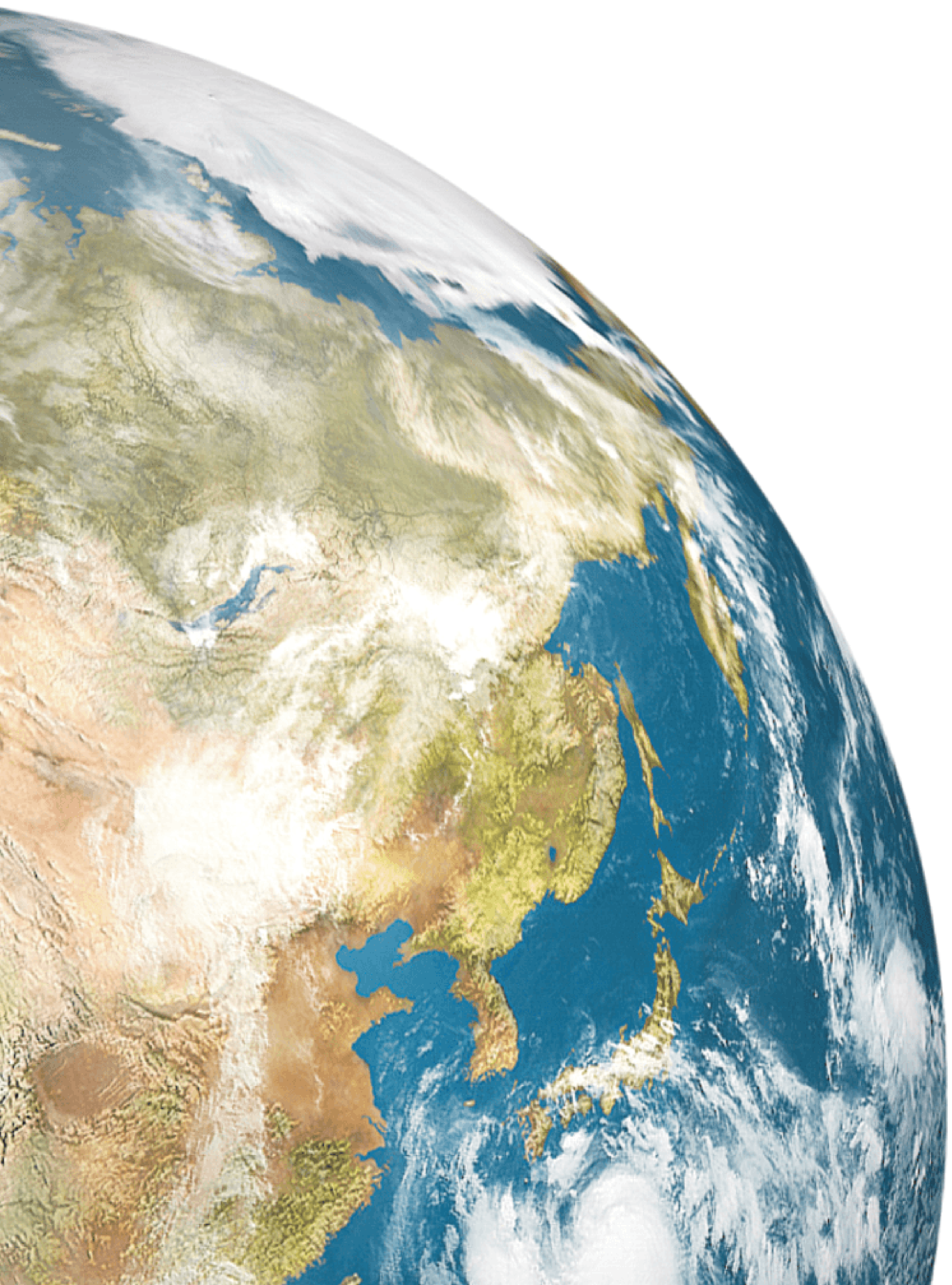
Chapter 3

Logarithms



Contents:

OPENING PROBLEM

In a plentiful springtime, a population of 1000 mice will double every week.

The population after t weeks is given by the exponential function $P(t) = 1000 \times 2^t$ mice.

Things to think about:

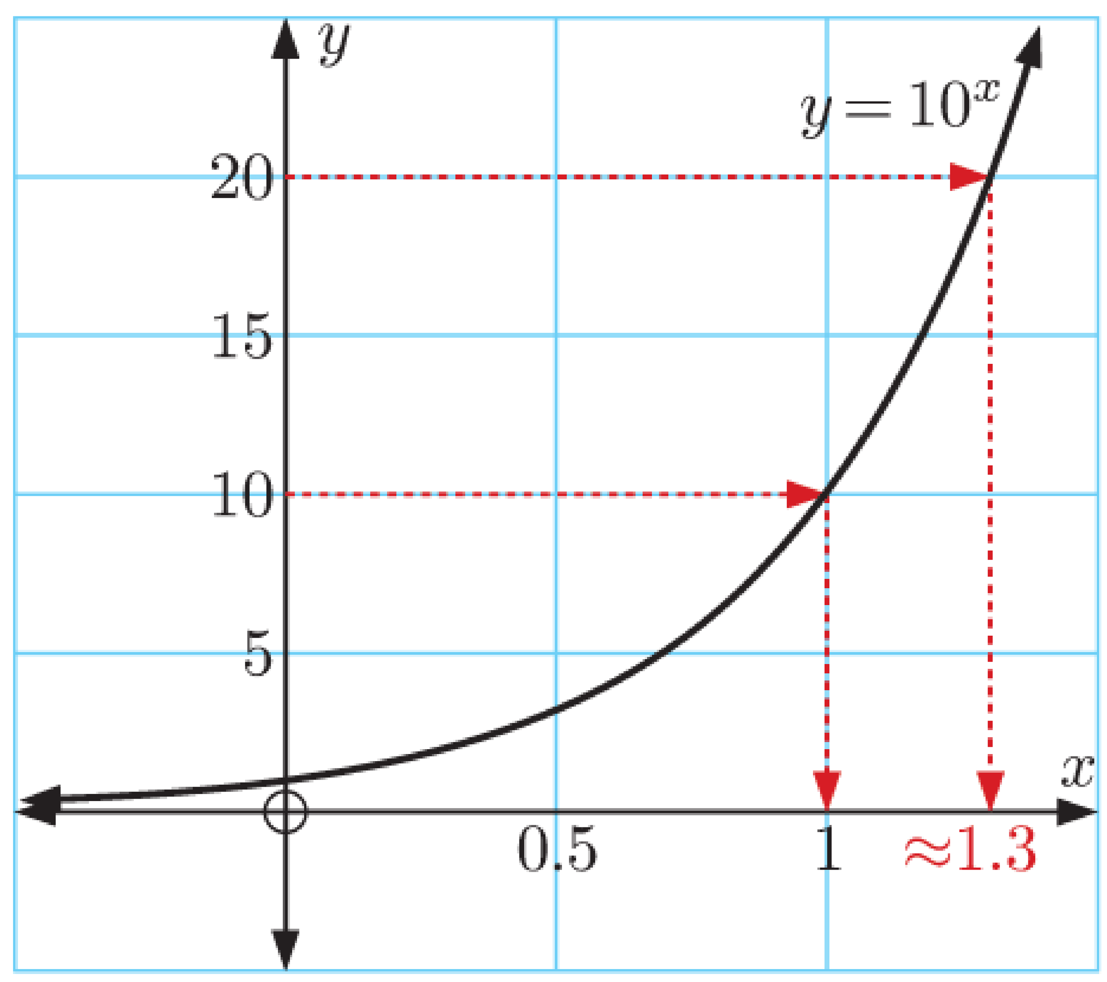
- **a** What does the graph of the population over time look like?
- **b** How long will it take for the population to reach 20 000 mice?
- **c** Can we write a function for t in terms of P, which determines the time at which the population P is reached?

