Mixed examination practice 15

Short questions

1. Express $z = 3\mathrm{i} - \frac{2}{\mathrm{i} + \sqrt{3}}$ in the form $x + \mathrm{i}y$. *[5 marks]*

2. If z and w are complex numbers, solve the simultaneous equations:

$$3z + w = 9 + 11\mathrm{i}$$
$$\mathrm{i}w - z = -8 - 2\mathrm{i}$$
[5 marks]

3. $f(z) = z^3 + az^2 + bz + c$ where a, b and c are real constants.
Two roots of $f(z) = 0$ are $z = 1$ and $z = 1 + 2\mathrm{i}$. Find a, b and c. *[6 marks]*

4. Find the complex number z such that $3z - 5z^* = 4 - 3\mathrm{i}$. *[4 marks]*

5. Find the exact value of $\frac{1}{(\sqrt{3} + \mathrm{i})^6}$. *[6 marks]*

6. The polynomial $z^3 + az^2 + bz - 65$ has a factor of $(z - 2 - 3\mathrm{i})$. Find the values of the real constants a and b. *[6 marks]*

7. If $w = 1 + \sqrt{3}\mathrm{i}$ and $z = 1 + \mathrm{i}$ show that $\mathrm{Re}\left(\frac{w + \sqrt{2}z}{w - \sqrt{2}z}\right) = 0$. *[6 marks]*

8. If $z = 4\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$ and $w = 2\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)$ evaluate $\left(\frac{z}{w}\right)^6$.
leaving your answer in a simplified form. *[6 marks]*

9. If $\arg\left((a + \mathrm{i})^3\right) = \pi$ and a is real and positive, find the exact value of a. *[6 marks]*

10. Let z and w be complex numbers satisfying $\frac{w + \mathrm{i}}{w - \mathrm{i}} = \frac{z + 1}{z - 1}$.
(a) Express w in terms of z.
(b) Show that if $\mathrm{Im}(z) = 0$ then $\mathrm{Re}(w) = 0$. *[6 marks]*

11. If $|z + 2\mathrm{i}| = |z - 6\mathrm{i}|$ find the imaginary part of z. *[6 marks]*

12. If $|z + 25| = 5|z + 1|$ find $|z|$. *[6 marks]*

13. (a) The equation $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$ has roots $\frac{1}{3}, \frac{2}{3}, 1, 1$ and 3. Find the value of a.
(b) Let $1, \omega_1, \omega_2, \omega_3, \omega_4$ be the roots of the equation $z^5 = 1$. Find the value of $\omega_1 + \omega_2 + \omega_3 + \omega_4$. *[5 marks]*

14. (a) Show that $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$.

(b) Let α and β be the roots of the quadratic equation $x^2+7x+2=0$. Find a quadratic equation with roots α^3 and β^3. *[7 marks]*

15. By considering the product $(2+i)(3+i)$ show that

$\arctan\frac{1}{2}+\arctan\frac{1}{3}=\frac{\pi}{4}$. *[6 marks]*

16. If $0<\theta<\frac{\pi}{2}$ and $z=(\sin\theta+i(1-\cos\theta))^2$ find in its simplest form $\arg z$. *[6 marks]*

17. Let z and w be complex numbers such that $w=\frac{1}{1-z}$ and $|z|^2=1$. Find the real part of w. *[6 marks]*

18. If $z=\text{cis}\,\theta$ prove that $\frac{z^2-1}{z^2+1}=i\tan\theta$. *[6 marks]*

19. $w=\frac{kz}{z^2+1}$ where $z^2\neq-1$. If $\text{Im}(w)=\text{Im}(k)=0$ and $\text{Im}(z)\neq0$

prove that $|z|=1$. *[6 marks]*

Long questions

1. Let $z_1=\frac{\sqrt{6}-i\sqrt{2}}{2}$, and $z_2=1-i$.

(a) Write z_1 and z_2 in the form $r(\cos\theta+i\sin\theta)$, where $r>0$ and $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$.

(b) Show that $\frac{z_1}{z_2}=\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}$.

(c) Find the value of $\frac{z_1}{z_2}$ in the form $a+bi$, where a and b are to be determined exactly in radical (surd) form. Hence or otherwise find the exact values of $\cos\frac{\pi}{12}$ and $\sin\frac{\pi}{12}$. *[8 marks]*

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2. (a) Express $\frac{\sqrt{3}}{2}-\frac{1}{2}i$ in the form $r(\cos\theta+i\sin\theta)$.

(b) Hence show that $\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)^9=ci$ where c is a real number to be found.

(c) Find one pair of possible values of positive integers m and n such that:

$\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)^m=\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)^n$ *[8 marks]*

3. Let $z = \cos\theta + \mathrm{i}\sin\theta$, for $-\frac{\pi}{4} < \theta \le \frac{\pi}{4}$.

(a) (i) Find z^3 using the binomial theorem.

