand, and since 5 is a constant, we can multiply and divide by 5 so that we group the 5 and dx to form the du required by the substitution:

$$\int \cos 5x \, dx = \frac{1}{5} \int \cos x \cdot 5dx = \frac{1}{5} \int \cos u \, du = \frac{1}{5} \sin u + c$$

$$= \frac{1}{5} \sin 5x + c$$

The main challenge in using the substitution rule is to think of an appropriate substitution. You should try to select u to be a part of the integrand whose differential is also included (except for the constant). In Example 14.4 (a), we selected u to be $(x^3 + 2)$ knowing that $du = 3x^2 dx$. Then we compensated for the absence of 3. Finding the right substitution is a subtle art, which you will acquire with practice. It is often the case that your first guess may not work.

In integration, multiplying by a constant inside the integral and compensating for that with the reciprocal outside the integral depends on formula 2 from Table 14.3.

However, we cannot do this with a variable.

For example,

$\int e^x \, dx = \frac{1}{2x}\int e^x 2x \, dx$

is **not** valid because $2x$ is not a constant.

(d) By letting $u = x^2$, $du = 2x\,dx$ and so

$$\int \cos x^2 \cdot x\,dx = \frac{1}{2}\int \cos x^2 \cdot 2x\,dx = \frac{1}{2}\int \cos u\,du$$

$$= \frac{1}{2}\sin u + c = \frac{1}{2}\sin x^2 + c$$

(e) $\int e^{3x+1}\,dx = \frac{1}{3}\int e^{3x+1}\,3dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + c = \frac{1}{3}e^{3x+1} + c$

Example 14.5

Evaluate each integral.

(a) $\int e^{-3x}\,dx$ (b) $\int \sin^2 x \cos x\,dx$ (c) $\int 2\sin(3x-5)\,dx$

(d) $\int e^{mx+n}\,dx$ (e) $\int x\sqrt{x}\,dx$ and $F(1) = 2$

Solution

(a) Let $u = -3x$, then $du = -3dx$

$$\int e^{-3x}\,dx = -\frac{1}{3}\int e^{-3x}(-3\,dx) = -\frac{1}{3}\int e^u\,du = -\frac{1}{3}e^u + c$$

$$= -\frac{1}{3}e^{-3x} + c$$

(b) Let $u = \sin x \Rightarrow du = \cos x\,dx$, and hence

$$\int \sin^2 x \cos x\,dx = \int u^2\,du = \frac{1}{3}u^3 + c = \frac{1}{3}\sin^3 x + c$$

(c) Let $u = 3x - 5$, then $du = 3dx$

$$\int 2\sin(3x-5)\,dx = 2\cdot\frac{1}{3}\int \sin(3x-5)\,3\,dx = \frac{2}{3}\int \sin u\,du$$

$$= -\frac{2}{3}\cos u + c = -\frac{2}{3}\cos(3x-5) + c$$

(d) Let $u = mx + n$, then $du = m\,dx$

$$\int e^{mx+n}\,dx = \frac{1}{m}\int e^{mx+n}\,m\,dx = \frac{1}{m}\int e^u\,du = \frac{1}{m}e^u + c = \frac{1}{m}e^{mx+n} + c$$

(e) $F(x) = \int x\sqrt{x}\,dx = \int x^{\frac{3}{2}}\,dx = \dfrac{x^{\frac{5}{2}}}{\left(\frac{5}{2}\right)} + c = \frac{2}{5}x^{\frac{5}{2}} + c$, but $F(1) = 2$

$$F(1) = \frac{2}{5}1^{\frac{5}{2}} + c = \frac{2}{5} + c = 2 \Rightarrow c = \frac{8}{5}$$

Therefore $F(x) = \frac{2}{5}x^{\frac{5}{2}} + \frac{8}{5}$

Examples 14.4 and 14.5 make it clear that Table 14.2 is limited in scope because we cannot use the integrals directly to evaluate composite functions. We therefore need to revise some of the derivative formulae.

	Derivative formula	Integration formula
1	$\frac{d}{dx}(u(x)) = u'(x) \Rightarrow du = u'(x)dx$	$\int du = u + c$
2	$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n u'(x), n \neq -1 \Rightarrow d\left(\frac{u^{n+1}}{n+1}\right) = u^n u'(x)\,dx$	$\int u^n\,du = \frac{u^{n+1}}{n+1} + c, n \neq -1$
3	$\frac{d}{dx}(\sin(u)) = \cos(u)u'(x) \Rightarrow d(\sin(u)) = \cos(u)u'(x)\,dx$	$\int \cos u\,du = \sin u + c$
4	$\frac{d}{dx}(\cos(u)) = -\sin(u)u'(x) \Rightarrow d(\cos(u)) = -\sin(u)u'(x)dx$	$\int \sin u\,du = -\cos u + c$
5	$\frac{d}{dt}(\tan u) = \sec^2 u\, u'(t) \Rightarrow d(\tan u) = \sec^2 u\, u'(t)\,dt$	$\int \sec^2 u\,du = \tan u + c$
6	$\frac{d}{dx}(e^u) = e^u u'(x)\,dx \Rightarrow d(e^u) = e^u u'(x)\,dx$	$\int e^u\,du = e^u + c$
7	$\frac{d}{dx}(\ln\lvert u\rvert) = \frac{1}{u}u'(x) \Rightarrow d(\ln\lvert u\rvert) = \frac{1}{u}u'(x)\,dx$	$\int \frac{1}{u}\,du = \ln\lvert u\rvert + c$
8	$\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}}u'(x) \Rightarrow d(\arcsin u) = \frac{1}{\sqrt{1-u^2}}u'(x)\,dx$	$\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + c$
9	$\frac{d}{dx}(\arctan u) = \frac{1}{1+u^2}u'(x) \Rightarrow d(\arctan u) = \frac{1}{1+u^2}u'(x)\,dx$	$\int \frac{du}{1+u^2} = \arctan u + c$

Table 14.4 More advanced derivative and integration formulae.

Example 14.6

Evaluate each integral.

(a) $\int \sqrt{6x+11}\,dx$

(b) $\int (5x^3+2)^8 x^2\,dx$

(c) $\int \frac{x^3-2}{\sqrt[5]{x^4-8x+13}}\,dx$

(d) $\int \sin^4(3x^2)\cos(3x^2)\,x\,dx$

Solution

(a) We let $u = 6x + 11$ and calculate du:

$$u = 6x + 11 \Rightarrow du = 6\,dx$$

Since du contains the factor 6, the integral is not in the form $\int f(u)\,du$. However, here we can use one of two approaches.

Introduce the factor 6, as we have done before; that is,

$$\int \sqrt{6x+11}\,dx = \frac{1}{6}\int \sqrt{6x+11}\,6\,dx$$

$$= \frac{1}{6}\int \sqrt{u}\,du = \frac{1}{6}\int u^{\frac{1}{2}}\,du$$

$$= \frac{1}{6}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{18}u^{\frac{3}{2}} + c$$

$$= \frac{1}{9}(6x + 11)^{\frac{3}{2}} + c$$

Or since $u = 6x + 11 \Rightarrow du = 6\,dx \Rightarrow dx = \frac{du}{6}$, then

$\int \sqrt{6x + 11}\,dx = \int \sqrt{u}\,\frac{du}{6} = \frac{1}{6}\int u^{\frac{1}{2}}du$, then we follow the same steps as before.

(b) We let $u = 5x^3 + 2$, so $du = 15x^2\,dx$. This means that we need to introduce the factor 15 into the integrand

$$\int (5x^3 + 2)^8 x^2\,dx = \frac{1}{15}\int (5x^3 + 2)^8\, 15x^2\,dx$$

$$= \frac{1}{15}\int u^8\,du = \frac{1}{15}\frac{u^9}{9} + c$$

$$= \frac{1}{135}(5x^3 + 2)^9 + c$$

(c) We let $u = x^4 - 8x + 13 \Rightarrow du = (4x^3 - 8)dx = 4(x^3 - 2)dx$

$$\int \frac{x^3 - 2}{\sqrt[5]{x^4 - 8x + 13}}dx = \frac{1}{4}\int \frac{4(x^3 - 2)dx}{\sqrt[5]{x^4 - 8x + 13}} = \frac{1}{4}\int \frac{du}{u^{\frac{1}{5}}}$$

$$= \frac{1}{4}u^{\frac{1}{5}}du = \frac{1}{4}\frac{u^{\frac{4}{5}}}{\frac{4}{5}} + c$$

$$= \frac{5}{16}(x^4 - 8x + 13)^{\frac{4}{5}} + c$$

(d) We let $u = \sin(3x^2) \Rightarrow du = \cos(3x^2)6x\,dx$ using the chain rule.

$$\int \sin^4(3x^2)\cos(3x^2)\,x\,dx = \frac{1}{6}\int \sin^4(3x^2)\cos(3x^2)\,6x\,dx$$

$$= \frac{1}{6}\int u^4\,du = \frac{1}{6}\frac{u^5}{5} + c$$

$$= \frac{1}{30}\sin^5(3x^2) + c$$

Exercise 14.1

1. Find the most general antiderivative of each function.

(a) $f(x) = x + 2$

(b) $f(t) = 3t^2 - 2t + 1$

(c) $g(x) = \frac{1}{3} - \frac{2}{7}x^3$

(d) $f(t) = (t - 1)(2t + 3)$

(e) $g(u) = u^{\frac{2}{5}} - 4u^3$

(f) $f(x) = 2\sqrt{x} - \frac{3}{2\sqrt{x}}$

(g) $h(\theta) = 3\sin\theta + 4\cos\theta$

(h) $f(t) = 3t^2 - 2\sin t$

(i) $f(x) = \sqrt{x}(2x - 5)$

(j) $g(\theta) = 3\cos\theta - 2\sec^2\theta$

(k) $h(t) = e^{3t-1}$

(l) $f(t) = \dfrac{2}{t}$

(m) $h(u) = \dfrac{t}{3t^2 + 5}$

(n) $h(\theta) = e^{\sin\theta}\cos\theta$

(o) $f(x) = (3 + 2x)^2$

2. Find f.

(a) $f''(x) = 4x - 15x^2$

(b) $f''(x) = 1 + 3x^2 - 4x^3, f'(0) = 2, f(1) = 2$

(c) $f''(t) = 8t - \sin t$

(d) $f'(x) = 12x^3 - 8x + 7, f(0) = 3$

(e) $f'(\theta) = 2\cos\theta - \sin(2\theta)$

3. Evaluate each integral.

(a) $\int x(3x^2 + 7)^5 dx$

(b) $\int \dfrac{x}{(3x^2 + 5)^4} dx$

(c) $\int 2x^2 \sqrt[4]{5x^3 + 2}\, dx$

(d) $\int \dfrac{(3 + 2\sqrt{x})^5}{\sqrt{x}} dx$

(e) $\int t^2 \sqrt{2t^3 - 7}\, dt$

(f) $\int \left(2 + \dfrac{3}{x}\right)^5 \left(\dfrac{1}{x^2}\right) dx$

(g) $\int \sin(7x - 3) dx$

(h) $\int \dfrac{\sin(2\theta - 1)}{\cos(2\theta - 1) + 3} d\theta$

(i) $\int \sec^2(5\theta - 2) d\theta$

(j) $\int \cos(\pi x + 3) dx$

(k) $\int \sec 2t \tan 2t\, dt$

(l) $\int x e^{x^2+1} dx$

(m) $\int \sqrt{t}\, e^{2t\sqrt{t}} dt$

(n) $\int \dfrac{2}{\theta}(\ln\theta)^2 d\theta$

4. Evaluate each integral.

(a) $\int t\sqrt{3 - 5t^2}\, dt$

(b) $\int \theta^2 \sec^2 \theta^3\, d\theta$

(c) $\int \dfrac{\sin\sqrt{t}}{2\sqrt{t}} dt$

(d) $\int \tan^5 2t \sec^2 2t\, dt$

(e) $\int \dfrac{dx}{\sqrt{x}(\sqrt{x} + 2)}$

(f) $\int \sec^5 2t \tan 2t\, dt$

(g) $\int \dfrac{x + 3}{x^2 + 6x + 7} dx$

(h) $\int \dfrac{k^3 x^3}{\sqrt{a^2 - a^4 x^4}} dx$

(i) $\int 3x\sqrt{x - 1}\, dx$

(j) $\int \csc^2 \pi t\, dt$

(k) $\int \sqrt{1 + \cos\theta}\, \sin\theta\, d\theta$

(l) $\int t^2\sqrt{1 - t}\, dt$

(m) $\int \dfrac{r^2 - 1}{\sqrt{2r - 1}} dr$

(n) $\int \dfrac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}} x dx$

14.2 Integration by parts

Although differentiation and integration are strongly linked, finding derivatives is very different from finding integrals. With the derivative rules available, we are able to find the derivative of just about any function we can think of. By contrast, we can compute antiderivatives for a rather small number of functions. Thus far, we have developed a set of basic integration formulas, most of which followed directly from the related differentiation formulas seen in Table 14.2.

In some cases, using substitution helps us reduce the difficulty of evaluating integrals by expressing them in familiar forms. However, there are many cases where simple substitution will not help. For example, we cannot evaluate

$$\int x \cos x \, dx$$

using the methods we have learned so far. However, we can evaluate this integral using integration by parts.

Recall the product rule for differentiation:

$$\frac{d}{dx}(u(x)v(x)) = u'(x)v(x) + u(x)v'(x),$$

which gives rise to the differential form

$$d(u(x)v(x)) = v(x)d(u(x)) + u(x)d(v(x))$$

and for convenience, we will write

$$d(uv) = v\,du + u\,dv$$

If we integrate both sides of this equation, we get

$$\int d(uv) = \int v\,du + \int u\,dv \Leftrightarrow uv = \int v\,du + \int u\,dv$$

Solving this equation for $\int u\,dv$, we get

$$\int u\,dv = uv - \int v\,du$$

This rule is **integration by parts**.

Example 14.7

Evaluate $\int x \cos x \, dx$

Solution

First, observe that we cannot evaluate this as it stands: it is not one of our basic integrals and no substitution can help either.

We need to make a clever choice of u and dv so that the integral on the right side is one that makes the evaluation easier.

We need to choose u (to differentiate) and dv (to integrate), thus we let

$u = x$, and $dv = \cos x\,dx$

Then $du = dx$, and $v = \sin x$. (We will introduce c at the end of the process.)

It can help to organise our work in table form:

$$\begin{array}{ll} u = x & du = dx \\ dv = \cos x\,dx & v = \sin x \end{array}$$

This gives us

$$\int \underbrace{x}_{u} \underbrace{\cos x\,dx}_{dv} = \int u\,dv = uv - \int v\,du$$

$$= x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + c$$

To verify the result, simply differentiate the right-hand side.

$$\frac{d}{dx}(x\sin x + \cos x + c) = \sin x + x\cos x - \sin x + 0 = x\cos x$$

There are three other choices of u and dv that we can make in this problem.

(i) Let

$$\left.\begin{array}{ll} u = \cos x & du = -\sin x\,dx \\ dv = x\,dx & v = \frac{x^2}{2} \end{array}\right] \Rightarrow \int x\cos x\,dx = \frac{x^2}{2}\cos x + \int \frac{x^2}{2}\sin x\,dx$$

This new integral is worse than the one we started with.

(ii) Let

$$\left.\begin{array}{ll} u = x\cos x & du = (\cos x - x\sin x)\,dx \\ dv = dx & v = x \end{array}\right] \Rightarrow$$

$$\int x\cos x\,dx = x^2\cos x - \int x(\cos x - x\sin x)\,dx$$

Again, this new integral is worse than the one we started with.

(iii) Let

$$\begin{array}{ll} u = 1 & du = 0 \\ dv = x\cos x\,dx & v = ?\,? \end{array}$$

This is obviously a bad choice since we still don't know how to integrate $dv = x\cos x\,dx$

The objective of integration by parts is to move from an integral $\int u\,dv$ that we can't see how to evaluate to an integral $\int v\,du$ that we can evaluate. So, keep in mind that integration by parts does not necessarily work all the time, and that we have to develop enough experience with such a process in order to make the correct choice for u and $v\,du$.

Example 14.8

Evaluate

(a) $\int xe^{-x}\,dx$ (b) $\int \ln x\,dx$ (c) $\int x^2\ln x\,dx$

Solution

(a) $\left.\begin{matrix} u = x & du = dx \\ dv = e^{-x}dx & v = -e^{-x} \end{matrix}\right] \Rightarrow \int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx$

$= -xe^{-x} - e^{-x} + c$

(b) $\left.\begin{matrix} u = \ln x & du = \frac{dx}{x} \\ dv = dx & v = x \end{matrix}\right] \Rightarrow \int \ln x\,dx = x\ln x - \int x\frac{dx}{x}$

$= x\ln x - x + c$

(c) Since x^2 is easier to integrate than $\ln x$, and the derivative of $\ln x$ is also simpler than $\ln x$ itself, we make the following substitutions.

$$\left.\begin{matrix} u = \ln x & du = \frac{dx}{x} \\ dv = x^2dx & v = \frac{x^3}{3} \end{matrix}\right] \Rightarrow \int x^2\ln x\,dx = \frac{x^3}{3}\ln x - \int \frac{x^{3}}{3}\frac{dx}{x}$$

$$= \frac{x^3}{3}\ln x - \int \frac{1}{3}x^2dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + c$$

Sometimes we need to use integrations by parts more than once.

Example 14.9

Evaluate $\int x^2\sin x\,dx$

Solution

Since $\sin x$ is equally easy to integrate or differentiate while x^2 is easier to differentiate, we make the following substitution

$$\left.\begin{matrix} u = x^2 & du = 2x\,dx \\ dv = \sin x\,dx & v = -\cos x \end{matrix}\right] \Rightarrow \int x^2\sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx$$

This first step simplified the original integral. However, the right-hand side still needs further integration. Here again, we use Integration by parts.

$$\left.\begin{matrix} u = 2x & du = 2\,dx \\ dv = \cos x\,dx & v = \sin x \end{matrix}\right] \Rightarrow \int 2x\cos x\,dx = 2x\sin x - 2\int \sin x\,dx$$

$$= 2x\sin x + 2\cos x + c$$

Combining the two results, we can now write

$$\int x^2\sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

When applying integration by parts more than once, we need to be careful not to change the nature of the substitution in successive applications. For instance, in Example 14.9, the first substitution was $u = x^2$ and $dv = \sin x\,dx$. In the second step, if we had switched the substitution to $u = \cos x$ and $dv = 2x\,dx$, we would have obtained

$\int x^2\sin x\,dx = -x^2\cos x + x^2\cos x + \int x^2\sin x\,dx = \int x^2\sin x\,dx$

This undoes the previous integration and returns to the original integral.

Example 14.10

Evaluate $\int x^2 e^x \, dx$

Solution

Since e^x is equally easy to integrate or differentiate, while x^2 is easier to differentiate, we make the following substitution

$$\left.\begin{matrix} u = x^2 & du = 2x\,dx \\ dv = e^x\,dx & v = e^x \end{matrix}\right] \Rightarrow \int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$$

This first step simplified the original integral. However, the right-hand side still needs further integration. Here again, we use integration by parts.

$$\left.\begin{matrix} u = 2x & du = 2\,dx \\ dv = e^x dx & v = e^x \end{matrix}\right] \Rightarrow \int 2xe^x dx = 2xe^x - 2\int e^x dx$$
$$= 2xe^x - 2e^x + c$$

Hence

$$\int x^2 e^x dx = x^2 e^x - \int 2xe^x dx$$
$$= x^2 e^x - 2xe^x + 2e^x + c$$

Using integration by parts to find unknown integrals

Integrals like the one in Example 14.11 often occur in electricity problems. Their evaluation requires repeated applications of integration by parts followed by algebraic manipulation.