In the previous Chapter we solved exponential equations by writing both sides with the same base, and by using graphs.

In this Chapter we study a more formal solution to exponential equations in which we use the **inverse** of the exponential function. We call this a **logarithm**.

A LOGARITHMS IN BASE 10

Consider the graph of $y = 10^x$ shown.



Notice that the range of the function is $\{y \mid y > 0\}$. This means that every positive number y can be written in the form 10^x.

For example:

- When $y = 10$, $x = 1$, so $10 = 10^1$.
- When $y = 20$, $x \approx 1.3$, so $20 \approx 10^{1.3}$.

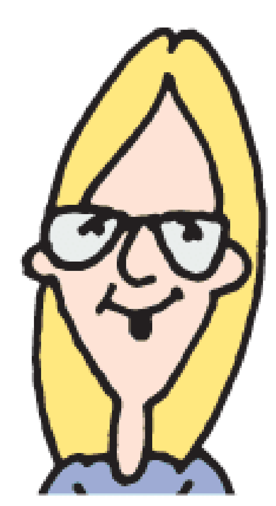
When we write a positive number y in the form 10^x, we say that x is the **logarithm in base 10**, of y.

> The **logarithm in base 10** of a positive number is the power that 10 must be raised to in order to obtain that number.

For example:

- The logarithm in base 10 of 1000 is 3, since $1000 = 10^3$.
  We write $\log_{10} 1000 = 3$ or simply $\log 1000 = 3$.
- $\log(0.01) = -2$ since $0.01 = 10^{-2}$.

> If no base is indicated we assume it means base 10.
> $\log b$ means $\log_{10} b$.



By observing that $\log 1000 = \log(10^3) = 3$ and $\log(0.01) = \log(10^{-2}) = -2$, we conclude that $\boldsymbol{\log 10^x = x}$ for any $x \in \mathbb{R}$.

Example 1 **Self Tutor**

Find: **a** $\log 100$ **b** $\log \sqrt[4]{10}$

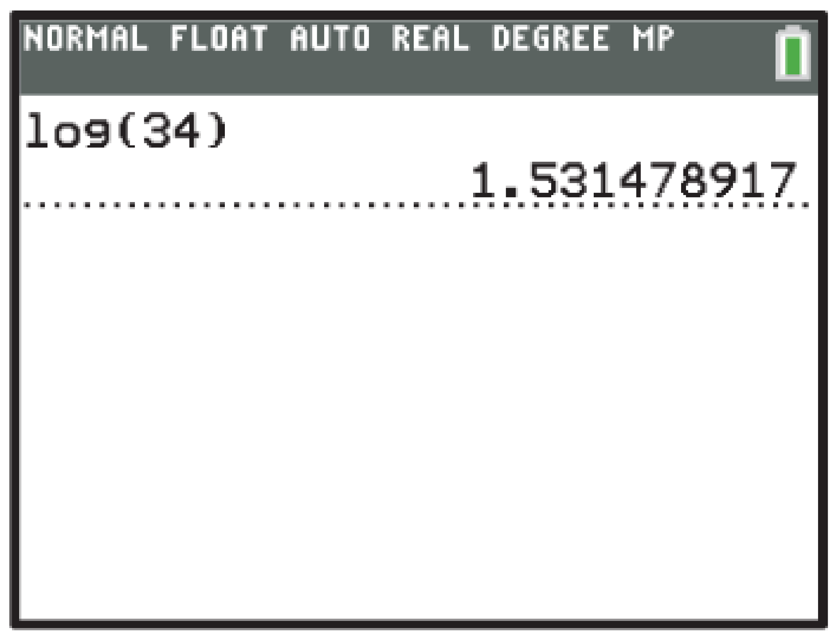
a $\log 100 = \log(10^2) = 2$

b $\log \sqrt[4]{10} = \log(10^{\frac{1}{4}}) = \frac{1}{4}$

The logarithms in **Example 1** can be found by hand because it is easy to write 100 and $\sqrt[4]{10}$ as powers of 10. The logarithms of most values, however, can only be found using a calculator.