(ii) Use De Moivre's theorem to show that:

$\cos 3\theta = 4\cos 3\theta - 3\cos\theta$ and $\sin 3\theta = 3\sin\theta - 4\sin 3\theta$.

(b) Hence prove that $\dfrac{\sin 3\theta - \sin\theta}{\cos 3\theta + \cos\theta} = \tan\theta$.

(c) Given that $\sin\theta = \frac{1}{3}$, find the exact value of $\tan 3\theta$.

[10 marks]

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4. If ω is a complex third root of unity and x and y are real numbers prove that:

(a) $1 + \omega + \omega^2 = 0$.

(b) $(\omega x + \omega^2 y)(\omega^2 x + \omega y) = x^2 - xy + y^2$. *[7 marks]*

5. (a) A cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots x_1, x_2, x_3.

(i) Write down the values of $x_1 + x_2 + x_3$ and $x_1 x_2 x_3$ in terms of a, b, c and d.

(ii) Show that $x_1x_2 + x_2x_3 + x_3x_1 = \frac{c}{a}$.

(b) The roots α, β and γ of the equation $2x^3 + bx^2 + cx + 16 = 0$ form a geometric progression.

(i) Show that $\beta = -2$.

(ii) Show that $c = 2b$. *[14 marks]*

6. Let $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) Show that $2\cos\theta = z + \frac{1}{z}$.

(b) Show that $2\cos n\theta = z^n + \frac{1}{z^n}$.

(c) Consider the equation $3z^4 - z^3 + 2z^2 - z + 3 = 0$.

(i) Show that the equation can be written as $6\cos 2\theta - 2\cos\theta + 2 = 0$.

(ii) Find all four complex roots of the original equation. *[7 marks]*

7. Let $\omega = \mathrm{e}^{\frac{2\mathrm{i}\pi}{5}}$.

(a) Write ω^2, ω^3 and ω^4 in the form $\mathrm{e}^{\mathrm{i}\theta}$.

(b) Explain why $\omega^1 + \omega^2 + \omega^3 + \omega^4 = -1$.

(c) Show that $\omega + \omega^4 = 2\cos\left(\frac{2\pi}{5}\right)$ and $\omega^2 + \omega^3 = 2\cos\left(\frac{4\pi}{5}\right)$.

(d) Form a quadratic equation in $\cos\left(\frac{2\pi}{5}\right)$ and hence show that

$\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}$. *[10 marks]*

8. (a) By considering $(\cos\theta + \mathrm{i}\sin\theta)^3$ find the expressions for $\cos(3\theta)$ and $\sin(3\theta)$.

(b) Show that $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$.

(c) Hence show that $\tan\left(\frac{\pi}{12}\right)$ is a root of the equation $x^3 - 3x^2 - 3x + 1 = 0$.

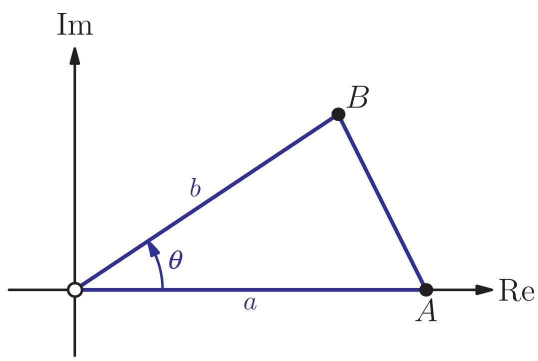
(d) Show that $(x-1)$ is a factor of $x^3 - 3x^2 - 3x + 1$ and hence find the exact solutions of the equation $x^3 - 3x^2 - 3x + 1 = 0$.

(e) By considering $\tan\left(\frac{\pi}{4}\right)$ explain why $\tan\left(\frac{\pi}{12}\right) < 1$.

(f) Hence state the exact value of $\tan\left(\frac{\pi}{12}\right)$. *[14 marks]*

9. (a) Points P and Q in the Argand diagram correspond to complex numbers $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$. Show that $PQ = |z_1 - z_2|$.

(b) The diagram shows a triangle with one vertex at the origin, one at the point A $(a, 0)$ and one at the point B such that $OB = b$ and $A\hat{O}B = \theta$.



(i) Write down the complex number corresponding to point A.

(ii) Write down the number corresponding to point B in polar form.

(iii) Write down an expression for the length of AB in terms of a, b and θ.

(iv) Hence prove the cosine rule for the triangle AOB:

$|AB|^2 = |OA|^2 + |OB|^2 + 2|OA||OB|\cos\theta$ *[13 marks]*