Example 14.11

Evaluate $\int \cos x \, e^x \, dx$

Solution

Let

$$\left.\begin{matrix} u = e^x & du = e^x dx \\ dv = \cos x\,dx & v = \sin x \end{matrix}\right] \Rightarrow \int \cos x e^x dx = e^x \sin x - \int \sin x \, e^x dx$$

The second integral is of the same nature, so use integration by parts again:

$$\left.\begin{matrix} u = e^x & du = e^x dx \\ dv = \sin x\,dx & v = -\cos x \end{matrix}\right] \Rightarrow \int \sin x \, e^x dx = -e^x \cos x + \int \cos x \, e^x dx$$

Hence:

$$\int \cos x\, e^x\, dx = e^x \sin x - \int \sin x e^x\, dx$$
$$= e^x \sin x - \left(-e^x \cos x + \int \cos x e^x\, dx\right)$$
$$= e^x \sin x + e^x \cos x - \int \cos x e^x\, dx$$

Now, the unknown integral appears on both sides of the equation, thus

$$\int \cos x\, e^x\, dx + \int \cos x\, e^x\, dx = e^x \sin x + e^x \cos x$$
$$\Rightarrow 2\int \cos x\, e^x\, dx = e^x \sin x + e^x \cos x$$
$$\Rightarrow \int \cos x\, e^x\, dx = \frac{e^x \sin x + e^x \cos x}{2} + c$$

Example 14.12

Evaluate $\int x \ln x\, dx$

Solution

$$\left.\begin{aligned} u &= \ln x & du &= \frac{dx}{x} \\ dv &= x\,dx & v &= \frac{x^2}{2} \end{aligned}\right] \Rightarrow \int x \ln x\, dx = \frac{x^2}{2}\ln x - \int \frac{x^{\cancel{2}1}}{2}\frac{dx}{\cancel{x}}$$
$$= \frac{x^2}{2}\ln x - \int \frac{x\,dx}{2} = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$$

Alternatively, we could have used a different substitution

$$\left.\begin{aligned} u &= x\ln x & du &= (\ln x + 1)dx \\ dv &= dx & v &= x \end{aligned}\right] \Rightarrow \int x \ln x\, dx = x^2 \ln x - \int x(\ln x + 1)dx$$
$$= x^2 \ln x - \int x \ln x\, dx - \int x\, dx$$

Adding $\int x \ln x\, dx$ to both sides and integrating $\int x\, dx$ we get

$$\int x \ln x\, dx + \int x \ln x\, dx = x^2 \ln x - \frac{x^2}{2} + c$$
$$\Rightarrow 2\int O{::}\, x \ln x\, dx = x^2 \ln x - \frac{x^2}{2} + c$$
$$\Rightarrow \int x \ln x\, dx = \frac{1}{2}\left(x^2 \ln x - \frac{x^2}{2} + c\right) = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c$$

The constant c is arbitrary, and hence it is unimportant whether we use $\frac{c}{2}$ or c in our final answer.

Exercise 14.2

1. Evaluate each integral.

(a) $\int x^2 e^{-x^3} dx$ **(b)** $\int x^2 e^{-x} dx$ **(c)** $\int x^2 \cos 3x \, dx$

(d) $\int x^2 \sin ax \, dx$ **(e)** $\int \cos x \ln(\sin x) \, dx$ **(f)** $\int x \ln x^2 \, dx$

(g) $\int x^2 \ln x \, dx$ **(h)** $\int x^2(e^x - 1) \, dx$ **(i)** $\int x \cos \pi x \, dx$

(j) $\int e^{3t} \cos 2t \, dt$ **(k)** $\int \arcsin x \, dx$ **(l)** $\int x^3 e^x \, dx$

(m) $\int e^{-2x} \sin 2x \, dx$ **(n)** $\int \sin(\ln x) dx$ **(o)** $\int \cos(\ln x) \, dx$

(p) $\int \ln(x + x^2) \, dx$ **(q)** $\int e^{kx} \sin x \, dx$ **(r)** $\int x \sec^2 x \, dx$

(s) $\int \sin x \sin 2x \, dx$ **(t)** $\int x \arctan x \, dx$ **(u)** $\int \frac{\ln x}{\sqrt{x}} dx$

2. In one scene of the movie *Stand and Deliver*, the teacher shows his students how to evaluate $\int x^2 \sin x \, dx$ by setting up a chart similar to this:

	$\sin x$	
x^2	$-\cos x$	+
$2x$	$-\sin x$	−
2	$\cos x$	+

(a) Multiply across each row and add the result.

(b) The integral is

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + c$$

Explain why the method works for this problem.

3. Use the result of question 2 to evaluate each integral.

(a) $\int x^4 \sin x \, dx$ **(b)** $\int x^5 \cos x \, dx$ **(c)** $\int x^4 e^x \, dx$

4. Show that the method used in question **2** will not work with $\int x^2 \ln x \, dx$

5. Show that

$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$, then use this reduction formula to show that $\int x^4 e^x dx = ax^4 e^x + bx^3 e^x + cx^2 e^x + dx e^x + fe^x + g$, where $a, b, c, \ldots, g$ are to be determined.

6. Show that $\int x^n \ln x \, dx = \frac{x^{n+1}}{n+1} \ln x - \frac{x^{n+1}}{(n+1)^2} + c$

7. Show that $\int e^{mx} \cos nx \, dx = \frac{e^{mx}(m \cos nx + n \sin nx)}{m^2 + n^2} + c$

8. Show that $\int e^{mx} \sin nx \, dx = \frac{e^{mx}(m \sin nx - n \cos nx)}{m^2 + n^2} + c$

14.3 More methods of integration

In Section 14.2, we looked at a very powerful method for integration that has a wide range of applications. However, integration by parts does not work for all situations. In some cases, even when it does work, it may not be the most efficient method. In this section we will consider a few trigonometric integrals and some substitutions related to trigonometric functions or their inverses.

These trigonometric identities will prove very helpful:

1. $\cos^2\theta + \sin^2\theta = 1$
2. $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$
3. $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$
4. $\sec^2\theta = 1 + \tan^2\theta$

Example 14.13

Evaluate

(a) $\int \sin^2 x\,dx$ (b) $\int \cos^4\theta\,d\theta$

Solution

(a) We can use identity 2 from the list above:

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int (1-\cos 2x)dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c$$

(b) Using identity 3:

$$\int \cos^4\theta\,d\theta = \int \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \frac{1}{4}\int (1 + 2\cos 2\theta + \cos^2 2\theta)d\theta$$

$$= \frac{1}{4}\int\left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right)d\theta$$

$$= \frac{1}{8}\left(2\theta + 2\sin 2\theta + \theta + \frac{1}{4}\sin 4\theta\right) + c$$

$$= \frac{1}{32}(12\theta + 8\sin 2\theta + \sin 4\theta) + c$$

Here is a list how to find some integrals.

Integral	How to find it
$\int \sin^m x \cos^n x \, dx$	If m is odd then break $\sin^m x$ into $\sin x$ and $\sin^{m-1} x$, use the substitution $u = \cos x$ and change the integral into the form $\int \cos^p x \, \sin x \, dx = \int u^p \, du$. Similarly if n is odd.
$\int \tan^m x \sec^n x \, dx$	If m and n are odd, break off a term for $\sec x \tan x$ and express the integrand in terms of $\sec x$ since $d(\sec x) = \sec x \tan x \, dx$
$\int \tan^n x \, dx$	Write the integrand as $\int \tan^{n-2} x \tan^2 x \, dx$, replace $\tan^2 x$ with $\sec^2 x - 1$ and then use $u = \tan x$
$\int \sec^n x \, dx$	If n is even, factor a $\sec^2 x$ out and write the rest in terms of $\tan^2 x + 1$. If n is odd, factor a $\sec^3 x$ out. Here, integration by parts may be useful.

Table 14.5 How to find some integrals

In exam papers, any non-standard cases will be accompanied by a recommended substitution.

Example 14.14

Evaluate $\int \sec x \, dx$

Solution

This integral is evaluated using a clever multiplication by an atypical factor:

$$\int \sec x \, dx = \int \sec x \frac{\tan x + \sec x}{\tan x + \sec x} \, dx = \int \frac{\sec x \tan x + \sec^2 x}{\tan x + \sec x} \, dx$$

Now use the substitution $u = \sec x + \tan x \Rightarrow du = (\sec x \tan x + \sec^2 x)dx$, hence

$$\int \sec x \, dx = \int \frac{\sec x \tan x + \sec^2 x}{\tan x + \sec x} \, dx = \int \frac{du}{u}$$

$$= \ln|u| + c = \ln|\tan x + \sec x| + c$$

Example 14.15

Evaluate $\int \sec^3 x \, dx$

Solution

This can be evaluated using integration by parts and some of the results we used earlier.

$$u = \sec x \qquad du = \sec x \tan x \, dx$$
$$dv = \sec^2 x \, dx \qquad v = \tan x$$

Hence,

$$\int \sec^3 x\,dx = \sec x \tan x - \int \tan x \sec x \tan x\,dx$$
$$= \sec x \tan x - \int \sec x \tan^2 x\,dx$$
$$= \sec x \tan x - \int \sec x \sec^2 x - 1\,dx$$
$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

Adding $\int \sec^3 x\,dx$ to both sides

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$
$$= \sec x \tan x + \ln|\sec x + \tan x|$$

And finally

$$\int \sec^3 x\,dx = \frac{\sec x \tan x + \ln|\sec x + \tan x|}{2} + c$$

Example 14.16

Evaluate $\int \sin^3 x \cos^3 x\,dx$

Solution

This integral can be evaluated by separating either a cosine or a sine, then writing the rest of the expression in terms of sine or cosine.

We will separate a cosine here

$$\int \sin^3 x \cos^3 x\,dx = \int \sin^3 x \cos^2 x \cos x\,dx$$
$$= \int \sin^3 x (1 - \sin^2 x) \cos x\,dx$$
$$= \int (\sin^3 x - \sin^5 x) \cos x\,dx$$

Now we let

$u = \sin x \Rightarrow du = \cos x\,dx$, and hence

$$\int \sin^3 x \cos^3 x\,dx = \int (\sin^3 x - \sin^5 x) \cos x\,dx$$
$$= \int (u^3 - u^5)\,du = \frac{u^4}{4} - \frac{u^6}{6} + c$$
$$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + c$$

Exercise 14.3

1. Evaluate each integral.

(a) $\int \sin^3 t \cos^2 t \, dt$

(b) $\int \sin^3 t \cos^3 t \, dt$

(c) $\int \sin^3 3\theta \cos 3\theta \, d\theta$

(d) $\int \frac{1}{t^2} \sin^5\left(\frac{1}{t}\right) \cos^2\left(\frac{1}{t}\right) dt$

(e) $\int \frac{\sin^3 x}{\cos^2 x} dx$

(f) $\int \tan^5 3x \sec^2 3x \, dx$

(g) $\int \theta \tan^3(\theta^2) \sec^4(\theta^2) \, d\theta$

(h) $\int \frac{1}{\sqrt{t}} \tan^3 \sqrt{t} \sec^3 \sqrt{t} \, dt$

(i) $\int \tan^4(5t) \, dt$

(j) $\int \frac{dt}{1 + \sin t}$

(k) $\int \frac{d\theta}{1 + \cos \theta}$

(l) $\int \frac{1 + \sin t}{\cos t} dt$

(m) $\int \frac{\sin x - 5 \cos x}{\sin x + \cos x} dx$

(n) $\int \frac{\sec \theta \tan \theta}{1 + \sec^2 \theta} d\theta$

(o) $\int \frac{\arctan t}{1 + t^2} dt$

(p) $\int \frac{1}{(1 + t^2)\arctan t} dt$

(q) $\int \frac{dx}{x\sqrt{1 - (\ln x)^2}}$

(r) $\int \sin^3 x \, dx$

(s) $\int \frac{\sin^3 x}{\sqrt{\cos x}} dx$

(t) $\int \frac{\sin^3 \sqrt{x}}{\sqrt{x}} dx$

(u) $\int \cos t \cos^3(\sin t) \, dt$

(v) $\int \frac{\cos \theta + \sin 2\theta}{\sin \theta} d\theta$

2. Evaluate each integral.

(a) $\int t \sec t \tan t \, dt$

(b) $\int \frac{\cos x}{2 - \sin x} dx$

(c) $\int e^{-2x} \tan(e^{-2x}) \, dx$

(d) $\int \frac{\sec(\sqrt{t})}{\sqrt{t}} dt$

(e) $\int \frac{dt}{1 + \cos 2t}$

(f) $\int \sqrt{1 - 9x^2} \, dx$

(g) $\int \frac{dx}{(x^2 + 4)^{3/2}}$

(h) $\int \sqrt{4 + t^2} \, dt$

(i) $\int \frac{3e^t \, dt}{4 + e^{2t}}$

(j) $\int \frac{1}{\sqrt{9 - 4x^2}} dx$

(k) $\int \frac{1}{\sqrt{4 + 9x^2}} dx$

(l) $\int \frac{\cos x}{\sqrt{1 + \sin^2 x}} dx$

(m) $\int \frac{x}{\sqrt{4 - x^2}} dx$

(n) $\int \frac{x}{x^2 + 16} dx$

(o) $\int \frac{\sqrt{4 - x^2}}{x^2} dx$

(p) $\int \frac{dx}{(9 - x^2)^{3/2}}$

! For part (j), multiply the integrand by $\frac{1 - \sin t}{1 - \sin t}$

! For part (m), find numbers a and b such that you can replace $\sin x - 5 \cos x$ with an expression involving terms $a(\sin x + \cos x)$ and $b(\cos x - \sin x)$

! Parts (f) to –(v) will need trigonometric substitution.

(q) $\int x\sqrt{1 + x^2}\,dx$

(r) $\int e^{2x}\sqrt{1 + e^{2x}}\,dx$

(s) $\int e^{x}\sqrt{1 - e^{2x}}\,dx$

(t) $\int \frac{e^{x}\,dx}{\sqrt{e^{2x} + 9}}$

(u) $\int \frac{\ln x}{\sqrt{x}}\,dx$

(v) $\int \frac{x^3}{(x + 2)^2}\,dx$

3. The integral $\int \frac{x}{x^2 + 9}\,dx$ can be evaluated either by trigonometric substitution or by direct substitution. Do both, and reconcile the results.

4. The integral $\int \frac{x^2}{x^2 + 9}\,dx$ can be evaluated either by trigonometric substitution or by rewriting the numerator as $(x^2 + 9) - 9$. Do it both ways and reconcile the results.

14.4 Area and the definite integral

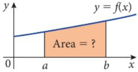


Figure 14.2 How do we find the area?

The function $f(x)$ is continuous and non-negative on an interval $[a, b]$. How do we find the area between the graph of $f(x)$ and the interval $[a, b]$ on the x-axis? (Figure 14.2)

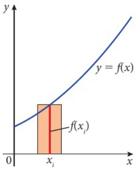


Figure 14.3 Dividing the base interval into subintervals

We divide the base interval $[a, b]$ into n equal subintervals, and over each subinterval construct a rectangle that extends from the x-axis to any point on the curve $y = f(x)$ that is above the subinterval; the particular point does not matter – it can be above the centre, above one endpoint, or above any other point in the subinterval. In Figure 14.3 it is at the centre.

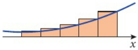


Figure 14.4 The total area of the rectangles can be viewed as an approximation

For each n, the total area of the rectangles can be viewed as an approximation to the exact area in question. As n increases, these approximations will get better and better and will eventually approach the exact area as a limit. See Figures 14.3–14.5

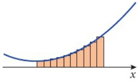


Figure 14.5 As n increases, the approximations get better

A traditional approach would be to study how the choice of where to put the rectangular strip does not affect the approximation as the number of intervals increases. We can construct inscribed rectangles that, at the start, give us an underestimate of the area (Figure 14.6). On the other hand we can construct circumscribed rectangles that, at the start, overestimate the area (Figure 14.7).

As the number of intervals increases, the difference between the overestimates and the underestimates will approach 0.

Figures 14.8 and 14.9 show n inscribed and circumscribed rectangles and Figure 14.10 shows the difference between the overestimates and underestimates.

Figure 14.10 shows that as the number n increases, the difference between the estimates will approach 0. Because we set up our rectangles by choosing a point inside the interval, the areas of the rectangles will lie between the overestimates and underestimates, and hence, as the difference between the extremes approaches zero, the rectangles we construct will give the area of the region required.

If we consider the width of each interval to be $\triangle x$, then the area of any rectangle is given as

$$A_i = f(x_i^*)\triangle x$$

The total area of the rectangles so constructed is

$$A_n = \sum_{i=0}^{n} f(x_i^*)\triangle x$$

where x_i^* is an arbitrary point within any subinterval $[x_{i-1}, x_i]$, $x_0 = a$, and $x_n = b$.

In the case of a function $f(x)$ that has both positive and negative values on $[a, b]$, it is necessary to consider the signs of the areas in the following sense.

On each subinterval, we have a rectangle with width $\triangle x$ and height $f(x^*)$. If $f(x^*) > 0$, then this rectangle is above the x-axis; if $f(x^*) < 0$, then this rectangle is below the x-axis. We will consider the sum defined above as the sum of the signed areas of these rectangles. That means the total area on the interval is the sum of the areas above the x-axis minus the sum of the areas of the rectangles below the x-axis.

We are now ready to look at a loose definition of the definite integral:

If $f(x)$ is a continuous function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into n subintervals of equal width $\triangle x = \dfrac{(b-a)}{n}$. We let $x_0 = a$, and $x_n = b$ and we choose $x_1^*, x_2^*, \ldots, x_n^*$ in these subintervals, so that x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the definite integral of $f(x)$ from a to b is

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\triangle x$$

In the notation $\int_a^b f(x)\,\mathrm{d}x$, a and b are called the limits of integration: a is the lower limit and b is the upper limit.

Because we have assumed that $f(x)$ is continuous, it can be proved that the limit definition above always exists and gives the same value no matter how we choose the points x_i^*. If we take these points at the centre, at two thirds the distance from the lower endpoint or at the upper endpoint, the value is the same. This why we will state the definition of the integral from now on as

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\triangle x$$

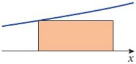


Figure 14.6 Underestimation of area

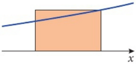


Figure 14.7 Overestimation of area

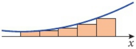


Figure 14.8 n inscribed rectangles

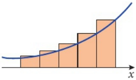


Figure 14.9 n circumscribed rectangles

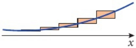


Figure 14.10 difference between over- and under-estimates

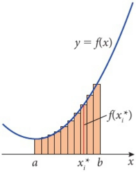


Figure 14.11 Area of each circumscribed rectangle

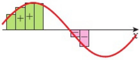


Figure 14.12 Areas above and below the x-axis

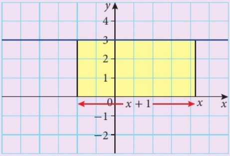
For a more rigorous treatment of the definition of definite integrals using Riemann sums, refer to university calculus books. Such a treatment is beyond the scope of the IB syllabus and this book.

Calling the area under the function an integral is no coincidence. To make the point, let us take the following example:

Example 14.17

Find the area $A(x)$ between the graph of the function $f(x) = 3$ and the interval $[-1, x]$, and find the derivative $A'(x)$ of this area function.

Solution



The area in question is

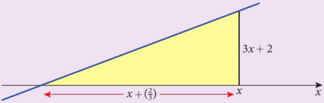
$$A(x) = 3(x - (-1)) = 3x + 3$$

$$A'(x) = 3 = f(x)$$

Example 14.18

Find the area $A(x)$ between the graph of the function $f(x) = 3x + 2$ and the interval $\left[-\frac{2}{3}, x\right]$, and find the derivative $A'(x)$ of this area function.

Solution



The area in question is

$$A(x) = \frac{1}{2}\left(x + \frac{2}{3}\right)(3x + 2) = \frac{1}{6}(3x + 2)^2$$

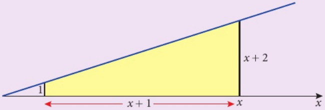
since this is the area of a triangle. Hence

$$A'(x) = \frac{1}{6} \times 2(3x + 2) \times 3 = 3x + 2 = f(x)$$

Example 14.19

Find the area $A(x)$ between the graph of the function $f(x) = x + 2$ and the interval $[-1, x]$, and find the derivative $A'(x)$ of this area function.

Solution



This is a trapezium, so the area is

$$A(x) = \frac{1}{2}(1 + (x + 2))(x + 1) = \frac{1}{2}(x^2 + 4x + 3), \text{ and}$$

$$A'(x) = \frac{1}{2} \times (2x + 4) = x + 2 = f(x)$$

Note that in every case, $A'(x) = f(x)$

That is, the derivative of the area function $A(x)$ is the function whose graph forms the upper boundary of the region. It can be shown that this relation is true not only for linear functions but for all continuous functions. Thus, to find the area function $A(x)$, we can look instead for a particular function whose derivative is $f(x)$. This is, of course, nothing but the antiderivative of $f(x)$.

So, intuitively, as we have seen above, we define the area function as

$$A(x) = \int_a^x f(t)\, dt, \text{ that is } A'(x) = f(x)$$

This is the trigger to the **fundamental theorem of calculus**.

We will now look at some of the properties of the definite integral.

Basic properties of the definite integral

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

When we defined the definite integral $\int_a^b f(x)\, dx$, we implicitly assumed that $a < b$. When we reverse a and b, then $\triangle x$ changes from $\frac{(b-a)}{n}$ to $\frac{(a-b)}{n}$. Therefore the result above follows.

$$\int_a^b f(x)\, dx = 0$$

When $a = b$, then $\triangle x = 0$, and so, the result above follows.

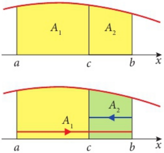


Figure 14.13 $A(x) = A_1 + A_2$

$$\int_a^b c\,dx = c(b-a)$$

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx, \text{ where } c \text{ is any constant.}$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

This property can be demonstrated as follows. The area from a to b is the sum of the two areas, that is $A(x) = A_1 + A_2$ (Figure 14.13). Additionally, even if $c > b$ the relationship holds because the area from c to b in this case will be negative.