For example, $\log 34 \approx 1.53$
so $34 \approx 10^{1.53}$

GRAPHICS CALCULATOR INSTRUCTIONS



Logarithms allow us to write any number as a power of 10. In particular:

$$x = 10^{\log x} \quad \text{for any } x > 0.$$

Example 2 **Self Tutor**

Use your calculator to write the following in the form 10^x where x is correct to 4 decimal places:

a 8 **b** 800 **c** 0.08

a $8 = 10^{\log 8}$
$\approx 10^{0.9031}$

b $800 = 10^{\log 800}$
$\approx 10^{2.9031}$

c $0.08 = 10^{\log 0.08}$
$\approx 10^{-1.0969}$

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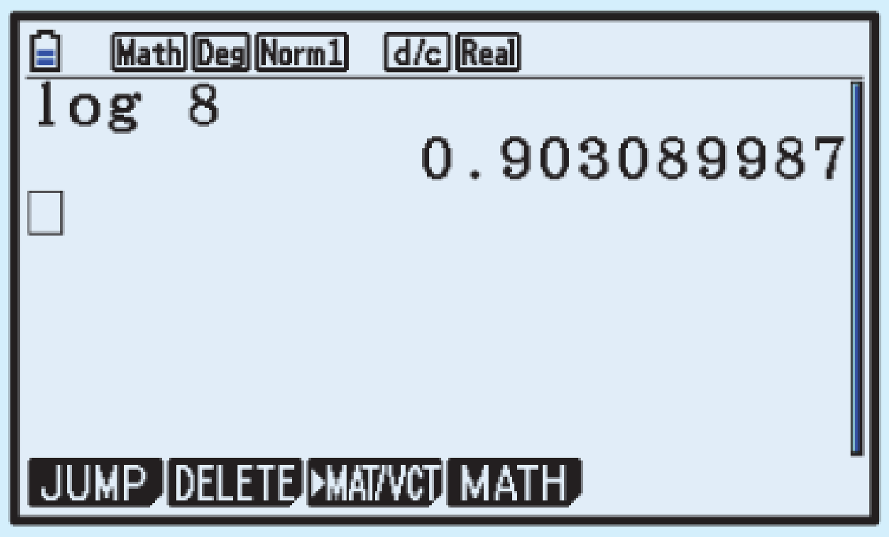

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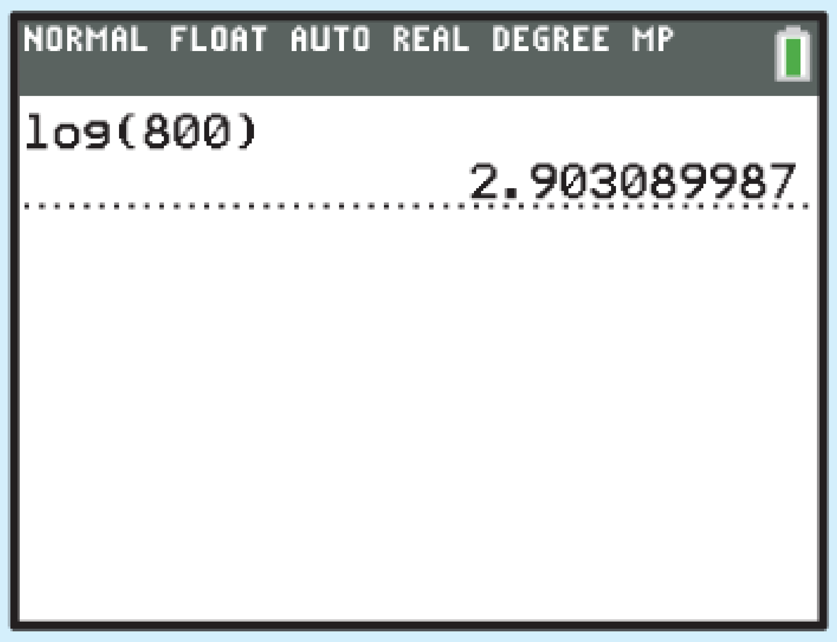
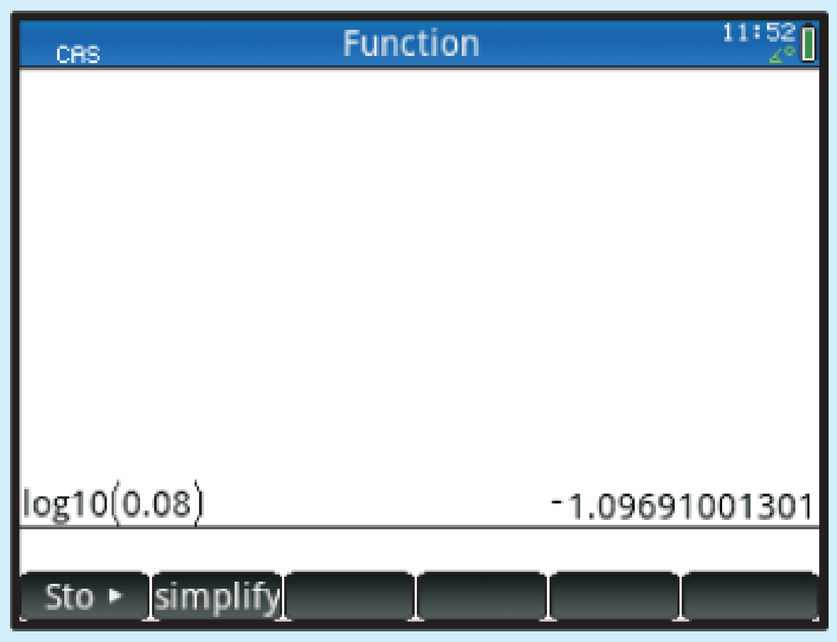


HP Prime



EXERCISE 3A

1 Without using a calculator, find:

a $\log 10\,000$ **b** $\log 0.001$ **c** $\log 10$ **d** $\log 1$

e $\log \sqrt{10}$ **f** $\log \sqrt[3]{10}$ **g** $\log\left(\frac{1}{\sqrt[4]{10}}\right)$ **h** $\log\left(10\sqrt{10}\right)$

i $\log \sqrt[3]{100}$ **j** $\log\left(\frac{100}{\sqrt{10}}\right)$ **k** $\log\left(10 \times \sqrt[3]{10}\right)$ **l** $\log\left(1000\sqrt{10}\right)$

Check your answers using your calculator.

2 Simplify:

a $\log(10^n)$ **b** $\log(10^a \times 100)$ **c** $\log\left(\frac{10}{10^m}\right)$ **d** $\log\left(\frac{10^a}{10^b}\right)$

3 **a** Explain why $\log 237$ must lie between 2 and 3.

b Use your calculator to evaluate $\log 237$ correct to 2 decimal places.

4 **a** Between which two consecutive whole numbers does $\log(0.6)$ lie?
b Check your answer by evaluating $\log(0.6)$ correct to 2 decimal places.

