Name:

Mathematics Analysis & Approaches
Higher level
Paper 3

October 8, 2020 (morning)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the Mathematics Analysis & Approaches formula booklet is required for this paper.
- The maximum mark for this examination paper is [**55 marks**].

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working

1. [Maximum mark: 28]

This question asks you to investigate some properties of a sequence of functions $f_n(x) = \tan(n \arctan x)$, where $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$.

Important: When sketching graphs in this question, you are not required to find the coordinates of any axes intercepts or the coordinates of any stationary points unless requested.

(a) Write down $f_1(x)$ in the simplified form. [1]

(b) Use the double angle identity to show that $f_2(x) = \dfrac{2x}{1-x^2}$. [2]

(c) Find a similar expression for $f_3(x)$. [4]

(d) Show that $f_4(x) = \dfrac{4x-4x^3}{1-6x^2+x^4}$. [4]

(e) Find the equations of any horizontal or oblique asymptotes of $f_2(x)$ and $f_4(x)$. [2]

(e) Sketch the graphs of $f_2(x)$ and $f_4(x)$ on separate axes. [2]

(f) For even values of $n > 2$ use your graphic display calculator to vary the value of n. Hence suggest and expression for even values of n, in terms of n for the number of: [4]

(i) x-intercepts of the graph,

(ii) vertical asymptotes.

(g) Suggest an equation of any horizontal or oblique asymptote of the graph $y = f_n(x)$ when n is even. [1]

(h) Using the expression you found in (c) find the equation of any horizontal or oblique asymptote of $f_3(x)$. [2]

It can be shown that:

$$f_5(x) = \frac{5x - 10x^3 + x^5}{1 - 10x^2 + 5x^4} \text{ and } f_7(x) = \frac{7x - 35x^3 + 21x^5 - x^7}{1 - 21x^2 + 35x^4 - 7x^6}.$$

(i) Find the equations of any horizontal or oblique asymptotes of the graphs of $y = f_5(x)$ and $y = f_7(x)$. [4]

(j) Using your answers to part (g) and (i) suggest an expression involving n for equations of any horizontal or oblique asymptotes of the graph $y = f_n(x)$ when n is odd. [2]

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2. [Maximum mark: 27]

This question asks you to investigate the reduction method for calculating exact value of definite integrals of the form $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx$ where $m, n \in \mathbb{Z}$.

Let $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx$, so that for example $I_{5,3} = \int_0^{\frac{\pi}{2}} \sin^5 x \cos^3 x dx$.

(a) Calculate the exact value of: [4]

(i) $I_{0,0}$,

(ii) $I_{1,0}$,

(iii) $I_{0,1}$,

(iv) $I_{1,1}$.

(b) By writing $\sin^m x \cos^n$ as $\sin^{m-1} x \sin x \cos^n x$ and using integration by parts show that: [10]

$$I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$$

(c) **Hence** find the exact value of: [7]

(i) $\int_0^{\frac{\pi}{2}} \sin^7 x \cos x dx$

(ii) $\int_0^{\frac{\pi}{2}} \sin^2 x dx$

(ii) $\int_0^{\frac{\pi}{2}} \sin^3 x dx$

It can also be shown that:

$$I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$$

(d) Use the above reduction methods to find the exact values of the following integrals: [6]

(i) $\int_0^{\frac{\pi}{2}} \sin^5 x \cos^4 x dx$

(ii) $\int_0^{\frac{\pi}{2}} \sin^6 x \cos^6 x dx$