Average value of a function

From statistics, the average value of a variable is $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$

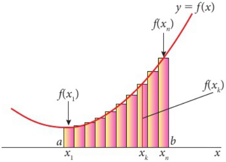


Figure 14.14 $y = f(x)$ partitioned into n subintervals

We can also think of the average value of a function in the same manner. Consider a continuous function $f(x)$ defined over a closed interval $[a, b]$. We partition this interval into n subintervals of equal length in a fashion similar to the previous discussion. Each interval has a length

$$\triangle x = \frac{b-a}{n}$$

The average value of $f(x)$ can be defined as

$$av(f) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}, \text{ or written in sigma notation}$$

$$av(f) = \frac{\sum_{k=1}^{n} f(x_k)}{n} = \frac{1}{n}\sum_{k=1}^{n} f(x_k)$$

However,

$$\triangle x = \frac{b-a}{n} \Rightarrow \frac{1}{n} = \frac{\triangle x}{b-a}, \text{ hence}$$

$$av(f) = \frac{1}{n}\sum_{k=1}^{n} f(x_k) = \frac{\Delta x}{b-a}\sum_{k=1}^{n} f(x_k)$$

This leads us to the following definition of the average value of a function $f(x)$ over an interval $[a, b]$:

The average (mean value) of an integrable function $f(x)$ over an interval $[a, b]$ is given by

$$av(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

Max–min inequality

If f_{max} and f_{min} represent the maximum and minimum values of a non-negative, continuous, differentiable function $f(x)$ over an interval $[a, b]$, then the area under the curve lies between the area of the rectangle with base $[a, b]$ and f_{min} as height, and the rectangle with f_{max} as height.

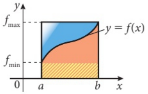


Figure 14.15 Max-min inequality

That is

$$(b - a)\, f_{min} \leqslant \int_a^b f(x)\,dx \leqslant (b - a)\, f_{max}$$

With the assumption that $b > a$, this in turn is equivalent to

$$f_{min} \leqslant \frac{1}{b - a}\int_a^b f(x)\,dx \leqslant f_{max}$$

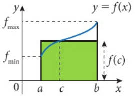


Figure 14.16 Average value

Using the intermediate value theorem, we can ascertain that there is at least one point $c \in [a, b]$ where

$$f(c) = \frac{1}{b - a}\int_a^b f(x)\,dx$$

The value $f(c)$ in this theorem is, in fact, the average value of the function.

The first fundamental theorem of integral calculus

Our understanding of the definite integral as the area under the curve for $f(x)$ helps us establish the basis for the fundamental theorem of integral calculus.

In the definition of definite integral, we'll make the upper limit a variable, say x. Then we will call the area between a and x, $A(x)$; that is,

$$A(x) = \int_a^x f(t)\,dt$$

Consequently,

$$A(x + h) = \int_a^{x+h} f(t)\,dt$$

Now, if we want to find the derivative of $A(x)$, we evaluate

$$\lim_{h \to 0} \frac{A(x + h) - A(x)}{h}$$

Using the properties of definite integrals discussed earlier, we have

$$\begin{aligned} A(x + h) - A(x) &= \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt \\ &= \int_x^a f(t)\,dt + \int_a^{x+h} f(t)\,dt \\ &= \int_x^{x+h} f(t)\,dt \end{aligned}$$

Therefore

$$\lim_{h\to 0}\frac{A(x+h)-A(x)}{h}=\lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,\mathrm{d}t}{h}=\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t$$

Looking at this result and what we established about the average value of $f(x)$ over the interval $[x, x+h]$, we can conclude that there is a point $c \in [x, x+h]$ such that

$$f(c)=\frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t$$

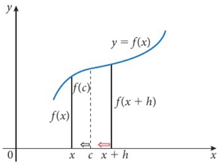


Figure 14.17 $f(c)$ approaching $f(x)$

What happens to c as h approaches 0? As h approaches 0, $x+h$ must approach x. This means, we are 'squeezing' c between x and a number approaching x. So, c must also approach x. That is

$f(c)=f(x)$, and consequently

$$\lim_{h\to 0}\frac{A(x+h)-A(x)}{h}=\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t=f(c)=f(x)$$

This last equation is stating that

$$\frac{\mathrm{d}}{\mathrm{d}x}(A(x))=A'(x)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)\,\mathrm{d}t\right)=f(x)$$

It is important to remember that $\int_a^x f(t)\,\mathrm{d}t$ is a function of x.

This very powerful statement is called the **first fundamental theorem of integral calculus**. In essence, it says that the processes of integration and differentiation are inverses of one another.

Example 14.20

Find each derivative.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\int_{-e}^{x}\sec^2 t\,\mathrm{d}t$ (b) $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{\mathrm{d}t}{1+t^4}$ (c) $\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{\pi}\frac{1}{1+t^4}\mathrm{d}t$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{2x+x^3}\frac{1}{1+t^4}\mathrm{d}t$ (e) $\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{2x+x^3}\frac{1}{1+t^4}\mathrm{d}t$

Solution

(a) This is a direct application of the fundamental theorem:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-e}^{x}\sec^2 t\,\mathrm{d}t=\sec^2 x$$

(b) This is also straightforward:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x\frac{\mathrm{d}t}{1+t^4}=\frac{1}{1+x^4}$$

(c) We need to rewrite the expression before we perform the calculation.

$$\frac{d}{dx}\int_x^{\pi}\frac{1}{1+t^4}dt = \frac{d}{dx}\int_{\pi}^{x} -\frac{1}{1+t^4}dt = -\frac{d}{dx}\int_x^{\pi}\frac{1}{1+t^4}dt = \frac{-1}{1+x^4}$$

(d) This is a function of x, and the upper limit is a function of x, which makes $\int_0^{2x+x^3}\frac{1}{1+t^4}dt$ a composite of $\int_0^{u}\frac{1}{1+t^4}dt$ and $u = 2x + x^3$.
So, we have to use the chain rule.

$$\frac{d}{dx}\int_0^{2x+x^3}\frac{1}{1+t^4}dt = \left(\frac{d}{du}\int_0^{u}\frac{1}{1+t^4}dt\right)\left(\frac{du}{dx}\right)$$

$$= \frac{1}{1+u^4}\cdot\frac{du}{dx}$$

$$= \frac{1}{1+(2x+x^3)^4}\cdot(2+3x^2)$$

$$= \frac{2+3x^2}{1+(2x+x^3)^4}$$

(e) Again, we need to rewrite the integral before evaluation

$$\frac{d}{dx}\int_x^{2x+x^3}\frac{1}{1+t^4}dt = \frac{d}{dx}\left(\int_x^{k}\frac{1}{1+t^4}dt + \int_k^{2x+x^3}\frac{1}{1+t^4}dt\right)$$

$$= \frac{2+3x^2}{1+(2x+x^3)^4} - \frac{1}{1+x^4}$$

The second fundamental theorem of integral calculus

Recall that $A(x) = \int_a^x f(t)\,dt$. If $F(x)$ is any antiderivative of $f(x)$, then applying what we learned earlier

$F(x) = A(x) + c$ where c is an arbitrary constant.

Now

$$F(b) = A(b) + c = \int_a^b f(t)\,dt + c, \text{ and}$$

$$F(a) = A(a) + c = \int_a^a f(t)\,dt + c = 0 + c, \text{ and hence}$$

$$F(b) - F(a) = \int_a^b f(t)\,dt + c - c$$

$$= \int_a^b f(t)\,dt$$

The second fundamental theorem of calculus states:
$\int_a^b f(t)\,dt = F(b) - F(a)$

The theorem is also known as the **evaluation theorem**. Also, since we know that $F'(x)$ is the rate of change in $F(x)$ with respect to x, and that $F(b) - F(a)$ is the change in y when x changes from a to b, we can reformulate the theorem in words to read:

The integral of a rate of change is the **total change**:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Notation

We will use the following notation in evaluating definite integrals. If we know that $F(x)$ is an antiderivative of $f(x)$, then we will write

$$\int_a^b f(t)\,dt = F(x)\Big|_a^b = F(b) - F(a)$$

Here are a few instances where this applies:

- If $V'(t)$ is the rate at which a liquid flows into or out of a container at time t, then $\int_{t_1}^{t_2} V'(t)\,dt = V(t_2) - V(t_1)$ is the change in the amount of liquid in the container between time t_1 and t_2.
- If the rate of growth of a population is $n'(t)$, then $\int_{t_1}^{t_2} n'(t)\,dt = n(t_2) - n(t_1)$ is the increase (or decrease) in population during the period from t_1 to t_2.

This theorem has many other applications in calculus and several other fields. It is a very powerful tool that allows us to deal with problems of area, volume, and work. In this book, we will apply it to finding areas between functions and volumes of revolution as well in displacement problems.

Example 14.21

Evaluate each integral

(a) $\int_{-1}^{3} x^5\,dx$ (b) $\int_0^4 \sqrt{x}\,dx$

(c) $\int_{\pi}^{2\pi} \cos\theta\,d\theta$ (d) $\int_1^2 \frac{4+u^2}{u^3}\,du$

Solution

(a) $\int_{-1}^{3} x^5\,dx = \frac{x^6}{6}\Big|_{-1}^{3} = \frac{3^6}{6} - \frac{1}{6} = \frac{364}{3}$

(b) $\int_0^4 \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^4 = \frac{2}{3}4^{\frac{3}{2}} - 0 = \frac{16}{3}$

(c) $\int_{\pi}^{2\pi} \cos\theta\,d\theta = \sin\theta\Big|_{\pi}^{2\pi} = 0 - 0 = 0$

(d) $\int_1^2 \frac{4+u^2}{u^3}\,du = \int_1^2 \left(\frac{4}{u^3} + \frac{1}{u}\right) du = 4\cdot\frac{u^{-2}}{-2} + \ln|u|\Big|_1^2$

$$= -2u^{-2} + \ln u\Big|_1^2$$

$$= (-2\cdot 2^{-2} + \ln 2) - (-2\cdot 1 + \ln 1)$$

$$= -\frac{1}{2} + \ln 2 + 2 = \frac{3}{2} + \ln 2$$

Using substitution with the definite integral

In Section 14.1, we discussed the use of substitution to evaluate integrals in cases that are not easily recognised. We established that

$$\int f(u(x)) \cdot u'(x)\,dx = \int f(u)\,du = F(u(x)) + c$$

When evaluating definite integrals by substitution, two methods are available.

- Evaluate the indefinite integral first, revert to the original variable, then use the fundamental theorem. For example, to evaluate

  $$\int_0^{\frac{\pi}{3}} \tan^5 x \sec^2 x\,dx$$

  we find the indefinite integral

  $$\int \tan^5 x \sec^2 x\,dx = \int u^5\,du = \frac{1}{6}u^6 = \frac{1}{6}\tan^6 x$$

  then we use the fundamental theorem

  $$\int_0^{\frac{\pi}{3}} \tan^5 x \sec^2 x\,dx = \frac{1}{6}\tan^6 x\Big|_0^{\frac{\pi}{3}} = \frac{1}{6}(\sqrt{3})^6 = \frac{27}{6} = \frac{9}{2}$$

- Or we can use the following substitution rule for definite integrals

  $$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

  Proof:

  If $F(x)$ is an antiderivative of $f(x)$, then by the fundamental theorem

  $$\int_a^b f(u(x))u'(x)\,dx = F(u(x))\Big|_a^b = F(u(b)) - F(u(a))$$

  Also

  $$\int_{u(a)}^{u(b)} f(u)\,du = F(u)\Big|_{u(a)}^{u(b)} = F(u(b)) - F(u(a))$$

Therefore, to evaluate

$$\int_0^{\frac{\pi}{3}} \tan^5 x \sec^2 x\,dx$$

let $u = \tan x \Rightarrow u\left(\frac{\pi}{3}\right) = \sqrt{3}$, $u(0) = 0$, and so

$$\int_0^{\frac{\pi}{3}} \tan^5 x \sec^2 x\,dx = \int_0^{\sqrt{3}} u^5\,du = \frac{1}{6}u^6\Big|_0^{\sqrt{3}} = \frac{9}{2}$$

Example 14.22

Evaluate $\int_2^6 \sqrt{4x+1}\,\mathrm{d}x$

Solution

Let $u = 4x + 1$, then $\mathrm{d}u = 4\mathrm{d}x$. The limits of integration are $u(2) = 9$, and $u(6) = 25$. Therefore

$$\int_2^6 \sqrt{4x+1}\,\mathrm{d}x = \frac{1}{4}\int_9^{25} \sqrt{u}\,\mathrm{d}u = \frac{1}{4}\left(\frac{2}{3}u^{3/2}\right)\Big|_9^{25}$$

$$= \frac{1}{6}(125 - 27) = \frac{49}{3}$$

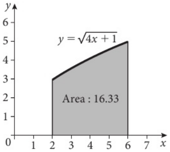


Figure 14.18 The area under the curve $y = \sqrt{4x+1}$ between $x = 2$ and $x = 6$

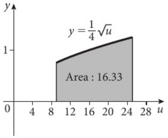


Figure 14.19 The area under the curve $y = \frac{1}{4}\sqrt{u}$ between $u = 9$ and $u = 25$

Note that, using this method, we do not return to the original variable of integration. We simply evaluate the new integral between the appropriate values of u.

Notice that the substitution $u = 4x + 1$ stretched the interval $[2, 6]$ by a factor of 4, and shifted it by 1 unit to the right. But the areas are the same.

Exercise 14.4

1. Evaluate each integral.

(a) $\int_{-2}^{1} (3x^2 - 4x^3)\,\mathrm{d}x$

(b) $\int_2^7 8\,\mathrm{d}x$

(c) $\int_1^5 \frac{2}{t^3}\,\mathrm{d}t$

(d) $\int_2^2 (\cos t - \tan t)\,\mathrm{d}t$

(e) $\int_1^7 \frac{2x^2 - 3x + 5}{\sqrt{x}}\,\mathrm{d}x$

(f) $\int_0^{\pi} \cos\theta\,\mathrm{d}\theta$

(g) $\int_0^{\pi} \sin\theta\,\mathrm{d}\theta$

(h) $\int_3^1 (5x^4 + 3x^2)\,\mathrm{d}x$

(i) $\int_1^3 \frac{u^5 + 2}{u^2}\,\mathrm{d}u$

(j) $\int_1^{\mathrm{e}} \frac{2\,\mathrm{d}x}{x}$

(k) $\int_1^3 \frac{2x}{x^2 + 2}\,\mathrm{d}x$

(l) $\int_1^3 (2 - \sqrt{x})^2\,\mathrm{d}x$

(m) $\int_0^{\frac{\pi}{4}} 3\sec^2\theta\,\mathrm{d}\theta$

(n) $\int_0^1 (8x^7 + \sqrt{\pi})\,\mathrm{d}x$

(o) **(i)** $\int_0^2 |3x|\,\mathrm{d}x$ **(ii)** $\int_{-2}^0 |3x|\,\mathrm{d}x$ **(iii)** $\int_{-2}^2 |3x|\,\mathrm{d}x$

(p) $\int_0^{\frac{\pi}{2}} \sin 2x\,\mathrm{d}x$

(q) $\int_1^9 \frac{1}{\sqrt{x}}\,\mathrm{d}x$

(r) $\int_{-2}^2 (\mathrm{e}^x - \mathrm{e}^{-x})\mathrm{d}x$

(s) $\int_{-1}^1 \frac{\mathrm{d}x}{1 + x^2}$

(t) $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$

(u) $\int_{-1}^{1} \frac{dx}{\sqrt{4-x^2}}$

(v) $\int_{-2}^{0} \frac{dx}{4+x^2}$

2. Evaluate each integral.

(a) $\int_0^4 \frac{x^3\,dx}{\sqrt{x^2+1}}$

(b) $\int_1^{\sqrt{e}} \frac{\sin(\pi \ln x)}{x} dx$

(c) $\int_e^{e^2} \frac{dt}{t \ln t}$

(d) $\int_{-1}^{2} 3x\sqrt{9-x^2}\,dx$

(e) $\int_{-\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x}{\sqrt{3+\cos x}} dx$

(f) $\int_e^{e^2} \frac{\ln x}{x} dx$

(g) $\int_1^{\sqrt{3}} \frac{\sqrt{\arctan x}}{1+x^2} dx$

(h) $\int_1^{\sqrt{e}} \frac{dx}{x\sqrt{1-(\ln x)^2}}$

(i) $\int_{-\ln 2}^{\ln 2} \frac{e^{2x}}{e^{2x}+9} dx$

(j) $\int_{\ln 2}^{\ln(2/\sqrt{3})} \frac{e^{-2x}\,dx}{\sqrt{1-e^{-4x}}}$

(k) $\int_0^{\frac{\pi}{4}} \sqrt{\tan x}\sec^2 x\,dx$

(l) $\int_0^{\sqrt{\pi}} 7x\cos x^2\,dx$

(m) $\int_{\pi^2}^{4\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}} dx$

(n) $\int_0^1 \frac{\sqrt{3}\,x}{\sqrt{4-3x^4}} dx$

(o) $\int_0^{\frac{2}{\sqrt{3}}} \frac{dx}{9+4x^2}$

(p) $\int_1^{\sqrt{2}} \frac{x\,dx}{3+x^4}$

(q) $\int_0^{\frac{\pi}{6}} (1-\sin 3t)\cos 3t\,dt$

(r) $\int_0^{\frac{\pi}{4}} e^{\sin 2\theta}\cos 2\theta\,d\theta$

(s) $\int_0^{\frac{\pi}{8}} (3+e^{\tan 2t})\sec^2 2t\,dt$

(t) $\int_0^{\sqrt{\ln \pi}} 4t\,e^{t^2}\sin(e^{t^2})dt$

3. Find the average value of each function over the given interval.

(a) x^4, $[1, 2]$

(b) $\cos x$, $\left[0, \frac{\pi}{2}\right]$

(c) $\sec^2 x$, $\left[\frac{\pi}{6}, \frac{\pi}{4}\right]$

(d) e^{-2x}, $[0, 4]$

(e) $\frac{e^{3x}}{1+e^{6x}}$, $\left[\frac{-\ln 3}{6}, 0\right]$

4. Find the indicated derivative.

(a) $\frac{d}{dx}\int_2^x \frac{\sin t}{t} dt$

(b) $\frac{d}{dt}\int_t^3 \frac{\sin x}{x} dx$

(c) $\frac{d}{dx}\int_{x^2}^0 \frac{\sin t}{t} dt$

(d) $\frac{d}{dx}\int_0^{x^2} \frac{\sin u}{u} du$

(e) $\dfrac{d}{dt}\displaystyle\int_{-\pi}^{t}\frac{\cos y}{1+y^2}\,dy$

(f) $\dfrac{d}{dx}\displaystyle\int_{ax}^{bx}\frac{dt}{5+t^4}$

(g) $\dfrac{d}{d\theta}\displaystyle\int_{\sin\theta}^{\cos\theta}\frac{1}{1-x^2}\,dx$

(h) $\dfrac{d}{dx}\displaystyle\int_{5}^{x^{\frac{1}{4}}}e^{t^4+3t^2}\,dt$

5. Does the function $F(x)=\displaystyle\int_0^{2x-x^2}\cos\left(\frac{1}{1+t^2}\right)dt$ have an extreme value?

6. **(a)** Find $\displaystyle\int_0^k\frac{dx}{3x+2}$, giving your answer in terms of k.

(b) Given that $\displaystyle\int_0^k\frac{dx}{3x+2}=1$, calculate the value of k.

7. Given that $p, q \in \mathbb{N}$, show that

$$\int_0^1 x^p(1-x)^q\,dx=\int_0^1 x^q(1-x)^p\,dx$$

Do not attempt to evaluate the integrals.

8. Given that $k \in \mathbb{N}$, evaluate each integral:

(a) $\displaystyle\int x(1-x)^k\,dx$

(b) $\displaystyle\int_0^1 x(1-x)^k\,dx$

9. Let $F(x)=\displaystyle\int_3^x\sqrt{5t^2+2}\,dt$, find:

(a) $F(3)$ **(b)** $F'(3)$ **(c)** $F''(3)$

10. Show that the function

$$f(x)=\int_x^{3x}\frac{dt}{t}$$

is constant over the set of positive real numbers.

14.5 Integration by method of partial fractions

In this section, we will integrate rational functions with polynomial denominators. For example, if we find the indefinite integral $\displaystyle\int\frac{x+1}{x^2+5x+6}\,dx$, we first decompose the integrand into partial fractions (Section 2.6) and then the integration process is straightforward.

$$\frac{x+1}{x^2+5x+6}\equiv\frac{a}{x+2}+\frac{b}{x+3}$$

After solving for a and b we can perform the integration:

$$\int \frac{x+1}{x^2+5x+6}\,dx = \int \left(\frac{-1}{x+2} + \frac{1}{x+3}\right) dx = -\ln|x+2| + \ln|x+3| + c$$

$$= \ln\left|\frac{x+3}{x+2}\right| + c$$

Example 14.23

Find the indefinite integral $\int \frac{3x-1}{x^2+4x+4}\,dx$

Solution

Using partial fractions will make integration easier.