5 Use your calculator to evaluate, correct to 2 decimal places:
a $\log 76$ **b** $\log 114$ **c** $\log 3$ **d** $\log 831$
e $\log(0.4)$ **f** $\log 3247$ **g** $\log(0.008)$ **h** $\log(-7)$

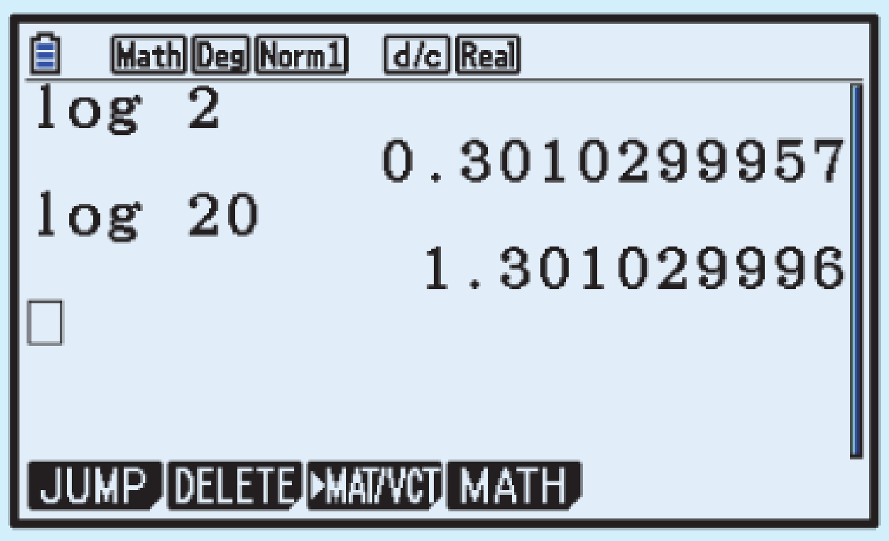
6 For what values of x is $\log x$:
a positive **b** zero **c** negative **d** undefined?

7 Use your calculator to write the following in the form 10^x where x is correct to 4 decimal places:
a 6 **b** 60 **c** 6000 **d** 0.6 **e** 0.006
f 15 **g** 1500 **h** 1.5 **i** 0.15 **j** 0.000 15

Example 3 **Self Tutor**

a Use your calculator to find: **i** $\log 2$ **ii** $\log 20$
b Explain why $\log 20 = \log 2 + 1$.

a



i $\log 2 \approx 0.3010$
ii $\log 20 \approx 1.3010$

b
$$\begin{aligned}\log 20 &= \log(2 \times 10)\\ &= \log(10^{\log 2} \times 10^1) \quad \{x = 10^{\log x}\}\\ &= \log(10^{\log 2 + 1}) \quad \{\text{adding indices}\}\\ &= \log 2 + 1\end{aligned}$$

8 **a** Use your calculator to find: **i** $\log 3$ **ii** $\log 300$
b Explain why $\log 300 = \log 3 + 2$.

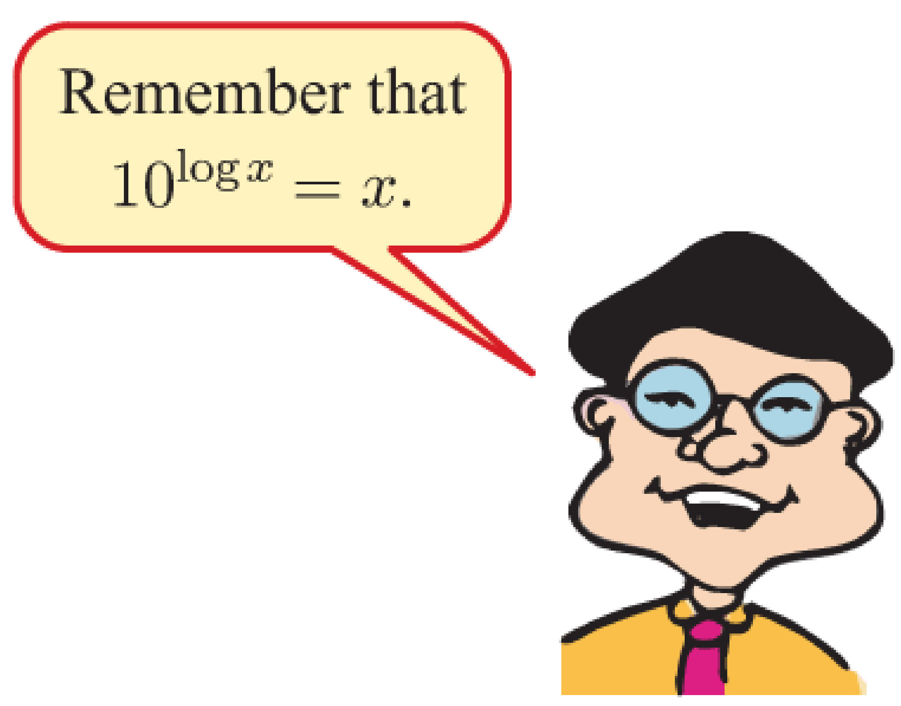
9 **a** Use your calculator to find: **i** $\log 5$ **ii** $\log(0.05)$
b Explain why $\log(0.05) = \log 5 - 2$.

Example 4 **Self Tutor**

Find x such that:
a $\log x = 3$ **b** $\log x \approx -0.271$

a
$$\begin{aligned}\log x &= 3\\ \therefore\ 10^{\log x} &= 10^3\\ \therefore\ x &= 1000\end{aligned}$$

b
$$\begin{aligned}\log x &\approx -0.271\\ \therefore\ 10^{\log x} &\approx 10^{-0.271}\\ \therefore\ x &\approx 0.536\end{aligned}$$