From Example 2.33 we know that

$$\frac{3x-1}{x^2+4x+4} \equiv \frac{3}{x+2} - \frac{7}{(x+2)^2}$$

Hence the integral can be rewritten as

$$\int \frac{3x-1}{x^2+4x+4}\,dx = \int \frac{3}{x+2}\,dx - \int \frac{7}{(x+2)^2}\,dx$$

These two integrals can be found by inspection, giving

$$\int \frac{3x-1}{x^2+4x+4}\,dx = 3\ln|x+2| + \frac{7}{x+2} + C$$

Example 14.24

Find the indefinite integral $\int \frac{2}{x^3+2x^2+2x}\,dx$

Solution

Factorising the denominator and separating fractions as we did in chapter 2, we have:

$$\frac{2}{x^3+2x^2+2x} \equiv \frac{1}{x} - \frac{x+2}{x^2+2x+2}$$

Hence, we can write the integral as

$$\int \frac{2}{x^3+2x^2+2x}\,dx = \int \frac{dx}{x} - \int \frac{x+2}{x^2+2x+2}\,dx$$

$$= \int \frac{dx}{x} - \int \frac{x+1+1}{x^2+2x+2}\,dx$$

$$= \int \frac{dx}{x} - \frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\,dx - \int \frac{dx}{(x+1)^2+1}$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+2x+2) - \arctan(x+1) + C$$

Example 14.25

Find the indefinite integral $\int \frac{5x^2 + 16x + 17}{2x^3 + 9x^2 + 7x - 6}\,dx$

Solution

From Example 2.32 we have

$$\int \frac{5x^2 + 16x + 17}{2x^3 + 9x^2 + 7x - 6}\,dx = \int \frac{3}{2x - 1}\,dx - \int \frac{1}{x + 2}\,dx + \int \frac{2}{x + 3}\,dx$$
$$= \frac{3}{2}\ln|2x - 1| - \ln|x + 2| + 2\ln|x + 3| + c$$

For IB Mathematics HL exams, the original fraction to be rewritten as partial fractions will be such that the degree of the denominator will not be greater than 2; factors will be at most two distinct linear terms, and the degree of the numerator will be less than the degree of the denominator.

The last case is unlikely since it would require factorisation of a cubic, unless it is given in factorised form.

A few cases of partial fractions

Denominator is quadratic – it factorises into two distinct linear factors, and the numerator $p(x)$ is a constant or linear

$$\frac{p(x)}{(ax + b)(cx + d)} \equiv \frac{A}{ax + b} + \frac{B}{cx + d}$$

Denominator is quadratic – it factorises into two repeated linear factors, and the numerator $p(x)$ is a constant or linear

$$\frac{p(x)}{(ax + b)^2} \equiv \frac{A}{ax + b} + \frac{B}{(ax + b)^2}$$

Denominator is cubic – it factorises into three distinct linear factors, and the numerator $p(x)$ is a constant, linear, or quadratic

$$\frac{p(x)}{(ax + b)(cx + d)(ex + f)} = \frac{A}{ax + b} + \frac{B}{cx + d} + \frac{C}{ex + f}$$

Remember that a consequence of the fundamental theorem of algebra is that any polynomial with real coefficients can only have factors that are linear or quadratic.

Exercise 14.5

1. Evaluate each integral.

(a) $\int \frac{5x + 1}{x^2 + x - 2}\,dx$

(b) $\int \frac{x + 4}{x^2 - 2x}\,dx$

(c) $\int \frac{x + 2}{x^2 + 4x + 3}\,dx$

(d) $\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}\,dx$

(e) $\int \frac{2x^2 + x - 12}{x^3 + 5x^2 + 6x}\,dx$

(f) $\int \frac{4x^2 + 2x - 1}{x^3 + x^2}\,dx$

(g) $\int \frac{3}{x^2 + x - 2}\,dx$

(h) $\int \frac{5 - x}{2x^2 + x - 1}\,dx$

(i) $\int \frac{3x+4}{(x+2)^2}\,dx$

(j) $\int \frac{12}{x^4 - x^3 - 2x^2}\,dx$

(k) $\int \frac{2}{x^3 + x}\,dx$

(l) $\int \frac{x+2}{x^3 + 3x}\,dx$

(m) $\int \frac{3x+2}{x^3 + 6x}\,dx$

(n) $\int \frac{2x+3}{x^3 + 8x}\,dx$

14.6 Areas

We have seen how the area between a curve defined by $y = f(x)$ and the x-axis can be computed by the integral $\int_a^b f(x)\,dx$ on an interval $[a, b]$ where $f(x) \geqslant 0$.

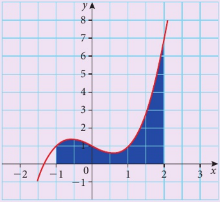
In this section, we shall use integration to find the area of more general regions between curves.

Areas between curves of functions of the form $y = f(x)$ and the x-axis

If the function $y = f(x)$ is always above the x-axis, finding the area is a straightforward computation of the integral $\int_a^b f(x)\,dx$.

Example 14.26

Find the area between the curve $f(x) = x^3 - x + 1$ and the x-axis over the interval $[-1, 2]$



Solution

This area is:

$$\int_{-1}^{2}(x^3 - x + 1)\,dx = \left[\frac{x^4}{4} - \frac{x^2}{2} + x\right]_{-1}^{2} = (4 - 2 + 2) - \left(\frac{1}{4} - \frac{1}{2} - 1\right) = 2\frac{1}{4}$$

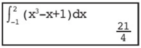


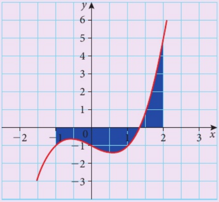
Figure 14.20 Using a GDC to find the area

Using our GDC, the area in Example 14.26 is found by simply choosing the MATH menu, then the $\int dx$ menu item, then typing in the function with the integration limits.

In some cases, we will have to adjust how to work. This is the case when the graph intersects the x-axis. Since we are interested in the area bounded by the curve and the interval $[a, b]$ on the x-axis, we do not want the two areas to cancel each other. This is why we have to split the process into subintervals where we take the absolute values of the areas found and add them.

Example 14.27

Find the area under the curve $f(x) = x^3 - x - 1$ and the x-axis over the interval $[-1, 2]$



Solution

As we see from the diagram, a part of the graph is below the x-axis, and its area will be negative. If we try to integrate this function without paying attention to the intersection with the x-axis, here is what we get:

$$\int_{-1}^{2}(x^3 - x - 1)\,\mathrm{d}x = \frac{x^4}{4} - \frac{x^2}{2} - x\Big|_{-1}^{2} = (4 - 2 - 2) - \left(\frac{1}{4} - \frac{1}{2} + 1\right)$$
$$= -\frac{3}{4}$$

This integration has to be split before we start. However, this is a function where we cannot find the intersection point. So, we either use a GDC to find the intersection or we just take the absolute values of the different parts of the region. This is done by integrating the absolute value of the function:

$$Area = \int_a^b |f(x)|\,\mathrm{d}x$$

As we said earlier, this is not easy to find given the difficulty with the x-intercept. It is best if we use a GDC.

Hence, $Area = \int_{-1}^{2}|x^3 - x - 1|\,\mathrm{d}x \approx 3.6145$

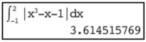
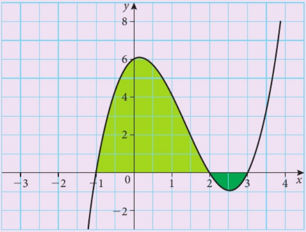


Figure 14.21 It is best to use a GDC

Example 14.28

Find the area enclosed by the graph of the function $f(x) = x^3 - 4x^2 + x + 6$ and the x-axis.



Solution

This function intersects the x-axis at three points where $x = -1$, 2, and 3. To find the area, we split it into two and then add the absolute values:

$$Area = \int_{-1}^{3} |f(x)|\,dx = \int_{-1}^{2} f(x)\,dx + \int_{2}^{3} (-f(x))\,dx$$

$$= \int_{-1}^{2} (x^3 - 4x^2 + x + 6)\,dx + \int_{2}^{3} (-x^3 + 4x^2 - x - 6)\,dx$$

$$= \frac{x^4}{4} - \frac{4x^3}{3} + \frac{x^2}{2} + 6x\Big|_{-1}^{2} + -\frac{x^4}{4} + \frac{4x^3}{3} - \frac{x^2}{2} - 6x\Big|_{2}^{3}$$

$$= \frac{45}{4} + \frac{7}{12} = \frac{71}{6}$$

Area between curves

In some practical problems, we may have to compute the area between two curves. Let $f(x)$ and $g(x)$ be functions such that $f(x) \geqslant g(x)$ on the interval $[a, b]$ (Figure 14.22). We do not insist that both functions are non-negative.

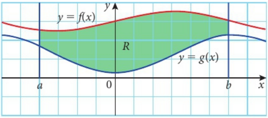


Figure 14.22 Area between two curves

To find the area of the region R between the curves from $x = a$ to $x = b$, we subtract the area between the lower curve $g(x)$ and the x-axis from the area between the upper curve $f(x)$ and the x-axis; that is

$$\text{Area of } R = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$$

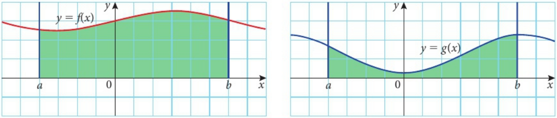


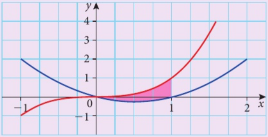
Figure 14.23 Areas under functions f and g

If $f(x)$ and $g(x)$ are functions such that $f(x) \geqslant g(x)$ on the interval $[a, b]$, then the area between the two curves is given by $A = \sum_a^b [f(x) - g(x)]\,dx$

This fact applies to all functions, not only positive functions. These facts are used to define the area between curves.

Example 14.29

Find the area of the region between the curves $y = x^3$ and $y = x^2 - x$ on the interval $[0, 1]$.



Solution

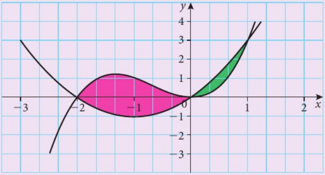
$y = x^3$ appears to be higher than $y = x^2 - x$ with one intersection at $x = 0$. Thus, the required area is

$$A = \int_0^1 [x^3 - (x^2 - x)]\,dx = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2}\Big|_0^1 = \frac{5}{12}$$

In some cases we must be very careful how we calculate the area. This is the case where the two functions intersect at more than one point.

Example 14.30

Find the area of the region bounded by the curves $y = x^3 + 2x^2$ and $y = x^2 + 2x$.



Solution

The two curves intersect when:

$$x^3 + 2x^2 = x^2 + 2x \Rightarrow x^3 + x^2 - 2x = 0 \Rightarrow x(x + 2)(x - 1) = 0$$

That is, when $x = -2, 0$, or 1

The area is equal to:

$$A = \int_{-2}^{0} [x^3 + 2x^2 - (x^2 + 2x)]\, dx + \int_{0}^{1} [x^2 + 2x - (x^3 + 2x^2)]\, dx$$

$$= \int_{-2}^{0} [x^3 + x^2 - 2x]\, dx + \int_{0}^{1} [-x^2 + 2x - x^3]\, dx$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{3} - x^2\right]_{-2}^{0} + \left[-\frac{x^4}{4} - \frac{x^3}{3} + x^2\right]_{0}^{1}$$

$$= 0 - \left[\frac{16}{4} - \frac{8}{3} - 4\right] + \left[-\frac{1}{4} - \frac{1}{3} + 1\right] - 0 = \frac{37}{12}$$

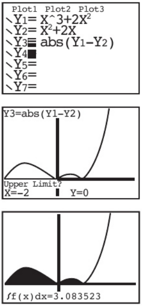


Figure 14.24 Using a GDC to find the area between two curves

This discussion leads us to stating the general expression we should use in evaluating areas between curves.

If $f(x)$ and $g(x)$ are functions that are continuous on the interval $[a, b]$, then the area between the two curves is given by

$$A = \int_{a}^{b} |f(x) - g(x)|\, dx$$

We can do this on our GDC.

Areas along the y-axis

To find the area enclosed by $y = 1 - x$ and $y^2 = x + 1$, it is best to treat the region between them by regarding x as a function of y (Figure 14.25).

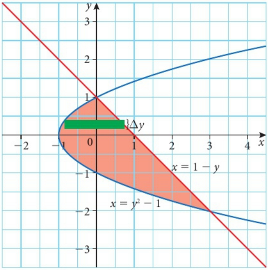


Figure 14.25 Area between two curves expressed by regarding x as a function of y

The area of the shaded region can be calculated using the integral:

$$A(y) = \int_{-2}^{1} \left|(1 - y) - (y^2 - 1)\right| dy$$

$$= \int_{-2}^{1} |2 - y - y^2|\, dy = \left|2y - \frac{y^2}{2} - \frac{y^3}{3}\right|_{-2}^{1} = \frac{9}{2}$$

If we used y as a function of x, then the calculation would involve calculating the area by dividing the interval into two: $[-1, 0]$ and $[0, 3]$.

In the first part, the area is enclosed between $y = \sqrt{x+1}$ and $y = -\sqrt{x+1}$, and the area in the second part is enclosed by $y = 1 - x$ and $y = -\sqrt{x+1}$:

$$A(x) = 2\int_{-1}^{0} \sqrt{x+1}\, dx + \int_{0}^{3} \left((1 - x) - (-\sqrt{x+1})\right) dx$$

Exercise 14.6

1. Find the area of the region bounded by the given curves. Sketch the region and then compute the required area.

(a) $y = x + 1, y = 7 - x^2$ **(b)** $y = \cos x, y = x - \frac{\pi}{2}, x = -\pi$

(c) $y = 2x, y = x^2 - 2$ **(d)** $y = x^3, y = x^2 - 2, x = 1$

(e) $y = x^6, y = x^2$ **(f)** $y = 5x - x^2, y = x^2$

(g) $y = 2x - x^3, y = x - x^2$ **(h)** $y = \sin x, y = 2 - \sin x$ (one period)

(i) $y = \frac{x}{2}, y = \sqrt{x}, x = 9$ **(j)** $y = \frac{x^4}{10}, y = 3x - x^3$

(k) $y = \frac{1}{x}, y = \frac{1}{x^3}, x = 8$

(l) $y = 2\sin x, y = \sqrt{3}\tan x, -\frac{\pi}{4} \leqslant x \leqslant \frac{\pi}{4}$

(m) $y = x - 1, y^2 = 2x + 6$ (n) $x = 2y^2, x = 4 + y^2$

(o) $4x + y^2 = 12, y = x$ (p) $x - y = 7, x = 2y^2 - y + 3$

(q) $x = y^2, x = 2y^2 - y - 2$

(r) $y = x^3 + 2x^2, y = x^3 - 2x, x = -3$, and $x = 2$

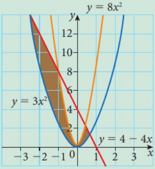
(s) $y = \sec^2 x, y = \sec x \tan x, x = -\frac{\pi}{3}$, and $x = \frac{\pi}{6}$

(t) $y = x^3 + 1, y = (x + 1)^2$

(u) $y = x^3 + x, y = 3x^2 - x$

(v) $y = 3 - \sqrt{x}, y = \dfrac{2\sqrt{x} + 1}{2\sqrt{x}}$

2. Find the area of the shaded region.



3. Find the area of the region enclosed by $y = e^x$, $x = 0$, and the tangent to $y = e^x$ at $x = 1$

4. Find the area of the inside of the 'loop' in the graph of the curve $y^2 = x^4(x + 3)$

5. Find the area enclosed by the curve $y^2 = 2x^2 - 4x^4$

6. Find the area of the region enclosed by $x = 3y^2$ and $x = 12y - y^2 - 5$

7. Find the area of the region enclosed by $y = (x - 2)^2$ and $y = x(x - 4)^2$

8. Find a value for $m > 0$ such that the area under the graph of $y = e^{2x}$ over the interval $[0, m]$ is 3 square units.

9. Find the area of the region bounded by $y = x^3 - 4x^2 + 3x$ and the x-axis.

14.7 Volumes with integrals

The underlying principle for finding the area of a plane region is to divide the region into thin strips, approximate the area of each strip by the area of a rectangle, and then add the approximations and take the limit of the sum to produce an integral for the area. The same strategy can be used to find the volume of a solid.

The idea is to divide the solid that stretches over an interval $[a, b]$ into thin slices, approximate the volume of each slice, add the approximations, and take the limit of the sum to produce an integral of the volume.

We start by taking cross-sections perpendicular to the x-axis, as shown in Figure 14.26. Each slice will be approximated by a solid whose volume will be equal to the product of its base times its height (Figure 14.27).

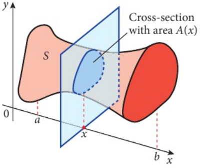


Figure 14.26 Taking a cross-section perpendicular to the x-axis

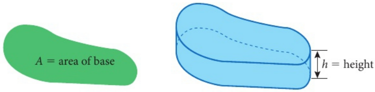


Figure 14.27 Volume = area of base × height

If we call the volume of the slice v_i and the area of its base $A(x)$, then

$$v_i = A(x_i) \cdot h = A(x_i) \cdot \Delta x_i$$

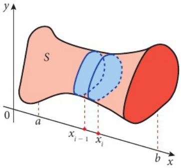


Figure 14.28 Subintervals of $[a, b]$

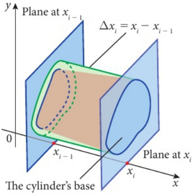


Figure 14.29 A cylindrical strip

Using this approximation, the volume of the whole solid can be found by

$$V \approx \sum_{i=1}^{n} A(x_i)\Delta x_i$$

Taking the limit as n increases and the widths of the subintervals approach zero yields the definite integral

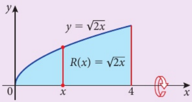
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\Delta x_i = \int_a^b A(x)\,dx$$

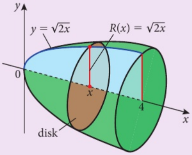
If we place the solid along the y-axis and take the cross-sections perpendicular to that axis, we will arrive at a similar expression for the volume of the solid:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(y_i)\Delta y_i = \int_a^b A(y)\,dy$$

Example 14.31

Consider the solid formed when the graph of the parabola $y = \sqrt{2x}$ over $[0, 4]$ is rotated around the x-axis through an angle of 2π radians as shown in the diagrams.





Solution

The cross-section is a circular disk whose radius is $y = \sqrt{2x}$. Therefore

$$A(x) = \pi R^2 = \pi(\sqrt{2x})^2 = 2\pi x$$

The volume is then

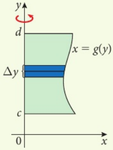
$$V = \int_0^4 A(x)\,dx = \int_0^4 2\pi x\,dx = 2\pi\left]\frac{x^2}{2}\right[_0^4 = 16\pi \text{ cubic units.}$$

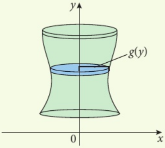
If the region bounded by a closed interval $[a, b]$ on the x-axis and a function $f(x)$ is rotated about the x-axis, the volume of the resulting **solid of revolution** is given by:

$$V = \int_a^b \pi(f(x))^2\,dx$$

If the region bounded by a closed interval $[c, d]$ on the y-axis and a function $g(y)$ is rotated about the y-axis, the volume of the resulting solid of revolution is given by:

$$V = \int_c^d \pi(g(y))^2\,dy$$





Example 14.32 is a special case of the general process for finding volumes of solids of revolution.

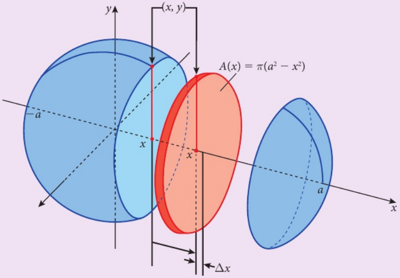
Example 14.32

Find the volume of a sphere with radius $R = a$.

Solution

If we place the sphere with its centre at the origin, then the equation of the circle is

$$x^2 + y^2 = a^2 \Rightarrow y = \pm\sqrt{a^2 - x^2}$$



The cross-section of the sphere, perpendicular to the x-axis, is a circular disk with radius y; so the area is

$$A(x) = \pi R^2 = \pi y^2 = \pi\left(\sqrt{a^2 - x^2}\right)^2 = \pi(a^2 - x^2)$$

So, the volume of the sphere is

$$V = \int_{-a}^{a} \pi(a^2 - x^2)\mathrm{d}x = \pi\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a}$$

$$= \pi\left(a^3 - \frac{a^3}{3}\right) - \pi\left(-a^3 + \frac{a^3}{3}\right) = \pi\left(2a^3 - 2\frac{a^3}{3}\right) = \frac{4\pi a^3}{3}$$

If we want to rotate the right-hand region of the circle around the y-axis, then the cross-section of the sphere, perpendicular to the y-axis, is a circular disk with radius x. Solving the equation for x instead:

$x^2 + y^2 = a^2 \Rightarrow x = \pm\sqrt{a^2 - y^2}$, and hence the area is

$A(y) = \pi R^2 = \pi x^2 = \pi\left(\sqrt{a^2 - y^2}\right)^2 = \pi(a^2 - y^2)$, and the volume of the sphere is

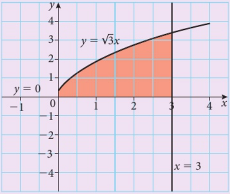
$$V = \int_{-a}^{a} \pi(a^2 - y^2)\mathrm{d}y = \pi\left[a^2y - \frac{y^3}{3}\right]_{-a}^{a} = \pi\left(a^3 - \frac{a^3}{3}\right) - \pi\left(-a^3 + \frac{a^3}{3}\right)$$

$$= \pi\left(2a^3 - 2\frac{a^3}{3}\right) = \frac{4\pi a^3}{3}$$

The same result as given in Example 14.32

Example 14.33

Find the volume of the solid generated when the region enclosed by $y = \sqrt{3x}$, $x = 3$, and $y = 0$ is



Solution

$$V = \int_0^3 \pi(f(x))^2\,dx$$

$$= \pi\int_0^3 (\sqrt{3x})^2\,dx$$

$$= 3\pi\left[\frac{x^2}{2}\right]_0^3 = \frac{27\pi}{2}$$

Example 14.34

Find the volume of the solid generated when the region enclosed by $y = \sqrt{3x}$, $y = 3$, and $x = 0$ is revolved about the y-axis.

Solution

Here, we first find x as a function of y.

$y = \sqrt{3x} \Rightarrow x = \frac{y^2}{3}$, the interval on the y-axis is [0, 3]

So, the volume required is

$$V = \int_0^3 \pi\left(\frac{y^2}{3}\right)^2 dy = \frac{\pi}{9}\int_0^3 y^4\,dy = \frac{\pi}{9}\left[\frac{y^5}{5}\right]_0^3 = \frac{27\pi}{5}$$

Washers

Consider the region R between two curves, $y = f(x)$ and $y = g(x)$, from $x = a$ to $x = b$ where $f(x) > g(x)$. Rotating R about the x-axis generates a solid of revolution S. How do we find the volume of S?

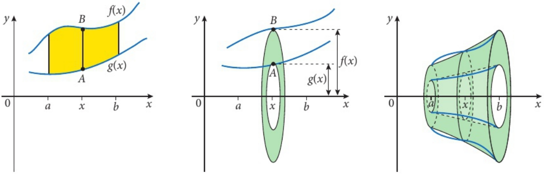


Figure 14.30 Generating washers

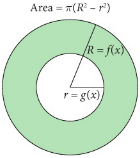


Figure 14.31 Area of a typical washer

Consider an arbitrary point x in the interval $[a, b]$. The segment AB represents the difference $f(x) - g(x)$. When we rotate this slice, the cross-section perpendicular to the x-axis is going to look like a washer whose area is:

$$A = \pi(R^2 - r^2) = \pi\left((f(x))^2 - (g(x))^2\right)$$

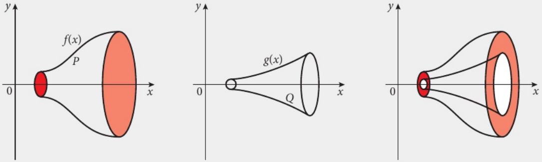
So, the volume of S is

$$V = \int_a^b A(x)\,\mathrm{d}x = \pi\int_a^b \left((f(x))^2 - (g(x))^2\right)\mathrm{d}x$$

If we are rotating about the y-axis, a similar formula applies.

$$V = \pi\int_c^d \left((p(y))^2 - (q(y))^2\right)\mathrm{d}y$$

To understand the washer more, we can think of it in the following manner. Let P be the solid generated by rotating the curve $y = f(x)$ and Q be the solid generated by rotating the curve $y = g(x)$. Then S can be found by removing the solid of revolution generated by $y = f(x)$ from the solid of revolution generated by $y = g(x)$



Volume of S = volume of P − volume of Q, which justifies the formula:

$$V = \pi\int_a^b (f(x))^2\,\mathrm{d}x - \pi\int_a^b (g(x))^2\,\mathrm{d}x = \pi\int_a^b \left((f(x))^2 - (g(x))^2\right)\mathrm{d}x$$

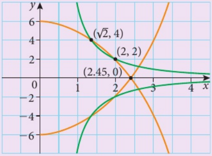
Example 14.35

The region in the first quadrant between $f(x) = 6 - x^2$ and $h(x) = \dfrac{8}{x^2}$ is rotated about the x-axis. Find the volume of the generated solid.

Solution

The rotated region is shown in the diagram. $f(x)$ is larger than $h(x)$ in this interval. Moreover, the two curves intersect at:

$$\frac{8}{x^2} = 6 - x^2 \Rightarrow x = \sqrt{2}, x = 2$$



Hence the volume of the solid of revolution is:

$$V = \pi \int_{\sqrt{2}}^{2} \left((6 - x^2)^2 - \left(\frac{8}{x^2} \right)^2 \right) \mathrm{d}x$$

$$= \pi \int_{\sqrt{2}}^{2} \left(x^4 - 12x^2 + 36 - \frac{64}{x^4} \right) \mathrm{d}x$$

$$= \pi \left[\frac{x^5}{5} - 4x^3 + 36x + \frac{64}{3x^3} \right]_{\sqrt{2}}^{2}$$

$$= \frac{736 - 512\sqrt{2}}{15} \pi$$

An alternative method: volumes by cylindrical shells

Consider the region R under the curve $y = f(x)$. Rotate R about the y-axis. We divide R into vertical strips, each of width $\triangle x$ (Figure 14.32). When we rotate a strip around the y-axis, we generate a cylindrical shell of thickness $\triangle x$ and height $f(x)$ (Figure 14.33). To understand how we get the volume, we cut the shell vertically as shown and unfold it. The resulting rectangular parallelepiped has length $2\pi x$, height $f(x)$, and thickness $\triangle x$ (Figure 14.34).

So, the volume of this shell is

$$\triangle v_i = \textit{length} \times \textit{height} \times \textit{thickness} = (2\pi x) \times f(x) \times \triangle x$$

The volume of the whole solid is the sum of the volumes of these shells as the number of shells increases, and consequently

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} \triangle v_i = \lim_{\triangle x \to 0} \sum (2\pi x) \times f(x) \times \triangle x = 2\pi \int_a^b x f(x) \mathrm{d}x$$

In many problems involving rotation about the y-axis, this would be more accessible than the disk-washer method.

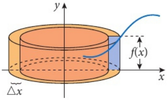


Figure 14.32 Divide R into vertical strips of width $\triangle x$

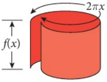


Figure 14.33 Cylindrical shell

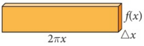
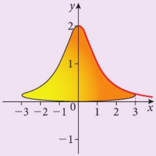


Figure 14.34 Resulting rectangular parallelepiped

Example 14.36

Find the volume of the solid generated when we rotate the region under $f(x) = \frac{2}{1+x^2}$, $x = 0$, and $x = 3$ about the y-axis.



Solution

Using the shell method, we have

$$V = 2\pi \int_0^3 x \times \frac{2}{1+x^2}\,dx$$

$$= 2\pi \int_0^3 \frac{2x}{1+x^2}\,dx = 2\pi \int_0^{10} \frac{du}{u}$$

$$= 2\pi[\ln u]_1^{10} = 2\pi \ln 10$$

Exercise 14.7

1. Find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis. Sketch the region, the solid, and a typical disk.

(a) $y = 3 - \frac{x}{3}, y = 0, x = 2, x = 3$

(b) $y = 2 - x^2, y = 0$

(c) $y = \sqrt{16 - x^2}, y = 0, x = 1, x = 3$

(d) $y = \frac{3}{x}, y = 0, x = 1, x = 3$

(e) $y = 3 - x, y = 0, x = 0$

(f) $y = \sqrt{\sin x}, y = 0, 0 \leqslant x \leqslant \pi$

(g) $y = \sqrt{\cos x}, y = 0, -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{3}$

(h) $y = 4 - x^2, y = 0$

(i) $y = x^3 + 2x + 1, y = 0, x = 1$

(j) $y = -4x - x^2, y = x^2$

(k) $y = \sec x, x = \frac{\pi}{4}, x = \frac{\pi}{3}, = 0$

(l) $y = 1 - x^2, y = x^3 + 1$

(m) $y = \sqrt{36 - x^2}, y = 4$

(n) $x = \sqrt{y}, y = 2x$

(o) $y = \sin x, y = \cos x, x = \frac{\pi}{4}, x = \frac{\pi}{2}$

(p) $y = 2x^2 + 4, y = x, x = 1, x = 3$

(q) $y = \sqrt{x^4 + 1}, y = 0, x = 1, x = 3$

(r) $y = 16 - x, y = 3x + 12, x = -1$

(s) $y = \frac{1}{x}, y = \frac{5}{2} - x$

2. Find the volume resulting from a rotation of the region shown in the diagram about:

(a) the x-axis

(b) the y-axis.

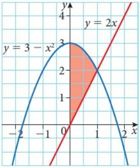


Figure 14.35 Diagram for question 2

3. Find the volume of the solid obtained by rotating the region bounded by the given curves about the y-axis. Sketch the region, the solid, and a typical disk/shell.

(a) $y = x^2, y = 0, x = 1, x = 3$

(b) $y = x, y = \sqrt{9 - x^2}, x = 0$

(c) $y = x^3 - 4x^2 + 4x, y = 0$

(d) $y = \sqrt{3x}, x = 5, x = 11, y = 0$

(e) $y = x^2, y = \frac{2}{1 + x^2}$

(f) $y = \sqrt{x^2 + 2}, x = 3, y = 0, x = 0$

(g) $y = \frac{7x}{\sqrt{x^3 + 7}}, x = 3, y = 0$

(h) $y = \sin x, y = \cos x, x = \frac{\pi}{4}, x = \frac{\pi}{2}$

(i) $y = 2x^2 + 4, y = x, x = 1, x = 3$

(j) $y = \sin(x^2), y = 0, x = 0, x = \sqrt{\pi}$

(k) $y = 5 - x^3, y = 5 - 4x$

14.8 Modelling linear motion

So far, our mathematical models considered the motion of an object only along a straight line. For example, projectile motion (e.g. a ball being thrown) is often modelled by a position function that simply gives the height (displacement) of the object. In this way, we are modelling the motion as if it was restricted to a vertical line.

In this section, we will again analyse the motion of an object as if its motion takes place along a straight line in space. This makes sense only if the mass (and thus, size) of the object is not taken into account. Hence, the object is modelled by a particle whose mass is considered to be zero. This study of motion, without reference either to the forces that cause it or to the mass of the object, is known as **kinematics**.

Displacement and total distance travelled

Recall from Chapter 13 that given time t, displacement s, velocity v, and acceleration a, we have:

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} \text{ and } a = \frac{\mathrm{d}v}{\mathrm{d}t}, \text{ also } a = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$$

It is important to understand the difference between displacement and distance travelled. Consider a couple of simple examples of an object moving along the x-axis.

Assume that the object does not change direction during the interval $0 \leqslant t \leqslant 5$. If the position of the object at $t = 0$ is $x = 2$, and at $t = 5$ its position is $x = -3$, then its displacement, or change in position, is -5 because the object changed its position by 5 units in the negative x-direction. This can be calculated by: (final position) − (initial position) $= -3 - 2 = -5$

However, the distance travelled would be the absolute value of displacement, calculated by |final position − initial position| $= |-3 - 2| = 5$.

Assume that another object's initial and final positions are the same as in the first example; that is, at $t = 0$ its position is $x = 2$, and at $t = 5$ its position is $x = -3$. However, the object changed direction in that it first travelled to the left (negative velocity) from $x = 2$ to $x = -5$ during the interval $0 \leqslant t \leqslant 3$, and then travelled to the right (positive velocity) from $x = -5$ to $x = -3$. The object's displacement is -5, the same as in the first example because its net change in position is just the difference between its final and initial positions. However, it's clear that the object has travelled further than in the first example. But, we cannot calculate it the same way as we did in the first example. We will have to make a separate calculation for each interval where the direction changed. Hence, total distance travelled $= |-5 - 2| + |-3 - (-5)| = 7 + 2 = 9$.

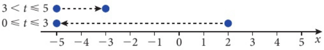


Figure 14.36 Travelled distances

- The **velocity** $v = \frac{ds}{dt}$ of a particle is a measure of how fast it is moving and of its direction of motion relative to a fixed point.
- The **speed** $|v|$ of a particle is a measure of how fast it is moving that does not indicate direction. Thus, speed is the magnitude of velocity and is always positive.
- The **acceleration** $a = \frac{dv}{dt}$ of a particle is a measure of how fast its velocity is changing.

Example 14.37

The displacement s of a particle on the x-axis, relative to the origin, is given by the position function $s(t) = -t^2 + 6t$ where s in centimetres and t is in seconds.

(a) Find a function for the particle's velocity $v(t)$ in terms of t. Graph the functions $s(t)$ and $v(t)$ on separate axes.

(b) Find the particle's position at the following times: $t = 0, 1, 3,$ and 6 seconds

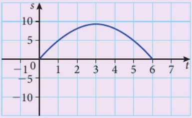
(c) Find the particle's displacement for the following intervals: $0 \leqslant t \leqslant 1$, $1 \leqslant t \leqslant 3$, $3 \leqslant t \leqslant 6$, and $0 \leqslant t \leqslant 6$

(b) Find the particle's total distance travelled for the following intervals: $0 \leqslant t \leqslant 1$, $1 \leqslant t \leqslant 3$, $3 \leqslant t \leqslant 6$, and $0 \leqslant t \leqslant 6$

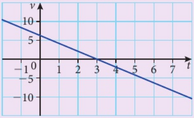
Solution

(a) $v(t) = \frac{d}{dt}(-t^2 + 6t) = -2t + 6$

Position function: $s(t) = -t^2 + 6t$



Velocity function: $v(t) = s'(t) = -2t + 6$



(b) The particle's position at:

$t = 0$ is $s(0) = -(0)^2 + 6(0) = 0$ cm

$t = 1$ is $s(1) = -(1)^2 + 6(1) = 5$ cm

$t = 3$ is $s(3) = -(3)^2 + 6(3) = 9$ cm

$t = 6$ is $s(6) = -(6)^2 + 6(6) = 0$ cm

(c) The particle's displacement for the interval:

$0 \leqslant t \leqslant 1$: $\triangle$ position $= s(1) - s(0) = 5 - 0 = 5$ cm

$1 \leqslant t \leqslant 3$: $\triangle$ position $= s(3) - s(1) = 9 - 5 = 4$ cm

$3 \leqslant t \leqslant 6$: $\triangle$ position $= s(6) - s(3) = 0 - 9 = -9$ cm

$0 \leqslant t \leqslant 6$: $\triangle$ position $= s(6) - s(0) = 0 - 0 = 0$ cm

This last result makes sense considering the particle moved to the right 9 cm then at $t = 3$ it turned around and moved to the left 9 cm, ending where it started – thus, no change in net position.

(d) The particle's total distance travelled for the interval:

$0 \leqslant t \leqslant 1$ is $|s(1) - s(0)| = |5 - 0| = 5$

$1 \leqslant t \leqslant 3$ is $|s(3) - s(1)| = |9 - 5| = 4$

$3 \leqslant t \leqslant 6$ is $|s(6) - s(3)| = |0 - 9| = |-9| = 9$

The object's motion changed direction (velocity= 0) at $t = 3$

$$0 \leqslant t \leqslant 6 \text{ is } |s(3) - s(0)| + |s(6) - s(3)| = |9 - 0| + |0 - 9| = 9 + 9 = 18$$

Since differentiation of the position function gives the velocity function $\left(\text{i.e. } v = \frac{ds}{dt}\right)$, we expect that the inverse of differentiation (integration) will lead us in the reverse direction – that is, from velocity to position. When velocity is constant, we can find the displacement with the formula:

displacement = velocity $\times$ change in time

If we drove a car at a constant velocity of 50 km h^{-1} for 3 hours, then our displacement (same as distance travelled in this case) is 150 km. If a particle travelled to the left on the x-axis at a constant rate of -4 units s^{-1} for 5 seconds, then the particle's displacement is -20 units.

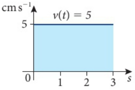


Figure 14.37 Velocity-time graph

The velocity–time graph (Figure 14.37) depicts an object's motion with a constant velocity of 5 cm s^{-1} for $0 \leqslant t \leqslant 3$. Clearly, the object's displacement is 5 cm s^{-1} $\times$ 3 s = 15 cm for this interval.

The area under the velocity curve for a certain interval is equal to the displacement for that interval. We can argue that just as the total area can be found by summing the areas of narrow rectangular strips, the displacement can be found by summing small displacements $(v \cdot \triangle t)$. Consider:

displacement = velocity $\times$ change in time $\Rightarrow s = v \cdot \triangle t \Rightarrow s = v \cdot dt$

We already know that when $f(x) \geqslant 0$, the definite integral $\int_a^b f(x)\,dx$ gives the area between $y = f(x)$ and the x-axis from $x = a$ to $x = b$. And if $f(x) \leqslant 0$, then $\int_a^b f(x)\,dx$ gives a number that is the opposite of the area between $y = f(x)$ and the x-axis from a to b.

Given that $v(t)$ is the velocity function for a particle moving along a line, then:

$\int_a^b v(t)\,dt$ gives the displacement from $t = a$ to $t = b$

$\int_a^b |v(t)|\,dt$ gives the total distance travelled from $t = a$ to $t = b$

Let's apply integration to find the displacement and distance travelled for the two intervals $3 \leqslant t \leqslant 6$ and $0 \leqslant t \leqslant 6$ in Example 14.37

For $3 \leqslant t \leqslant 6$:

$$\text{Displacement} = \int_3^6 (-2t + 6)\text{d}t = -t^2 + 6t\Big|_3^6 = 0 - 9 = -9$$

$$\text{Distance travelled} = \int_3^6 |(-2t + 6)|\text{d}t = |-t^2 + 6t|\Big|_3^6 = |0 - 9| = 9$$

For $0 \leqslant t \leqslant 6$:

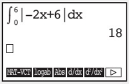
$$\text{Displacement} = \int_0^6 (-2t + 6)\text{d}t = -t^2 + 6t\Big|_0^6 = 0$$

$$\text{Distance travelled} = \int_0^3 |(-2t + 6)|\text{d}t + \int_3^6 |(-2t + 6)|\text{d}t$$

(particle changed direction at $t = 3$)

$$= |-t^2 + 6t|\Big|_0^3 + |-t^2 + 6t|\Big|_3^6 = 9 + 9 = 18$$

Note when using a GDC or your computer, you do not need to separate the integrals as we did here.



Example 14.38

The function $v(t) = \sin(\pi t)$ gives the velocity in m s^{-1} of a particle moving along the x-axis.

(a) Determine when the particle is moving to the right, to the left, and stopped. If it stops, determine if it changes direction at that time.

(b) Find the particle's displacement for the time interval $0 \leqslant t \leqslant 3$.

(c) Find the particle's total distance travelled for the time interval $0 \leqslant t \leqslant 3$.

Solution

(a) $v(t) = \sin(\pi t) = 0 \Rightarrow \sin(k \cdot \pi) = 0$ for $k \in \mathbb{Z} \Rightarrow \pi t = k\pi \Rightarrow t = k, k \in \mathbb{Z}$ for $0 \leqslant t \leqslant 3$, $t = 0, 1, 2, 3$. Therefore, the particle is stopped at $t = 0, 1, 2, 3$.