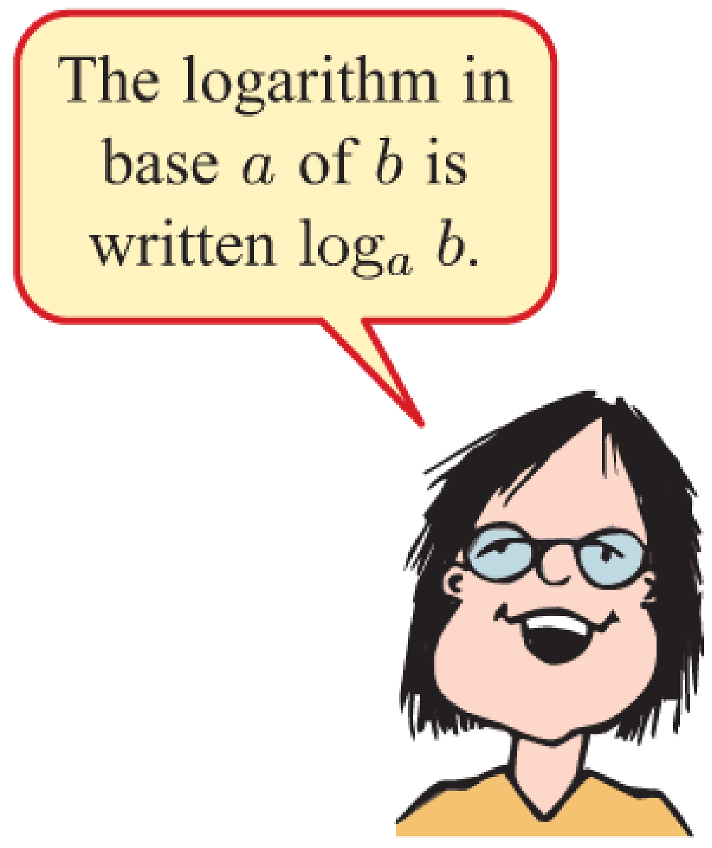
10 Find x such that:
a $\log x = 2$ **b** $\log x = 1$ **c** $\log x = 0$ **d** $\log x = -1$
e $\log x = \frac{1}{2}$ **f** $\log x = -\frac{1}{2}$ **g** $\log x = 4$ **h** $\log x = -5$
i $\log x \approx 0.8351$ **j** $\log x \approx 2.1457$ **k** $\log x \approx -1.378$ **l** $\log x \approx -3.1997$

B LOGARITHMS IN BASE a

In the previous Section we defined the logarithm in base 10 of a number as the power that 10 must be raised to in order to obtain that number.

We can use the same principle to define logarithms in other bases:

> The **logarithm in base a of b** is the power that a must be raised to in order to obtain b.



For example, to find $\log_2 8$, we ask "What power must 2 be raised to in order to obtain 8?". We know that $2^3 = 8$, so $\log_2 8 = 3$.

$a^x = b$ and $x = \log_a b$ are *equivalent* statements.

> For any $b > 0$, $a^x = b \Leftrightarrow x = \log_a b$

Example 5 — Self Tutor

a Write an equivalent exponential statement for $\log_{10} 1000 = 3$.

b Write an equivalent logarithmic statement for $3^4 = 81$.

a From $\log_{10} 1000 = 3$ we deduce that $10^3 = 1000$.

b From $3^4 = 81$ we deduce that $\log_3 81 = 4$.

EXERCISE 3B

1 Write an equivalent exponential statement for:

a $\log_{10} 100 = 2$ **b** $\log_{10} 10\,000 = 4$ **c** $\log_{10}(0.1) = -1$

d $\log_{10} \sqrt{10} = \frac{1}{2}$ **e** $\log_2 8 = 3$ **f** $\log_3 9 = 2$

g $\log_2 \left(\frac{1}{4}\right) = -2$ **h** $\log_3 \sqrt{27} = 1.5$ **i** $\log_5 \left(\frac{1}{\sqrt{5}}\right) = -\frac{1}{2}$

2 Write an equivalent logarithmic statement for:

a $4^3 = 64$ **b** $5^2 = 25$ **c** $7^2 = 49$ **d** $2^6 = 64$

e $2^{-3} = \frac{1}{8}$ **f** $10^{-2} = 0.01$ **g** $2^{-1} = \frac{1}{2}$ **h** $3^{-3} = \frac{1}{27}$

Example 6 — Self Tutor

Find: **a** $\log_2 16$ **b** $\log_5(0.2)$ **c** $\log_{10} \sqrt[5]{100}$ **d** $\log_2 \left(\frac{1}{\sqrt{2}}\right)$

a $\log_2 16$
$= \log_2(2^4)$
$= 4$

b $\log_5(0.2)$
$= \log_5\left(\frac{1}{5}\right)$
$= \log_5(5^{-1})$
$= -1$

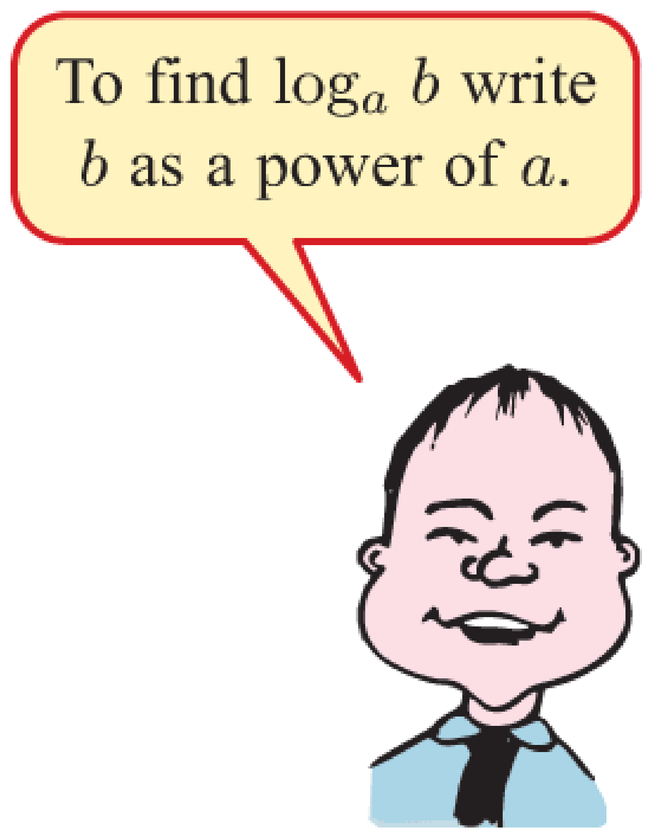
c $\log_{10} \sqrt[5]{100}$
$= \log_{10}((10^2)^{\frac{1}{5}})$
$= \log_{10}(10^{\frac{2}{5}})$
$= \frac{2}{5}$

d $\log_2 \left(\frac{1}{\sqrt{2}}\right)$
$= \log_2(2^{-\frac{1}{2}})$
$= -\frac{1}{2}$

3 Without using a calculator, find:

a $\log_{10} 100\,000$ **b** $\log_{10}(0.01)$ **c** $\log_3 \sqrt{3}$
d $\log_2 4$ **e** $\log_2 64$ **f** $\log_2 128$
g $\log_5 25$ **h** $\log_5 125$ **i** $\log_2(0.125)$
j $\log_9 3$ **k** $\log_4 16$ **l** $\log_{36} 6$
m $\log_3 243$ **n** $\log_2 \sqrt[3]{2}$ **o** $\log_8 2$
p $\log_6\left(6\sqrt{6}\right)$ **q** $\log_4 1$ **r** $\log_9 9$
s $\log_3\left(\frac{1}{3}\right)$ **t** $\log_{10} \sqrt[4]{1000}$ **u** $\log_7\left(\frac{1}{\sqrt{7}}\right)$
v $\log_5\left(25\sqrt{5}\right)$ **w** $\log_3\left(\frac{1}{\sqrt{27}}\right)$ **x** $\log_4\left(\frac{1}{2\sqrt{2}}\right)$

Check your answers using technology.