Since $t = 0$ and $t = 3$ are endpoints of the interval, the particle can change direction only at $t = 1$ or $t = 2$.

$$v\left(\frac{1}{2}\right) = \sin\left(\pi \cdot \frac{1}{2}\right) = 1;\ v\left(\frac{3}{2}\right) = \sin\left(\pi \cdot \frac{3}{2}\right) = -1$$

$\Rightarrow$ direction changes at $t = 1$

$$v\left(\frac{3}{2}\right) = \sin\left(\pi \cdot \frac{3}{2}\right) = -1;\ v\left(\frac{5}{2}\right) = \sin\left(\pi \cdot \frac{5}{2}\right) = 1$$

$\Rightarrow$ direction changes again at $t = 2$

(b) displacement $= \int_0^3 \sin(\pi t)\,dt = -\frac{1}{\pi}\cos(\pi t)\Big|_0^3$

$= -\frac{1}{\pi}\cos(3\pi) - \left(-\frac{1}{\pi}\cos(0)\right) = \frac{2}{\pi} \approx 0.637$ metres

(c) total distance travelled $= \int_0^1 |\sin(\pi t)|\,dt + \int_1^2 |\sin(\pi t)|\,dt + \int_2^3 |\sin(\pi t)|\,dt$

$= \left|\frac{2}{\pi}\right| + \left|-\frac{2}{\pi}\right| + \left|\frac{2}{\pi}\right| = \frac{6}{\pi} \approx 1.91$ metres

Note that in Example 14.39, the position function is not known precisely. The position function can be obtained by finding the antiderivative of the velocity function.

$$s(t) = \int v(t)\,dt = \int \sin(\pi t)\,dt = -\frac{1}{\pi}\cos(\pi t) + C$$

We can determine the constant of integration c only if we know the particle's initial position (or position at any other specific time). However, the particle's initial position will not affect displacement or distance travelled for any interval.

Position and velocity from acceleration

If we can obtain position from velocity by applying integration, then we can also obtain velocity from acceleration by integrating. Consider the next example.

Example 14.39

The motion of a falling parachutist is modelled as linear motion by considering that the parachutist is a particle moving along a line whose positive direction is vertically downwards. The parachute is opened at $t = 0$, at which time the parachutist's position is $s = 0$. According to the model, the acceleration function for the parachutist's motion for $t > 0$ is given by:

$$a(t) = -54e^{-1.5t}$$

(a) At the moment the parachute opens, the parachutist has a velocity of $42\ \text{m s}^{-1}$. Find the velocity function of the parachutist for $t > 0$. What does the model say about the parachutist's velocity as $t \to \infty$?

(b) Find the position function of the parachutist for $t > 0$.

Solution

(a) $v(t) = \int a(t)\,dt = \int (-54e^{-1.5t})\,dt$

$= -54\left(\frac{1}{-1.5}\right)e^{-1.5t} + C$

$= 36e^{-1.5t} + C$

Since $v = 42$ when $t = 0$, then $42 = 36e^0 + C \Rightarrow 42 = 36 + C \Rightarrow C = 6$

Therefore, after the parachute opens ($t > 0$) the velocity function is $v(t) = 36e^{-1.5t} + 6$

Since $\lim\limits_{t\to\infty} e^{-1.5t} = \lim\limits_{t\to\infty} \frac{1}{e^{1.5t}} = 0$, then as $t \to 0$, $\lim\limits_{t\to\infty} v(t) = 6\text{ m s}^{-1}$

(b) $s(t) = \int v(t)\, dt = \int (36e^{-1.5t} + 6)\, dt$

$$= 36\left(\frac{1}{-1.5}\right)e^{-1.5t} + 6t + C$$

$$= -24e^{-1.5t} + 6t + C$$

Since $s = 0$ when $t = 0$, then $0 = -24e^0 + 6(0) + C \Rightarrow 0 = -24 + C$ $\Rightarrow C = 24$

Therefore, after the parachute opens ($t > 0$), the position function is $s(t) = -24e^{-1.5t} + 6t + 24$

The limit of the velocity as $t \to \infty$, for a falling object, is called the **terminal velocity** of the object. While the limit $t \to \infty$ is never attained as the parachutist eventually lands on the ground, the velocity gets close to the terminal velocity very quickly. For example, after just 8 seconds, the velocity is $v(8) = 36e^{-1.5(8)} + 6 \approx 6.0002\text{ m s}^{-1}$

Uniformly accelerated motion

Motion under the effect of gravity in the vicinity of Earth (or other planets) is an important case of rectilinear motion. This is called **uniformly accelerated motion**.

If a particle moves with constant acceleration along the s-axis, and if we know the initial speed and position of the particle, then it is possible to have specific formulas for the position and speed at any time t.

Assume acceleration is constant; that is, $a(t) = a$, $v(0) = v_0$ and $s(0) = s_0$.

$v(t) = \int a(t)dt = at + c$; however, we know that $v(0) = v_0$, so

$v(0) = v_0 = a \times 0 + c \Rightarrow c = v_0$, hence $v(t) = at + v_0$

$s(t) = \int v(t)dt = \int (at + v_0)dt = \frac{1}{2}at^2 + v_0t + c$, and, as above, substituting $s(0) = s_0$ into the equation, we have

$$s(t) = \frac{1}{2}at^2 + v_0t + s_0$$

When this is applied to the free-fall model (s-axis vertical), then

$v(t) = -gt + v_0$ and

$s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$, where $g = 9.8\text{ m s}^{-2}$

Example 14.40

A ball is hit directly upwards from a point 2 m above the ground with initial velocity of 45 m s^{-1}. How high will the ball travel?

Solution

$$v(t) = -9.8t + 45$$

$$s(t) = -\frac{1}{2}(9.8)t^2 + 45t + 2 = -4.9t^2 + 45t + 2$$

The ball will rise until $v(t) = 0, \Rightarrow 0 = -9.8t + 45, \Rightarrow t \approx 4.6$ s

At this time

$$s(4.6) = -4.9(4.6)^2 + 45(4.6) + 2 \approx 105.32 \text{ m}$$

Exercise 14.8

1. The velocity of a particle along a rectilinear path is given by each equation for $v(t)$ in m s^{-1}. Find both the net distance and the total distance it travels between the times $t = a$ and $t = b$.

(a) $v(t) = t^2 - 11t + 24, a = 0, b = 10$

(b) $v(t) = t - \frac{1}{t^2}, a = 0.1, b = 1$

(c) $v(t) = \sin 2t, a = 0, b = \frac{\pi}{2}$

(d) $v(t) = \sin t + \cos t, a = 0, b = \pi$

(e) $v(t) = t^3 - 8t^2 + 15t, a = 0, b = 6$

(f) $v(t) = \sin\left(\frac{\pi t}{2}\right) + \cos\left(\frac{\pi t}{2}\right), a = 0, b = 1$

2. The acceleration of a particle along a rectilinear path is given by each equation for $a(t)$ in m s^{-2} and the initial velocity v_0 in m s^{-1} is also given. Find the velocity of the particle as a function of t, and both the net distance and the total distance travelled between times $t = a$ and $t = b$.

(a) $a(t) = 3, v_0 = 0, a = 0, b = 2$

(b) $a(t) = 2t - 4, v_0 = 3, a = 0, b = 3$

(c) $a(t) = \sin t, v_0 = 0, a = 0, b = \frac{3\pi}{2}$

(d) $a(t) = \frac{-1}{\sqrt{t+1}}, v_0 = 2, a = 0, b = 4$

(e) $a(t) = 6t - \frac{1}{(t+1)^3}, v_0 = 2, a = 0, b = 2$

3. The velocity and initial position of an object moving along a coordinate line are given. Find the position of the object at time t.

(a) $v = 9.8t + 5, s(0) = 10$

(b) $v = 32t - 2, s(0.5) = 4$

(c) $v = \sin \pi t, s(0) = 0$

(d) $v = \frac{1}{t+2}, t > -2, s(-1) = \frac{1}{2}$

4. The acceleration is given as well as the initial velocity and initial position of an object moving on a coordinate line. Find the position of the object at time t.

(a) $a = e^t$, $v(0) = 20$, $s(0) = 5$

(b) $a = 9.8$, $v(0) = -3$, $s(0) = 0$

(c) $a = -4\sin 2t$, $v(0) = 2$, $s(0) = -3$

(d) $a = \frac{9}{\pi^2}\cos\frac{3t}{\pi}$, $v(0) = 0$, $s(0) = -1$

5. An object moves with a speed of $v(t)$ m s^{-1} along the s-axis. Find the displacement and the distance travelled by the object during the given time interval.

(a) $v(t) = 2t - 4;\ 0 \leqslant t \leqslant 6$

(b) $v(t) = |t - 3|;\ 0 \leqslant t \leqslant 5$

(c) $v(t) = t^3 - 3t^2 + 2t;\ 0 \leqslant t \leqslant 3$

(d) $v(t) = \sqrt{t} - 2,\ 0 \leqslant t \leqslant 3$

6. An object moves with an acceleration $a(t)$ m s^{-2} along the s-axis. Find the displacement and the distance travelled by the object during the given time interval.

(a) $a(t) = t - 2,\ v_0 = 0,\ 1 \leqslant t \leqslant 5$

(b) $a(t) = \frac{1}{\sqrt{5t + 1}},\ v_0 = 2,\ 0 \leqslant t \leqslant 3$

(c) $a(t) = -2,\ v_0 = 3,\ 1 \leqslant t \leqslant 4$

7. The velocity of an object moving along the s-axis is $v = 9.8t - 3$

(a) Find the object's displacement between $t = 1$ and $t = 3$ given that $s(0) = 5$

(b) Find the object's displacement between $t = 1$ and $t = 3$ given that $s(0) = -2$

(c) Find the object's displacement between $t = 1$ and $t = 3$ given that $s(0) = s_0$

8. The displacement s metres of a moving object from a fixed point O at time t seconds is given by $s(t) = 50\text{t} - 10t^2 + 1000$.

(a) Find the velocity of the object in m s^{-1}.

(b) Find its maximum displacement from O.

9. A particle moves along a line so that its speed v at time t is given by

$$v(t) = \begin{cases} 5t & 0 \leqslant t < 1 \\ 6\sqrt{t} - \frac{1}{t} & t \geqslant 1 \end{cases}$$

where t is in seconds and v is in cm s^{-1}. Estimate the time(s) at which the particle is 4 cm from its starting position.

10. A projectile is fired vertically upwards with an initial velocity of 49 m s^{-1} from a platform 150 m high.

(a) How long will it take the projectile to reach its maximum height?

(b) What is the maximum height of the projectile?

(c) How long will it take the projectile to pass its starting point on the way down?

(d) What is the velocity of the projectile when it passes the starting point on the way down?

(e) How long will it take the projectile to hit the ground?

(f) What will its speed be at impact?

Chapter 10 practice questions

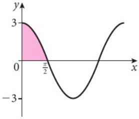


Figure 14.38 Graph for question 1

1. The graph in Figure 14.38 represents the function

$f: x \mapsto p \cos x, p \in \mathbb{N}$.

Find:

(a) the value of p

(b) the area of the shaded region.

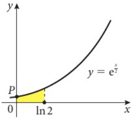


Figure 14.39 Diagram for question 2

2. The diagram in Figure 14.39 shows part of the graph of $y = e^{\frac{x}{2}}$.

(a) Find the coordinates of the point P, where the graph meets the y-axis.

The shaded region between the graph and the x-axis, bounded by $x = 0$ and $x = \ln 2$, is rotated through 360° about the x-axis.

(b) Write down an integral that represents the volume of the solid obtained.

(c) Show that this volume is π.

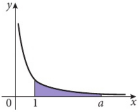


Figure 14.40 Diagram for question 3

3. The diagram in Figure 14.40 shows part of the graph of $y = \frac{1}{x}$.

The area of the shaded region is 2 units.

Find the exact value of a.

4. **(a)** Find the equation of the tangent to the curve $y = \ln x$ at the point (e, 1), and verify that the origin is on this line.

   (b) Show that $(x \ln x - x)' = \ln x$

   (c) The diagram shows the region enclosed by the curve $y = \ln x$, the tangent in part **(a)**, and the line $y = 0$.
   Use the result of part **(b)** to show that the area of this region is $\frac{1}{2}e - 1$

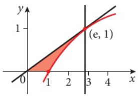
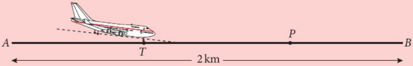


Figure 14.41 Diagram for question 4

5. The main runway at Concordville airport is 2 km long. An aeroplane, landing at Concordville touches down at point T and immediately starts to slow down. The point A is at the southern end of the runway. A marker is located at point P on the runway.



   As the aeroplane slows down, its distance, s, from A, is given by

   $$s = c + 100t - 4t^2$$

   where t is the time in seconds after touchdown, and c metres is the distance of T from A.

   (a) The aeroplane touches down 800 m from A, (i.e. $c = 800$).

   (i) Find the distance travelled by the aeroplane in the first 5 seconds after touchdown.

   (ii) Write down an expression for the velocity of the aeroplane at time t seconds after touchdown, and hence find the velocity after 5 seconds.

   The aeroplane passes the marker at P with a velocity of $36\ \text{m s}^{-1}$. Find:

   (iii) how many seconds after touchdown it passes the marker

   (iv) the distance from P to A.

   (b) Show that if the aeroplane touches down before reaching point P, it can stop before reaching the northern end, B, of the runway.

6. **(a)** Sketch the graph of $y = \pi \sin x - x$, $-3 \leqslant x \leqslant 3$, on millimetre square paper, using a scale of 2 cm per unit on each axis.
   Label and number both axes and indicate clearly the approximate positions of the x-intercepts and the local maximum and minimum points.

   (b) Find the solution of the equation $\pi \sin x - x = 0$, $x > 0$.

   (c) Find the indefinite integral $\int (\pi \sin x - x)\mathrm{d}x$ and hence, or otherwise, calculate the area of the region enclosed by the graph, the x-axis, and the line $x = 1$.

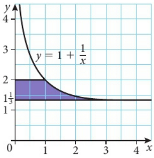


Figure 14.42 Diagram for question 7

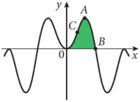


Figure 14.43 Diagram for question 9

7. Figure 14.42 shows the graph of the function $y = 1 + \frac{1}{x}$, $0 < x \leqslant 4$.

Find the exact value of the area of the shaded region.

8. Note that radians are used throughout this question.

(a) **(i)** Sketch the graph of $y = x^2 \cos x$, for $0 \leqslant x \leqslant 2$, making clear the approximate positions of the positive intercept, the maximum point, and the endpoints.

(ii) Write down the approximate coordinates of the positive x-intercept, the maximum point and the endpoints.

(b) Find the exact value of the positive x-intercept for $0 \leqslant x \leqslant 2$.

Let R be the region in the first quadrant enclosed by the graph and the x-axis.

(c) **(i)** Shade R on your sketch.

(ii) Write down an integral which represents the area of R.

(d) Evaluate the integral in part **(c) (ii)**, either by using a graphic display calculator or by using:

$$\frac{d}{dx}(x^2 \sin x + 2x \cos x - 2 \sin x) = x^2 \cos x.$$

9. Note that radians are used throughout this question.
The function f is given by $f(x) = (\sin x)^2 \cos x$

Figure 14.43 shows part of the graph of $y = f(x)$.

The point A is a maximum point, the point B lies on the x-axis, and the point C is a point of inflection.

(a) Give the period of f.

(b) From consideration of the graph of $y = f(x)$, find the range of f, accurate to 1 significant figure.

(c) **(i)** Find $f'(x)$

(ii) Hence, show that at the point A, $\cos x = \sqrt{\frac{1}{3}}$

(iii) Find the exact maximum value.

(d) Find the exact value of the x-coordinate at the point B.

(e) **(i)** Find $\int f(x)dx$

(ii) Find the area of the shaded region in the diagram.

(f) Given that $f''(x) = 9(\cos x)^3 - 7 \cos x$, find the x-coordinate at the point C.

10. Note that radians are used throughout this question.

 (a) Draw the graph of $y = \pi + x\cos x$, $0 \leqslant x \leqslant 5$, on millimetre square graph paper, using a scale of 2 cm per unit. Make clear:

  (i) the integer values of x and y on each axis

  (ii) the approximate positions of the x-intercepts and the turning points.

 (b) Without the use of a calculator, show that π is a solution of the equation $\pi + x\cos x = 0$

 (c) Find another solution of the equation $\pi + x\cos x = 0$ for $0 \leqslant x \leqslant 5$, giving your answer to 6 significant figures.

 (d) Let R be the region enclosed by the graph and the axes for $0 \leqslant x \leqslant \pi$. Shade R on your diagram, and write down an integral which represents the area of R.

 (e) Evaluate the integral in part **(d)** to an accuracy of 6 significant figures. If considered necessary, you can make use of the result $\frac{d}{dx}(x\sin x + \cos x) = x\cos x$

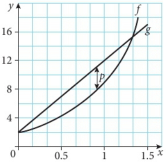


Figure 14.44 Diagram for question 11

11. Figure 14.44 shows the graphs of $f(x) = 1 + e^{2x}$ and $g(x) = 10x + 2$, $0 \leqslant x \leqslant 1.5$

 (a) (i) Write down an expression for the vertical distance p between the graphs of f and g.

  (ii) Given that p has a maximum value for $0 \leqslant x \leqslant 1.5$, find the value of x at which this occurs.

 The graph of $y = f(x)$ only is shown Figure 14.45.
 When $x = a$, $y = 5$.

 (b) (i) Find $f^{-1}(x)$

  (ii) Hence, show that $a = \ln 2$

 (c) The region shaded in Figure 14.45 is rotated through 360° about the x-axis. Write down an expression for the volume obtained.

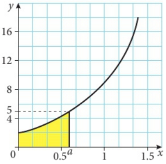


Figure 14.45 Second diagram for question 11

12. The area of the enclosed region shown in Figure 14.46 is defined by

$$y \geqslant x^2 + 2,\ y \leqslant ax + 2, \text{ where } a > 0$$

 This region is rotated through 360° about the x-axis to form a solid of revolution. Find, in terms of a, the volume of this solid of revolution.

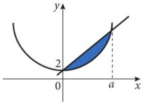


Figure 14.46 Diagram for question 12

13. Using the substitution $u = \frac{1}{2}x + 1$, or otherwise, find the integral

$$\int x\sqrt{\frac{1}{2}x + 1}\,dx$$

14. A particle moves along a straight line. When it is a distance s from a fixed point, where $s > 1$, the velocity v is given by $v = \frac{3s + 2}{2s - 1}$

Find the acceleration when $s = 2$.

15. The area between the graph of $y = e^x$ and the x-axis from $x = 0$ to $x = k$ $(k > 0)$ is rotated through 360° about the x-axis. In terms of k and e, find the volume of the solid generated.

16. Find the real number $k > 1$ for which $\int_1^k \left(1 + \frac{1}{x^2}\right)dx = \frac{3}{2}$

17. The acceleration, $a(t)$ m s^{-2}, of a fast train during the first 80 seconds of motion is given by

$$a(t) = -\frac{1}{20}t + 2$$

where t is the time in seconds. If the train starts from rest at $t = 0$, find the distance travelled by the train in the first minute.

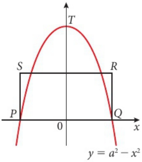


Figure 14.47 Diagram for question 18

18. In Figure 14.47, PTQ is an arc of the parabola $y = a^2 - x^2$, where a is a positive constant and $PQRS$ is a rectangle. The area of rectangle $PQRS$ is equal to the area between the arc PTQ of the parabola and the x-axis.

Find, in terms of a, the dimensions of the rectangle.

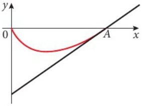


Figure 14.48 Diagram for question 19

19. Consider the function $f_k(x) = \begin{cases} x\ln x - kx & x > 0 \\ 0 & x = 0 \end{cases}$, where $k \in \mathbb{N}$

(a) Find the derivative of $f_k(x)$, $x > 0$.

(b) Find the interval over which $f(x)$ is increasing.

The graph of the function $f_k(x)$ is shown in Figure 14.48.

(c) **(i)** Show that the stationary point of $f_k(x)$ is at $x = e^{k-1}$.

(ii) One x-intercept is at (0, 0). Find the coordinates of the other x-intercept.

(d) Find the area enclosed by the curve and the x-axis.

(e) Find the equation of the tangent to the curve at A.

(f) Show that the area of the triangular region created by the tangent and the coordinate axes is twice the area enclosed by the curve and the x-axis.

(g) Show that the x-intercepts of $f_k(x)$ for consecutive values of k form a geometric sequence.

20. Consider the graphs of the functions $f(x) = a - |x - a|$ and $g(x) = |x - a|$, where $a > 0$. Find the value of a if the two graphs enclose an area of 12.5 square units.

21. The equation of motion of a particle with mass m subjected to a force kx can be written as $kx = mv\dfrac{dv}{dx}$, where x is the displacement and v is the velocity. When $x = 0$, $v = v_0$. Find v, in terms of v_0, k, and m, when $x = 2$.

22. **(a)** Sketch and label the graphs of $f(x) = e^{-x^2}$ and $g(x) = e^{-x^2} - 1$ for $0 \leqslant x \leqslant 1$, and shade the region A that is bounded by the graphs and the y-axis.

(b) Let the x-coordinate of the point of intersection of the curves $y = f(x)$ and $y = g(x)$ be p. Without finding the value of p, show that $\dfrac{p}{2} <$ area of region $A < p$.

(c) Find the value of p correct to 4 decimal places.

(d) Express the area of region A as a definite integral and calculate its value.

23. Let $f(x) = x \cos 3x$

(a) Use integration by parts to show that
$\int f(x)\,dx = \dfrac{1}{3}x \sin 3x + \dfrac{1}{9}\cos 3x + c$

(b) Use your answer to part **(a)** to calculate the exact area enclosed by $f(x)$ and the x-axis in each of the following cases. Give your answers in terms of π.

(i) $\dfrac{\pi}{6} \leqslant x \leqslant \dfrac{3\pi}{6}$ **(ii)** $\dfrac{3\pi}{6} \leqslant x \leqslant \dfrac{5\pi}{6}$ **(iii)** $\dfrac{5\pi}{6} \leqslant x \leqslant \dfrac{7\pi}{6}$

(c) Given that the above areas are the first three terms of an arithmetic sequence, find an expression for the total area enclosed by $f(x)$ and the x-axis for $\dfrac{\pi}{6} \leqslant x \leqslant \dfrac{(2n+1)\pi}{6}$, where $n \in \mathbb{Z}$. Give your answers in terms of n and π.

24. A particle is moving along a straight line so that t seconds after passing through a fixed point O on the line, its velocity $v(t)\ \text{m s}^{-1}$ is given by $v(t) = t \sin\left(\dfrac{\pi}{3}t\right)$.

(a) Find the values of t for which $v(t) = 0$, given that $0 \leqslant t \leqslant 6$.

(b) **(i)** Write down a mathematical expression for the total distance travelled by the particle in the first six seconds after passing through O.

(ii) Find this distance.

25. A particle is projected along a straight-line path. After t seconds, its velocity v in metres per second is given by $v = \frac{1}{2 + t^2}$

(a) Find the distance travelled in the first second.

(b) Find an expression for the acceleration at time t.

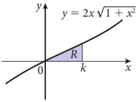


Figure 14.49 Diagram for question 26

26. Figure 14.49 shows the shaded region R enclosed by the graph of $y = 2x\sqrt{1 + x^2}$, the x-axis, and the vertical line $x = k$.

(a) Find $\frac{dy}{dx}$

(b) Using the substitution $u = 1 + x^2$ or otherwise, show that

$$\int 2x\sqrt{1 + x^2}\,dx = \frac{2}{3}(1 + x^2)^{\frac{3}{2}} + c$$

(c) Given that the area of R equals 1, find the value of k.

27. A particle moves in a straight line with velocity in metres per second, at time t seconds, given by $v(t) = 6t^2 - 6t, t \geqslant 0$.

Calculate the total distance travelled by the particle in the first two seconds of motion.

28. A particle moves in a straight line. Its velocity v m s^{-1} after t seconds is given by $v = e^{-\sqrt{t}} \sin t$.

Find the total distance travelled in the time interval $[0, 2\pi]$.

29. The temperature T °C of an object in a room, after t minutes, satisfies the differential equation $\frac{dT}{dt} = k(T - 22)$, where k is a constant.

(a) Solve the differential equation showing that $T = Ae^{kt} + 22$, where A is a constant.

(b) When $t = 0$, $T = 100$, and when $t = 15$, $T = 70$.

(i) Use this information to find the values of A and k.

(ii) Hence, find the value of t when $T = 40$.

30. Consider the function $f(x) = \frac{1}{x^2 + 5x + 4}$

(a) Sketch the graph of the function, indicating the equations of the asymptotes, intercepts, and extreme values.

(b) Find $\int_0^1 f(x)\,dx$ and express it in the form $\ln k$.

(c) Sketch the graph of $f(|x|)$ and hence determine the area of the region between this graph, the x-axis, and the lines $x = -1$, and $x = 1$.

31. Use the substitution $u = x + 2$ to find $\int \frac{x^3\,dx}{(x + 2)^2}$

32. (a) On the same axes, sketch the graphs of the functions, $f(x)$ and $g(x)$, where

$$f(x) = 4 - (1 - x)^2, \text{ for } -2 \leqslant x \leqslant 4$$
$$g(x) = \ln(x + 3) - 2, \text{ for } -3 \leqslant x \leqslant 5$$

(b) (i) Write down the equation of any vertical asymptotes.

(ii) State the x-intercept and y-intercept of $g(x)$.

(c) Find the values of x for which $f(x) = g(x)$.

(d) Let A be the region where $f(x) \geqslant g(x)$ and $x \geqslant 0$.

(i) On your graph, shade the region A.

(ii) Write down an integral that represents the area of A.

(iii) Evaluate this integral.

(e) In the region A, find the maximum vertical distance between $f(x)$ and $g(x)$.

33. Consider the equation $\frac{dy}{d\theta} = \frac{y}{e^{2\theta} + 1}$

(a) Use the substitution $x = e^{\theta}$ to show that $\int \frac{dy}{y} = \int \frac{dx}{x(x^2 + 1)}$

(b) Find $\int \frac{dx}{x(x^2 + 1)}$

(c) Hence, find y in terms of θ, if $y = \sqrt{2}$ when $\theta = 0$

34. Figure 14.50 shows part of the graph of $y = \frac{(\ln x)^2}{x}$, $x > 0$

(a) Find the extreme points of the curve.

(b) The region R is enclosed by the curve, the x-axis, and the line $x = e$. Find the area of the region R.

(c) Find the volume of the solid formed when the region R is rotated through 2π about the x-axis.

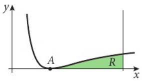


Figure 14.50 Diagram for question 34

35. (a) The functions f and g are defined by:

$$f(x) = \frac{e^x + e^{-x}}{2}, x \in \mathbb{R} \qquad g(x) = \frac{e^x - e^{-x}}{2}, x \in \mathbb{R}$$

(i) Show that $\frac{1}{4f(x) - 2g(x)} = \frac{e^x}{e^{2x} + 3}$

(ii) Use the substitution $u = e^x$ to find $\int_0^{\ln 3} \frac{1}{4f(x) - 2g(x)}\,dx$

Give your answer in the form $\frac{\pi\sqrt{a}}{b}$, where $a, b \in \mathbb{Z}^+$.

(b) Let $h(x) = nf(x) + g(x)$ where $n \in \mathbb{R}$, $n > 1$

(i) By forming a quadratic equation in e^x, solve the equation $h(x) = k$, where $k \in \mathbb{R}^+$.

(ii) Hence, or otherwise, show that the equation $h(x) = k$ has two real solutions, provided that $k > \sqrt{n^2 - 1}$ and $k \in \mathbb{R}^+$.

(c) Let $t(x) = \dfrac{g(x)}{f(x)}$

(i) Show that $t'(x) = \dfrac{[f(x)]^2 - [g(x)]^2}{[f(x)]^2}$ for $x \in \mathbb{R}$.

(ii) Hence show that $t'(x) > 0$ for $x \in \mathbb{R}$.

43. (a) $f'(x) = \frac{3}{3x+1}$ (b) $y = -\frac{7}{3}x + \frac{14}{3} + \ln 7$

44. See Worked Solutions

45. $\frac{dy}{dx} = \frac{e-1}{e}$

46. (a) See Worked Solutions
 (b) $b = -\frac{1}{\sqrt{6}}$

47. (a) $\frac{dy}{dx} = \frac{2-k}{2k-1}$ (b) $k = 2$

48. (a) $5\sqrt{5}\sqrt{x^2+4} + 5(2-x)$ minutes
 (b) See Worked Solutions
 (c) (i) $x = 1$
 (ii) 30 minutes
 (iii) $\frac{d^2T}{dx^2} > 0$ for $x = 1$, therefore it's a minimum

49. $\frac{72}{\pi}\arccos\frac{8}{13}$ cm

Chapter 14

Exercise 14.1

1. (a) $\frac{x^2}{2} + 2x + c$ (b) $t^3 - t^2 + t + c$
 (c) $\frac{x}{3} - \frac{x^4}{14} + c$ (d) $\frac{2t^3}{3} + \frac{t^2}{2} - 3t + c$
 (e) $\frac{5u^{\frac{7}{5}}}{7} - u^4 + c$ (f) $\frac{4x\sqrt{x}}{3} - 3\sqrt{x} + c$
 (g) $-3\cos\theta + 4\sin\theta + c$ (h) $t^3 + 2\cos t + c$
 (i) $\frac{4x^2\sqrt{x}}{5} - \frac{10x\sqrt{x}}{3} + c$ (j) $3\sin\theta - 2\tan\theta + c$
 (k) $\frac{1}{3}e^{3t-1} + c$ (l) $2\ln|t| + c$
 (m) $\frac{1}{6}\ln(3t^2+5) + c$ (n) $e^{\sin\theta} + c$
 (o) $\frac{(2x+3)^3}{6} + c$

2. (a) $-\frac{5x^4}{4} + \frac{2x^3}{3} + cx + k$
 (b) $-\frac{x^5}{5} + \frac{x^4}{4} + \frac{x^2}{2} + 2x - \frac{11}{20}$
 (c) $\frac{4t^3}{3} + \sin t + ct + k$
 (d) $3x^4 - 4x^2 + 7x + 3$
 (e) $2\sin\theta + \frac{1}{2}\cos 2\theta + c$

3. (a) $\frac{(3x^2+7)^6}{36} + c$ (b) $-\frac{1}{18(3x^2+5)^3} + c$
 (c) $\frac{8\sqrt[4]{(5x^3+2)^5}}{75} + c$ (d) $\frac{(2\sqrt{x}+3)^6}{6} + c$
 (e) $\frac{\sqrt{(2t^3-7)^3}}{9} + c$ (f) $-\frac{(2x+3)^6}{18x^6} + c$
 (g) $-\frac{\cos(7x-3)}{7} + c$
 (h) $-\frac{1}{2}\ln(\cos(2\theta-1)+3) + c$
 (i) $\frac{1}{5}\tan(5\theta-2) + c$ (j) $\frac{1}{\pi}\sin(\pi x+3) + c$
 (k) $\frac{1}{2}\sec 2t + c$ (l) $\frac{1}{2}e^{x^2+1} + c$
 (m) $\frac{1}{3}e^{2t\sqrt{t}} + c$ (n) $\frac{2}{3}(\ln\theta)^3 + c$

4. (a) $-\frac{1}{15}\sqrt{(3-5t^2)^3} + c$ (b) $\frac{1}{3}\tan\theta^3 + c$
 (c) $-\cos\sqrt{t} + c$ (d) $\frac{1}{12}\tan^6 2t + c$
 (e) $2\ln(\sqrt{x}+2) + c$ (f) $\frac{1}{10}\sec^5 2t + c$
 (g) $\frac{1}{2}\ln(x^2+6x+7) + c$
 (h) $-\frac{k^3}{2a^4}\sqrt{a^2-a^4x^4} + c = -\frac{k^3}{2|a|^3}\sqrt{1-a^2x^4} + c$
 (i) $\frac{2}{5}(3x^2-x-2)\sqrt{x-1} + c$
 (j) $-\frac{1}{\pi}\cot\pi t + c$
 (k) $-\frac{2}{3}\sqrt{(1+\cos\theta)^3} + c$
 (l) $\frac{2}{105}(15t^3-3t^2-4t-8)\sqrt{1-t} + c$
 (m) $\frac{1}{15}(3r^2+2r-13)\sqrt{2r-1} + c$
 (n) $\frac{1}{2}\ln(e^{x^2}+e^{-x^2}) + c$

Exercise 14.2

1. (a) $-\frac{1}{3}e^{-x^3} + c$
 (b) $-e^{-x}(x^2+2x+2) + c$
 (c) $\frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + \frac{1}{3}x^2\sin 3x + c$
 (d) $\frac{1}{a^3}(2\cos ax - a^2x^2\cos ax + 2ax\sin ax) + c$
 (e) $\sin x(\ln(\sin x) - 1) + c$
 (f) $\frac{1}{2}x^2(\ln x^2 - 1) + c$
 (g) $\frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + c$
 (h) $2e^x + x^2e^x - 2xe^x - \frac{1}{3}x^3 + c$
 (i) $\frac{1}{\pi^2}(\cos\pi x + \pi x\sin\pi x) + c$
 (j) $\frac{3}{13}\cos 2t\, e^{3t} + \frac{2}{13}e^{3t}\sin 2t + c$
 (k) $\sqrt{1-x^2} + x\arcsin x + c$
 (l) $e^x(x^3-3x^2+6x-6) + c$
 (m) $-\frac{1}{4}e^{-2x}(\cos 2x + \sin 2x) + c$
 (n) $\frac{1}{2}x(\sin(\ln x) - \cos(\ln x)) + c$
 (o) $\frac{1}{2}x(\sin(\ln x) + \cos(\ln x)) + c$
 (p) $\ln(x+1) - 2x + x\ln(x^2+x) + c$
 (q) $\frac{e^{kx}(k\sin x - \cos x)}{k^2+1} + c$
 (r) $\ln(\cos x) + x\tan x + c$
 (s) $\frac{1}{2}\sin x - \frac{1}{6}\sin 3x + c$
 (t) $\frac{1}{2}\arctan x(1+x^2) - \frac{1}{2}x + c$
 (u) $2\sqrt{x}(\ln x - 2) + c$

2. Verification: First column represents u in "repeated by parts" and second column represents v.

3. (a) $-x^4\cos x + 4x^3\sin x + 12x^2\cos x - 24x\sin x - 24\cos x + c$
(b) $x^5\sin x + 5x^4\cos x - 20x^3\sin x - 60x^2\cos x + 120x\sin x + 120\cos x + c$
(c) $e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) + c$
4. No pattern in the second column
5. Use repeated "by parts" with $u = x^n$ and $dv = e^x dx$
6. "By parts" with $u = \ln x$
7. Repeated "by parts" to find the unknown integral
8. Repeated "by parts" to find the unknown integral

Exercise 14.3

1. (a) $\frac{\cos^5 t}{5} - \frac{\cos^3 t}{3} + c$
(b) $\frac{1}{192}\cos 6t - \frac{3}{64}\cos 2t + c$
(c) $\frac{\sin^4 3\theta}{12} + c$
(d) $\frac{1}{3}\cos^3\frac{1}{t} - \frac{2}{5}\cos^5\frac{1}{t} + \frac{1}{7}\cos^7\frac{1}{t} + c$
(e) $\sec x + \cos x + c$
(f) $\frac{1}{18}\tan^6 3x + c$
(g) $\frac{1}{24}(3\tan^4\theta^2 + 2\tan^6\theta^2) + c$
(h) $\frac{2\sec^5\sqrt{t}}{5} - \frac{2\sec^3\sqrt{t}}{3} + c$
(i) $\frac{1}{15}(\tan^3 5t - 3\tan 5t + 15t) + c$
(j) $\tan t - \sec t + c$
(k) $\csc t - \cot t + c$
(l) $-\ln(1 - \sin t) + c$
(m) $-2x - 3\ln(\sin x + \cos x) + c$
(n) $\arctan(\sec\theta) + c$
(o) $\frac{1}{2}(\arctan t)^2 + c$
(p) $\ln|\arctan| + c$
(q) $\arcsin(\ln x) + c$
(r) $\frac{-\cos x}{3}(\sin^2 x + 2) + c$ or $\frac{\cos^3 x}{3} - \cos x + c$
(s) $\frac{2}{5}(\cos^2 x\sqrt{\cos x} - 10\sqrt{\cos x}) + c$
(t) $\frac{-\cos\sqrt{x}}{3}(2\sin^2\sqrt{x} + 4) + c$ or $2\left(\frac{\cos^3\sqrt{x}}{3} - \cos\sqrt{x}\right) + c$
(u) $\frac{\sin(\sin t)}{3}(\cos^2(\sin t) + 2) + c$
or $\sin(\sin x) - \frac{\sin^3(\sin x)}{3} + c$
(v) $\ln(\sin\theta) + 2\sin\theta + c$
2. (a) $t\sec t - \ln|\sec t + \tan t| + c$
(b) $-\ln(2 - \sin x) + c$
(c) $\frac{1}{2}\ln(\cos(e^{-2x})) + c$
(d) $2\ln|\sec\sqrt{x} + \tan\sqrt{x}| + c$
(e) $\frac{1}{2}\tan x + c$
(f) $\frac{1}{6}(\arcsin 3x + 3x\sqrt{1 - 9x^2}) + c$
(g) $\frac{x}{4\sqrt{x^2 + 4}} + c$
(h) $2\ln(t + \sqrt{t^2 + 4}) + \frac{1}{2}t\sqrt{t^2 + 4} + c$
(i) $\frac{3}{2}\arctan\left(\frac{1}{2}e^t\right) + c$
(j) $\frac{1}{2}\arcsin\left(\frac{2}{3}x\right) + c$
(k) $\frac{1}{3}\ln\left|\frac{\sqrt{4 + 9x^2} + 3x}{2}\right| + c$
(l) $\ln(\sqrt{\sin^2 + 1} + \sin x) + c$
(m) $-\sqrt{4 - x^2} + c$
(n) $\frac{1}{2}\ln(x^2 + 16) + c$
(o) $-\arcsin\left(\frac{x}{2}\right) - \frac{\sqrt{4 - x^2}}{x} + c$
(p) $\frac{1}{9}\frac{x}{\sqrt{9 - x^2}} + c$
(q) $\frac{(x^2 + 1)^{\frac{3}{2}}}{3} + c$
(r) $\frac{(e^{2x} + 1)^{\frac{3}{2}}}{3} + c$
(s) $\frac{1}{2}(\sin^{-1}e^x + e^x\sqrt{1 + e^x}) + c$
(t) $\ln\left(\frac{1}{3}e^x + \frac{1}{3}\sqrt{e^{2x} + 9}\right) + c$
(u) $2\sqrt{x}(\ln x - 2) + c$
(v) $12\ln(x + 2) + \frac{8}{x + 2} + \frac{x^2}{2} - 4x + c$
3. $\frac{1}{2}\ln(x^2 + 9) + c_1$;$x = 3\tan\theta$ yields $\ln\left(\frac{\sqrt{x^2 + 9}}{3}\right) + c_2$; they differ by a constant
4. $x - 3\arctan\left(\frac{x}{3}\right) + c_1$; $x = 3\tan\theta$ yields $3(\tan\theta - \theta) + c_2 = 3\left(\frac{x}{3} - \arctan\frac{x}{3}\right) + c_2$

Exercise 14.4

1. (a) 24 (b) 40 (c) $\frac{24}{25}$ (d) 0
(e) $\frac{176\sqrt{7} - 44}{5}$ (f) 0 (g) 2
(h) -268 (i) $\frac{64}{3}$ (j) 2
(k) $\ln\left(\frac{11}{3}\right)$ (l) $\frac{44}{3} - 8\sqrt{3}$ (m) 3
(n) $\sqrt{\pi} + 1$
(o) (i) 6 (ii) 6 (iii) 12
(p) 1 (q) 4 (r) 0 (s) $\frac{\pi}{2}$
(t) $\frac{\pi}{6}$ (u) $\frac{\pi}{3}$ (v) $\frac{\pi}{8}$
2. (a) $\frac{14\sqrt{17} + 2}{3}$ (b) $\frac{1}{\pi}$ (c) $\ln(2)$
(d) $16\sqrt{2} - 5\sqrt{5}$ (e) $\sqrt{14} - \sqrt{10}$ (f) $\frac{3}{2}$
(g) $\pi^{3/2}\left(\frac{2\sqrt{3}}{27} - \frac{1}{12}\right)$ (h) $\frac{\pi}{6}$
(i) $-\frac{1}{2}\ln\left(\frac{37}{52}\right)$ (j) $-\arctan\left(\frac{\sqrt{15} - \sqrt{7}}{4}\right)$
(k) $\frac{2}{3}$ (l) 0 (m) -4 (n) $\frac{\pi}{6}$
(o) $\frac{1}{6}\arctan\left(\frac{4\sqrt{3}}{9}\right)$ (p) $\frac{\pi\sqrt{3} - 3\sqrt{3}\arctan\left(\frac{\sqrt{3}}{2}\right)}{18}$
(q) $\frac{1}{6}$ (r) $\frac{e - 1}{2}$
(s) $1 + \frac{e}{2}$ (t) $2\cos(1) + 2$

3. (a) $\frac{31}{5}$ (b) $\frac{2}{\pi}$ (c) $\frac{12 - 4\sqrt{3}}{\pi}$
(d) $\frac{e^8 - 1}{8e^8}$ (e) $\frac{\pi}{6\ln 3}$

4. (a) $\frac{\sin x}{x}$ (b) $-\frac{\sin x}{x}$ (c) $-2x\frac{\sin x^2}{x^2}$
(d) $2x\frac{\sin x^2}{x^2}$ (e) $\frac{\cos t}{1 + t^2}$ (f) $\frac{b - a}{5 + x^4}$
(g) $-\csc\theta - \sec\theta$ (h) $\frac{1}{4x^{\frac{3}{4}}}(e^{x+3x^{\frac{1}{2}}})$

5. Yes

6. (a) $\frac{1}{3}\ln\left(\frac{3k + 2}{2}\right)$ (b) $k = \frac{2(e^3 - 1)}{3}$

7. Substitute $u = 1 - x$

8. (a) $-(1 - x)^{k+1}\left(\frac{1}{k + 1} + \frac{x - 1}{k + 2}\right)$
(b) $\frac{1}{(k + 1)(k + 2)}$

9. (a) 0; (b) $\sqrt{47}$; (c) $\frac{15\sqrt{47}}{47}$

10. $f(x) = 0$.