GRAPHICS CALCULATOR INSTRUCTIONS

4 Simplify:

a $\log_x(x^2)$ **b** $\log_t\left(\frac{1}{t}\right)$ **c** $\log_x \sqrt{x}$
d $\log_m(m^3)$ **e** $\log_k \sqrt[4]{k}$ **f** $\log_x(x\sqrt{x})$
g $\log_a\left(\frac{1}{a^2}\right)$ **h** $\log_a\left(\frac{1}{\sqrt{a}}\right)$ **i** $\log_m \sqrt{m^5}$

Example 7 **Self Tutor**

Solve for x: $\log_3 x = 5$

$$\log_3 x = 5$$
$$\therefore\ x = 3^5$$
$$\therefore\ x = 243$$

5 Solve for x:

a $\log_2 x = 3$ **b** $\log_4 x = \frac{1}{2}$ **c** $\log_5 x = -3$
d $\log_x 81 = 4$ **e** $\log_2(x-6) = 3$ **f** $\log_2(\log_3 x) = -1$

6 Suppose $\log_a b = x$. Find, in terms of x, the value of $\log_b a$.

7 If $y = \log_2 \sqrt{5x-1}$, write x in terms of y.

HISTORICAL NOTE

Acharya Virasena was an 8th century Indian mathematician. Among other areas, he worked with the concept of *ardhaccheda*, which is how many times a number of the form 2^n can be divided by 2. The result is the integer n, and is the logarithm of the number 2^n in base 2.

In 1544, the German **Michael Stifel** published *Arithmetica Integra* which contains a table expressing many other integers as powers of 2. In effect, he had created an early version of a logarithmic table.

C LAWS OF LOGARITHMS

INVESTIGATION 1 DISCOVERING THE LAWS OF LOGARITHMS

What to do:

1 **a** Use your calculator to find:

i $\log 2 + \log 3$ **ii** $\log 3 + \log 7$ **iii** $\log 4 + \log 20$
iv $\log 6$ **v** $\log 21$ **vi** $\log 80$

b From your answers, suggest a possible simplification for $\log m + \log n$.

2 **a** Use your calculator to find:

i $\log 6 - \log 2$ **ii** $\log 12 - \log 3$ **iii** $\log 3 - \log 5$
iv $\log 3$ **v** $\log 4$ **vi** $\log(0.6)$

b From your answers, suggest a possible simplification for $\log m - \log n$.

3 **a** Use your calculator to find:

i $3 \log 2$ **ii** $2 \log 5$ **iii** $-4 \log 3$
iv $\log(2^3)$ **v** $\log(5^2)$ **vi** $\log(3^{-4})$

b From your answers, suggest a possible simplification for $m \log b$.

From the **Investigation**, you should have discovered the three important **laws of logarithms**:

- $\log m + \log n = \log(mn)$ for $m, n > 0$
- $\log m - \log n = \log\left(\frac{m}{n}\right)$ for $m, n > 0$
- $m \log b = \log(b^m)$ for $b > 0$

More generally, in any base a where $a \neq 1$, $a > 0$, we have these **laws of logarithms**:

- $\log_a m + \log_a n = \log_a(mn)$ for $m, n > 0$
- $\log_a m - \log_a n = \log_a\left(\frac{m}{n}\right)$ for $m, n > 0$
- $m \log_a b = \log_a(b^m)$ for $b > 0$

Proof:

- $$\begin{aligned} &\log_a(mn) \\ &= \log_a\left(a^{\log_a m} \times a^{\log_a n}\right) \\ &= \log_a\left(a^{\log_a m + \log_a n}\right) \\ &= \log_a m + \log_a n \end{aligned}$$

- $$\begin{aligned} &\log_a\left(\frac{m}{n}\right) \\ &= \log_a\left(\frac{a^{\log_a m}}{a^{\log_a n}}\right) \\ &= \log_a\left(a^{\log_a m - \log_a n}\right) \\ &= \log_a m - \log_a n \end{aligned}$$

- $$\begin{aligned} &\log_a(b^m) \\ &= \log_a\left(\left(a^{\log_a b}\right)^m\right) \\ &= \log_a\left(a^{m \log_a b}\right) \\ &= m \log_a b \end{aligned}$$

Example 8 — Self Tutor

Use the laws of logarithms to write as a single logarithm or as an integer:

a $\log 5 + \log 3$ **b** $\log_3 24 - \log_3 8$ **c** $\log_2 5 - 1$

a $\log 5 + \log 3$
$= \log(5 \times 3)$
$= \log 15$

b $\log_3 24 - \log_3 8$
$= \log_3\left(\frac{24}{8}\right)$
$= \log_3 3$
$= 1$

c $\log_2 5 - 1$
$= \log_2 5 - \log_2(2^1)$
$= \log_2\left(\frac{5}{2}\right)$

EXERCISE 3C

1 Write as a single logarithm or as an integer:

a $\log 8 + \log 2$ **b** $\log 4 + \log 5$ **c** $\log 40 - \log 5$
d $\log p - \log m$ **e** $\log_4 8 - \log_4 2$ **f** $\log 5 + \log(0.4)$
g $\log 250 + \log 4$ **h** $\log_5 100 - \log_5 4$ **i** $\log 2 + \log 3 + \log 4$
j $\log 5 + \log 4 - \log 2$ **k** $\log_3 6 - \log_3 2 - \log_3 3$ **l** $\log\left(\frac{4}{3}\right) + \log 3 + \log 7$