Exercise 14.5

1. (a) $\int\frac{5x + 1}{x^2 + x - 2}dx = \int\frac{3}{x + 2}dx + \int\frac{2}{x - 1}dx = 3\ln|x + 2| + 2\ln|x - 1| + c$

(b) $\int\frac{x + 4}{x^2 - 2x}dx = \int\frac{3}{x - 2}dx - \int\frac{2}{x}dx = 3\ln|x - 2| - 2\ln|x| + c$

(c) $\int\frac{x + 2}{x^2 + 4x + 3}dx = \int\frac{1}{2(x + 3)}dx + \int\frac{1}{2(x + 1)}dx = \frac{1}{2}\ln|x + 3| + \frac{1}{2}\ln|x + 1| + c$ $= \frac{1}{2}\ln|x^2 + 4x + 3| + c$

(d) $\int\frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x}dx = \int\frac{9}{(x + 1)^2}dx - \int\frac{1}{x + 1}dx + \int\frac{6}{x}dx = -\frac{9}{x + 1} - \ln|x + 1| + 6\ln|x| + c$

(e) $\int\frac{2x^2 + x - 12}{x^3 + 5x^2 + 6x}dx = \int\frac{1}{x + 3}dx + \int\frac{3}{x + 2}dx - \int\frac{2}{x}dx = \ln|x + 3| + 3\ln|x + 2| - 2\ln|x| + c$

(f) $\int\frac{4x^2 + 2x - 1}{x^3 + x^2}dx = \int\frac{1}{x + 1}dx - \int\frac{1}{x^2}dx + \int\frac{3}{x}dx = \ln|x + 1| + \frac{1}{x} + 3\ln|x| + c$

(g) $\int\frac{3}{x^2 + x - 2}dx = \int\frac{1}{x - 1}dx - \int\frac{1}{x + 2}dx = -\ln|x + 2| + \ln|x - 1| + c = \ln\left|\frac{x - 1}{x + 2}\right| + c$

(h) $\int\frac{5 - x}{2x^2 + x - 1}dx = \int\frac{3}{2x - 1}dx - \int\frac{2}{x + 1}dx = \frac{3\ln|2x - 1|}{2} - 2\ln|x + 1| + c$

(i) $\int\frac{3x + 4}{(x + 2)^2}dx = \int\frac{3}{x + 2}dx - \int\frac{2}{(x + 2)^2}dx = 3\ln|x + 2| + \frac{2}{x + 2} + c$

(j) $\int\frac{12}{x^4 - x^3 - 2x^2}dx = \int\frac{1}{x - 2}dx - \int\frac{4}{x + 1}dx - \int\frac{6}{x^2}dx + \int\frac{3}{x}dx = \ln|x - 2| - 4\ln|x + 1| + \frac{6}{x} + 3\ln|x| + c$

(k) $\int\frac{2}{x^3 + x}dx = \int\frac{2}{x}dx - \int\frac{2x}{x^2 + 1}dx = 2\ln|x| - \ln(x^2 + 1) + c$

(l) $\int\frac{x + 2}{x^3 + 3x}dx = \int\frac{2}{3x}dx + \int\frac{dx}{(x^2 + 3)} - \int\frac{2x}{3(x^2 + 3)}dx = \frac{2\ln|x|}{3} + \frac{\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}x}{3}\right) - \frac{\ln(x^2 + 3)}{3} + c$

(m) $\int\frac{3x + 2}{x^3 + 6x}dx = \int\frac{1}{3x}dx + \int\frac{3}{x^2 + 6}dx - \int\frac{x}{3(x^2 + 6)}dx = \frac{\ln|x|}{3} + \frac{\sqrt{6}}{2}\arctan\left(\frac{\sqrt{6}x}{6}\right) - \frac{\ln(x^2 + 6)}{6} + c$

(n) $\int\frac{2x + 3}{x^3 + 8x}dx = \int\frac{3}{8x}dx + \int\frac{2}{x^2 + 8}dx - \int\frac{3x}{8(x^2 + 8)}dx = \frac{3}{8}\ln|x| + \frac{\sqrt{2}}{2}\arctan\left(\frac{\sqrt{2}x}{4}\right) - \frac{3}{16}\ln(x^2 + 8) + c$

Exercise 14.6

1. (a) $\frac{125}{6}$ (b) $\frac{9\pi^2}{8} + 1$ (c) $4\sqrt{3}$
(d) $\frac{10}{3}$ (e) $\frac{8}{21}$ (f) $\frac{125}{24}$
(g) $\frac{13}{12}$ (h) 4π (i) $\frac{59}{12}$
(j) 4.65 (k) $3\ln 2 - \frac{63}{128}$
(l) (between $-\frac{\pi}{2}$and $\frac{\pi}{6}$) $\sqrt{3}\ln\left(\frac{3}{4}\right) - 2\sqrt{3} + 4$
(m) 18 (n) $\frac{32}{3}$ (o) $\frac{64}{3}$
(p) 9 (q) $\frac{9}{2}$ (r) 19
(s) $\frac{2\sqrt{3}}{3} + 2$ (t) $\frac{37}{12}$ (u) $\frac{1}{2}$
(v) $\frac{2\sqrt{2}}{3}$

2. $\frac{269}{54}$
3. $\frac{e}{2} - 1$
4. $\frac{288\sqrt{3}}{35}$
5. $\frac{2\sqrt{2}}{3}$
6. $\frac{16}{3}$
7. 25.36
8. $m = 0.973$
9. $\frac{37}{12}$

Exercise 14.7

1. **(a)** $\frac{127\pi}{27}$ **(b)** $\frac{64\sqrt{2}\,\pi}{15}$ **(c)** $\frac{70\pi}{3}$
   (d) 6π **(e)** 9π **(f)** 2π
   (g) $\left(\frac{\sqrt{3}}{2} + 1\right)\pi$ **(h)** $\frac{512\pi}{15}$
   (i) approx. 5.937π **(j)** $\frac{32\pi}{3}$
   (k) $\pi(\sqrt{3} - 1)$ **(l)** $\frac{23\pi}{210}$ **(m)** $\frac{160\pi\sqrt{5}}{3}$
   (n) $\frac{64}{15}\pi$ **(o)** $\frac{\pi}{2}$ **(p)** $\frac{1778}{5}\pi$
   (q) $\frac{252}{5}\pi$ **(r)** $\frac{656}{3}\pi$ **(s)** $\frac{9}{8}\pi$
2. **(a)** $\frac{88}{15}\pi$ **(b)** $\frac{7}{6}\pi$
3. **(a)** $\frac{\pi}{2}$ **(b)** $2\pi(18 - 9\sqrt{2})$ **(c)** $\frac{32}{15}\pi$
   (d) $\frac{4}{5}\pi(121\sqrt{33} - 25\sqrt{15})$
   (e) $2\pi\left(\ln 2 - \frac{1}{4}\right)$ **(f)** $2\pi\left(\frac{11}{3}\sqrt{11} - \frac{2}{3}\sqrt{2}\right)$
   (g) $\frac{28}{3}\pi(\sqrt{34} - \sqrt{7})$ **(h)** $\pi\left(\frac{1}{2}\sqrt{2}\,\pi - \pi + 2\right)$
   (i) $\frac{284}{3}\pi$ **(j)** 2π **(k)** $\frac{256}{15}\pi$

Exercise 14.8

1. **(a)** $\frac{70}{3}$ m, 65 m **(b)** 8.5 m to the left, 8.5 m.
   (c) 1 m, 1 m **(d)** 2 m, $2\sqrt{2}$m.
   (e) 18 m, 28.67 m **(f)** $\frac{4}{\pi}m, \frac{4}{\pi}m$
2. **(a)** $3t$, 6 m, 6 m **(b)** $t^2 - 4t + 3$, 0, 2.67 m
   (c) $1 - \cos t, \left(\frac{3\pi}{2} + 1\right)m, \left(\frac{3\pi}{2} + 1\right)m$
   (d) $4 - 2\sqrt{t+1}$, 2.43 m, 2.91 m
   (e) $3t^2 + \frac{1}{2(1+t)^2} + \frac{3}{2}$, 11.3 m, 11.3 m
3. **(a)** $4.9t^2 + 5t + 10$ **(b)** $16t^2 - 2t + 1$
   (c) $\frac{1}{\pi} - \frac{\cos \pi t}{\pi}$ **(d)** $\ln(t + 2) + \frac{1}{2}$
4. **(a)** $e^t + 19t + 4$ **(b)** $4.9t^2 - 3t$
   (c) $\sin(2t) - 3$ **(d)** $-\cos\left(\frac{3t}{\pi}\right)$
5. **(a)** 12; 20 **(b)** $\frac{13}{2}; \frac{13}{2}$
   (c) $\frac{9}{4}; \frac{11}{4}$ **(d)** $2\sqrt{3} - 6; 6 - 2\sqrt{3}$
6. **(a)** $-\frac{10}{3}; \frac{17}{3}$ **(b)** $\frac{204}{25}$ **(c)** $-6; \frac{13}{2}$
7. **(a)** $\frac{166}{5}$ **(b)** $\frac{166}{5}$ **(c)** $\frac{166}{5}$
8. **(a)** $50 - 20t$ **(b)** 1187.5
9. 1.0041 s
10. **(a)** 5 s **(b)** 272.5 m **(c)** 10 s
    (d) -49 m s^{-1} **(e)** 12.46 s **(f)** -73.08 m s^{-1}

Chapter 14 practice questions

1. **(a)** $p = 3$ **(b)** 3 square units
2. **(a)** (0, 1) **(b)** $V = \pi\int_0^{\ln 2}\left(e^{\frac{x}{2}}\right)^2 dx$
   (c) See Worked Solutions
3. $a = e^2$
4. **(a)** $y = \frac{x}{e}$ **(b)** $\ln x + 1 - 1$
   (c) $\frac{1}{2} \cdot e \cdot 1 - \int_1^e \ln x dx$
5. **(a)** **(i)** 400 m
   (ii) $v = 100 - 8t$, 60 m/s
   (iii) 8 s
   (iv) 1344 m
   (b) Distance needed 625
6. **(a)** See Worked Solutions **(b)** 2.31
   (c) $-\pi \cos x - \frac{x^2}{2} + c$; 0.944
7. ln 3
8. **(a)** **(i)** See Worked Solutions
   (ii) (1.57, 0); (1.1, 0.55); (0, 0), (2, −1.66)
   (b) $x = \frac{\pi}{2}$
   (c) **(i)** See Worked Solutions
   (ii) $\int_0^{\frac{\pi}{2}} x^2 \cos x\, dx$
   (d) $\frac{\pi^2}{4} - 2 \approx 0.4674$
9. **(a)** 2π
   (b) range: $\{y \mid -0.4 < y < 0.4\}$
   (c) **(i)** $-3\sin^3 x + 2\sin x$ **(iii)** $\frac{2\sqrt{3}}{9}$
   (d) $\frac{\pi}{2}$
   (e) **(i)** $\frac{1}{3}\sin^3 x + c$ **(ii)** $\frac{1}{3}$
   (f) $\arccos\frac{\sqrt{7}}{3} \approx 0.491$
10. **(a)** **(i)** See Worked Solutions **(ii)** See Worked Solutions
    (b) See Worked Solutions **(c)** 3.69672
    (d) $\int_0^{\pi}(\pi + x\cos x)dx$ **(e)** $\pi^2 - 2 \approx 7.86960$
11. **(a)** **(i)** $10x - 1 - e^{2x}$ **(ii)** $\frac{\ln 5}{2} \approx 0.805$
    (b) **(i)** $f^{-1}(x) = \frac{\ln(x-1)}{2}$ **(ii)** See Worked Solutions
    (c) $v = \pi\int_0^{\ln 2}(1 + e^{2x})^2 dx$
12. $\pi\left(\frac{2}{15}a^5 + \frac{2}{3}a^3\right)$

13. $4\left(\frac{2}{5}\left(\frac{1}{2}x+1\right)^{\frac{5}{2}}-\frac{2}{3}\left(\frac{1}{2}x+1\right)^{\frac{3}{2}}\right)+c$

14. $a=-\frac{56}{27}$

15. $\frac{\pi}{2}(e^{2k}-1)$

16. $k=2$

17. 1800 m

18. $2a$ by $\frac{2}{3}a^2$

19. **(a)** $\ln x+1-k$ **(b)** $x>\frac{1}{e}$

(c) **(i)** See Worked Solutions

(ii) $(e^k,0)$

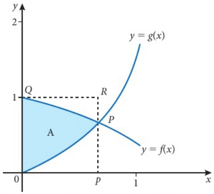
(d) $\frac{e^{2k}}{4}$ **(e)** $y=x-e^k$

(f) See Worked Solutions **(g)** Common ratio = e

20. $a=5$

21. $v=\sqrt{v_0^2+\frac{4k}{m}}$

22. **(a)**



(b) See Worked Solutions **(c)** 0.6937

(d) $\int_0^p (e^{-x^2}-(e^{-x^2}-1))dx \approx \sim 0.467$

23. **(a)** See Worked Solutions

(b) **(i)** $\frac{2\pi}{9}$ **(ii)** $\frac{4\pi}{9}$ **(iii)** $\frac{6\pi}{9}$

(c) $\frac{n\pi}{9}(n+1)$

24. **(a)** $t=0, 3,$ or 6

(b) **(i)** $\int_0^6\left|t\sin\left(\frac{\pi}{3}\right)tdt\right|$ **(ii)** 11.5 m

25. **(a)** 0.435 **(b)** $\frac{-2t}{(2+t^2)^2}$

26. **(a)** $\frac{dy}{dx}=\frac{2x^2}{\sqrt{1+x^2}}+2\sqrt{1+x^2}$

(b) See Worked Solutions **(c)** $k=0.918.$

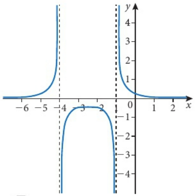
27. 6 m

28. 0.852

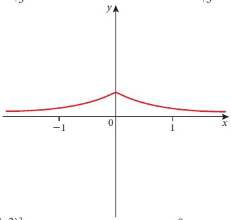
29. **(a)** See Worked Solutions

(b) **(i)** $A=78; k=\frac{1}{15}\ln\frac{48}{78}$ **(ii)** 45.3

30. **(a)**

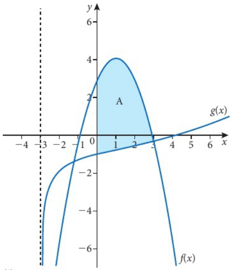


(b) $\ln\sqrt[3]{\frac{8}{5}}$ **(c)** area $=2\ln\sqrt[3]{\frac{8}{5}}$



31. $\frac{(x+2)^2}{2}-6(x+2)+12\ln|x+2|+\frac{8}{x+2}+c$

32. **(a)**



(b) **(i)** $x=23$

(ii) $x-\text{int}=e^2-3; y-\text{int}=\ln 3-2$

(c) $-1.34; 3.05$

(d) **(ii)** $\int_0^{3.05}(4-(1-x)^2-(\ln(x+3)-2))dx$

(iii) 10.6

(e) 4.63

33. **(a)** See Worked Solutions

(b) $\ln x-\frac{1}{2}\ln(x^2+1)+c$ **(c)** $y=\frac{2e^{\theta}}{\sqrt{e^{2u}\,1\,1}}$

34. **(a)** $(1, 0), \left(e^2, \frac{4}{e^2}\right)$ **(b)** $\frac{1}{3}$ **(c)** $\frac{\pi}{e}(24e - 65)$

35. **(a)** **(i)** See Worked Solutions

(ii) $\frac{1}{2(e^x + e^{-x}) - (e^x - e^{-x})} = \dots = \frac{e^x}{e^{3x} + 3}; \frac{\pi\sqrt{3}}{18}$

(b) **(i)** $x = \ln\left(\frac{k \pm \sqrt{k^2 - n^2 + 1}}{2(n + 1)}\right)$

(ii) 2 solutions $\Rightarrow k > \sqrt{k^2 - n^2 + 1}$ and $k^2 - n^2 + 1 > 0$

(c) **(i)** See Worked Solutions

(ii) Use quotient rule $f(x) > g(x)$ and result follows.

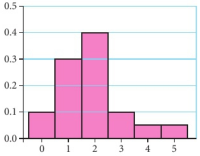
Chapter 15

Exercise 15.1

1. **(a)** discrete **(b)** continuous **(c)** continuous
(d) discrete **(e)** continuous **(f)** continuous
(g) discrete **(h)** continuous **(i)** continuous
(j) discrete **(k)** continuous **(l)** continuous
(m) discrete

2. **(a)** 0.4

(b)



(c) 1.85, 1.19 **(e)** 2.85, 1.19

(f) $E(Z) = E(Y + b) = E(Y) + b$ and $V(Y) = V(Y + b) = V(Y)$

3. **(a)** 0.26 **(b)** 0.37 **(c)** 0.77
(d) 16.29 **(e)** 8.1259
(f) 4.145; 2.031475
(g) $E(aX + b) = aE(X) + b$ and $V(aX + b) = a^2V(X)$

4. **(a)** 0.969 **(b)** 0.163 **(c)** 3.5
(d) $\sum(x - 3.5)^2 \cdot P(x) = 1.048 \Rightarrow \sigma = \sqrt{1.048} \approx 1.02$
(e) Empirical: 0.68, 0.95; approximately 0.68, approximately 0.90

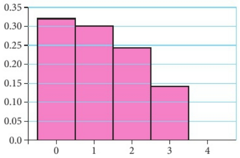
5. $k = \frac{1}{30}$

x	12	14	16	18
$P(X = x)$	$6k$	$7k$	$8k$	$9k$

6. **(a)** $k = \frac{1}{10}$ **(b)** $\frac{37}{60}$ **(c)** $\frac{19}{30}$
(d) $E(X) = 16$, SD $= 7$
(e) $E(Y) = \frac{11}{5}$; $V(Y) = \frac{49}{25}$

7. **(a)** $\frac{1}{50}$

(b)



(c) $\frac{17}{25}$ **(d)** $\mu = 1.2$; Var $= 1.08$

8. **(a)** $P(X = 18) = 0.2$, $P(X = 19) = 0.1$, Symmetric distribution.
(b) $\mu = 17$, SD $= 1.095$

9. **(a)** $\mu = 1.9$, SD $= 1.338$ **(b)** between 0 and 5.

10. $k = 0.667$, $E(X) = 5.444$

11. **(a)** $k = 0.3$ or 0.7
(b) for $k = 0.3$: $E(X) = 2.18$; for $k = 0.7$: $E(X) = 1.78$

12. **(a)**

x	0	1	2	3
$P(X = x)$	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$

(b) 2

13. **(a)** $k = \frac{1}{10}$ **(b)** $\frac{1}{2}$

14. **(a)** See table below. **(b)** 0.85 **(c)** 0.15
(d) 48.87 **(e)** 2.057 **(f)** 0.72

x	45	46	47	48	49	50	51	52	53	54	55
CDF	0.05	0.13	0.25	0.4	0.65	0.85	0.9	0.94	0.97	0.99	1

15. **(a)**

x	0	1	2	3	4	5	6
CDF	0.08	0.23	0.45	0.72	0.92	0.97	1

(b) 0.72 **(c)** 0.97 **(d)** 2.63 **(e)** 1.440

16. **(a)** 0.9 **(b)** 0.09 **(c)** 0.009
(d) **(i)** unacceptable **(ii)** acceptable
(e) $p(x) = (0.1^{x-1}) \times 0.9$

17. **(a)** 0 **(b)** 0.81 **(c)** 0.162
(d) **(i)** either **(ii)** acceptable
(e) $(x - 1)(0.1^{x-2}) \times 0.9^2, x > 1.$

18. $n = 132$

19. **(a)** **(i)** $\frac{1}{9}$ **(ii)** $\frac{1}{81}$
(b) **(i)** $\frac{73}{648}$ **(ii)** $\frac{575}{1296}$
(c) **(i)** See Worked Solutions
(ii)

X	1	2	3	4	5	6
CDF	$\frac{1}{1296}$	$\frac{15}{1296}$	$\frac{65}{1296}$	$\frac{175}{1296}$	$\frac{369}{1296}$	$\frac{671}{1296}$

(iii) $\frac{6797}{1296}$

20. 9.3

Exercise 15.2

1. **(a)**

x	0	1	2	3	4	5
$P(X = x)$	0.01024	0.0768	0.2304	0.3456	0.2592	0.07776