2 Write as a single logarithm:

a $\log 7 + 2$ **b** $\log 4 - 1$ **c** $1 + \log_2 3$
d $\log_3 5 - 2$ **e** $2 + \log 2$ **f** $\log 50 - 4$
g $t + \log w$ **h** $\log_m 40 - 2$ **i** $3 - \log_5 50$

Example 9 — Self Tutor

Simplify by writing as a single logarithm or as a rational number:

a $2\log 7 - 3\log 2$ **b** $2\log 3 + 3$ **c** $\dfrac{\log 8}{\log 4}$

a $2\log 7 - 3\log 2$
$= \log(7^2) - \log(2^3)$
$= \log 49 - \log 8$
$= \log\left(\frac{49}{8}\right)$

b $2\log 3 + 3$
$= \log(3^2) + \log(10^3)$
$= \log 9 + \log 1000$
$= \log 9000$

c $\dfrac{\log 8}{\log 4} = \dfrac{\log(2^3)}{\log(2^2)}$
$= \dfrac{3\log 2}{2\log 2}$
$= \frac{3}{2}$

3 Write as a single logarithm or integer:

a $5\log 2 + \log 3$ **b** $2\log 3 + 3\log 2$ **c** $3\log 4 - \log 8$
d $2\log_3 5 - 3\log_3 2$ **e** $\frac{1}{2}\log_6 4 + \log_6 3$ **f** $\frac{1}{3}\log\left(\frac{1}{8}\right)$
g $3 - \log 2 - 2\log 5$ **h** $1 - 3\log 2 + \log 20$ **i** $2 - \frac{1}{2}\log_n 4 - \log_n 5$

4 Simplify without using a calculator:

a $\dfrac{\log 4}{\log 2}$ **b** $\dfrac{\log_5 27}{\log_5 9}$ **c** $\dfrac{\log 8}{\log 2}$
d $\dfrac{\log 3}{\log 9}$ **e** $\dfrac{\log_3 25}{\log_3(0.2)}$ **f** $\dfrac{\log_4 8}{\log_4(0.25)}$

Example 10 **Self Tutor**

Show that:

a $\log\left(\frac{1}{9}\right) = -2\log 3$ **b** $\log 500 = 3 - \log 2$

a $\log\left(\frac{1}{9}\right)$
$= \log(3^{-2})$
$= -2\log 3$

b $\log 500$
$= \log\left(\frac{1000}{2}\right)$
$= \log 1000 - \log 2$
$= \log(10^3) - \log 2$
$= 3 - \log 2$

5 Show that:

a $\log 9 = 2\log 3$ **b** $\log\sqrt{2} = \frac{1}{2}\log 2$ **c** $\log\left(\frac{1}{8}\right) = -3\log 2$

d $\log\left(\frac{1}{5}\right) = -\log 5$ **e** $\log 5 = 1 - \log 2$ **f** $\log 5000 = 4 - \log 2$

6 The number $a \times 10^k$ where $1 \leqslant a < 10$, $k \in \mathbb{Z}$ is written in standard form. Show that $\log(a \times 10^k) = \log a + k$.

7 Suppose $p = \log_b 2$, $q = \log_b 3$, and $r = \log_b 5$. Write in terms of p, q, and r:

a $\log_b 6$ **b** $\log_b 45$ **c** $\log_b 108$

d $\log_b\left(\frac{5\sqrt{3}}{2}\right)$ **e** $\log_b\left(\frac{5}{32}\right)$ **f** $\log_b\left(\frac{2}{9}\right)$

8 Suppose $\log_2 P = x$, $\log_2 Q = y$, and $\log_2 R = z$. Write in terms of x, y, and z:

a $\log_2(PR)$ **b** $\log_2(RQ^2)$ **c** $\log_2\left(\frac{PR}{Q}\right)$

d $\log_2\left(P^2\sqrt{Q}\right)$ **e** $\log_2\left(\frac{Q^3}{\sqrt{R}}\right)$ **f** $\log_2\left(\frac{R^2\sqrt{Q}}{P^3}\right)$

9 If $\log_t M = 1.29$ and $\log_t N^2 = 1.72$, find:

a $\log_t N$ **b** $\log_t(MN)$ **c** $\log_t\left(\frac{N^2}{\sqrt{M}}\right)$

10 Suppose $\log_a(x+2) = \log_a x + 2$ and $a > 1$. Find x in terms of a.

11 In **factorial notation** we use $n!$ to denote the product $1 \times 2 \times 3 \times .... \times n$.

a Write $\log(8!) - \log(7!) + \log(6!) - \log(5!) + \log(4!) - \log(3!) + \log(2!) - \log(1!)$ as a single logarithm.

b Write $\log_2(6!)$ in the form $a + \log_2 b$, where $a, b \in \mathbb{Z}$ and b is as small as possible.

12 Write $\log x^4 + \log\left(\frac{x^4}{y}\right) + \log\left(\frac{x^4}{y^2}\right) + ... + \log\left(\frac{x^4}{y^9}\right)$ in the form $\log\left(\frac{x^m}{y^n}\right)$.

13 Evaluate the infinite series $\log\sqrt{3} - \log\sqrt[4]{3} + \log\sqrt[8]{3} - \log\sqrt[16]{3} + ...$

14 Suppose $x^2 + y^2 = 52xy$ where $0 < y < x$. Show that $\log\left(\frac{x-y}{5}\right) = \frac{1}{2}(\log x + \log 2y)$.

D NATURAL LOGARITHMS

The logarithm in base e is called the **natural logarithm**.

We use $\ln x$ to represent $\log_e x$, and call $\ln x$ the natural logarithm of x.

$$\ln e^x = x \quad \text{and} \quad e^{\ln x} = x.$$

Example 11 — Self Tutor

Find: **a** $\ln e^3$ **b** $\ln \sqrt{e}$ **c** $e^{2\ln 5}$

a $\ln e^3 = 3$

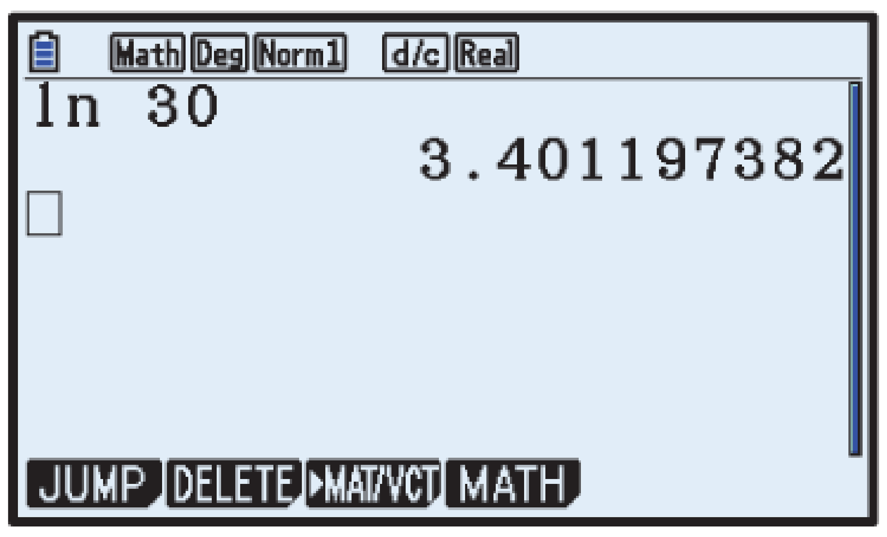
b $\ln \sqrt{e} = \ln(e^{\frac{1}{2}})$
$= \frac{1}{2}$

c $e^{2\ln 5} = (e^{\ln 5})^2$
$= 5^2$
$= 25$

As with base 10 logarithms, we can use our calculator to find natural logarithms.

For example, $\ln 30 \approx 3.40$, which means that $30 \approx e^{3.40}$.

GRAPHICS CALCULATOR INSTRUCTIONS



Example 12 — Self Tutor

Use your calculator to write the following in the form e^k where k is correct to 4 decimal places:

a 50 **b** 0.005

a $50 = e^{\ln 50}$ $\{x = e^{\ln x}\}$
$\approx e^{3.9120}$

b $0.005 = e^{\ln 0.005}$
$\approx e^{-5.2983}$

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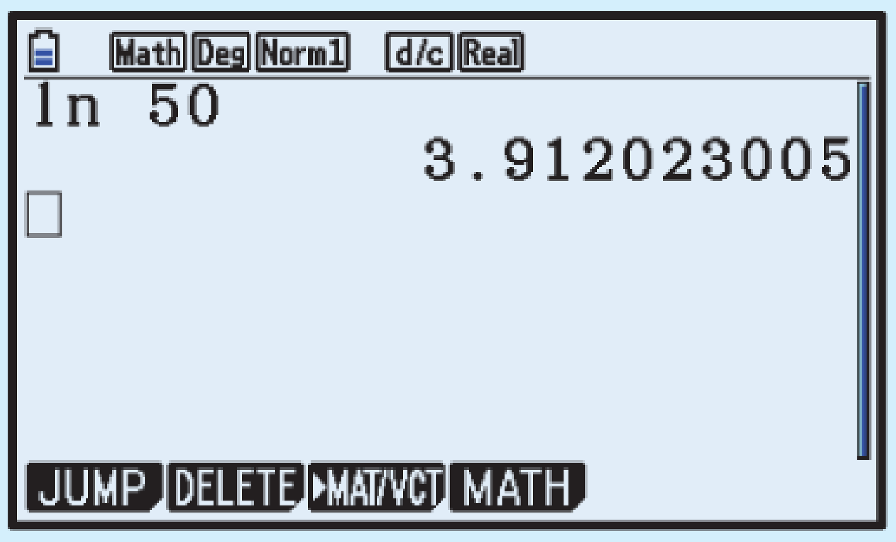

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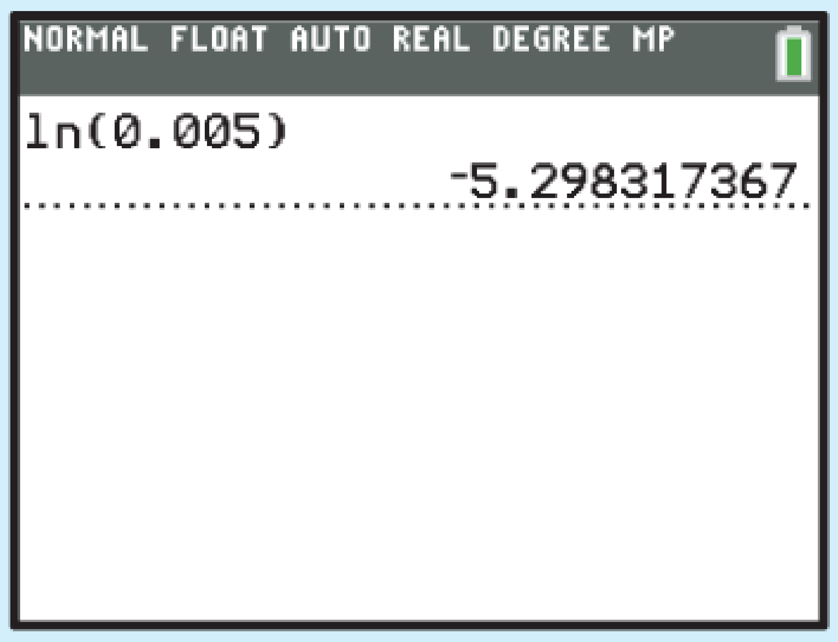


LAWS OF NATURAL LOGARITHMS

The laws for natural logarithms are the laws for logarithms written in base e:

- $\ln m + \ln n = \ln(mn)$ for $m, n > 0$
- $\ln m - \ln n = \ln\left(\frac{m}{n}\right)$ for $m, n > 0$
- $m \ln b = \ln(b^m)$ for $b > 0$

EXERCISE 3D

1 Without using a calculator find:

a $\ln(e^2)$ **b** $\ln(e^4)$ **c** $\ln\left((\sqrt{e})^3\right)$ **d** $\ln 1$

e $\ln\left(\frac{1}{e}\right)$ **f** $\ln \sqrt[3]{e}$ **g** $\ln\left(\frac{1}{e^2}\right)$ **h** $\ln\left(\frac{1}{\sqrt{e}}\right)$

Check your answers using a calculator.

2 Simplify:

a $e^{\ln 3}$ **b** $e^{2\ln 3}$ **c** $e^{-\ln 5}$ **d** $e^{-2\ln 2}$

e $\ln e^a$ **f** $\ln(e \times e^a)$ **g** $\ln(e^a \times e^b)$ **h** $\ln\left((e^a)^b\right)$

3 Use your calculator to find, correct to 3 decimal places:

a $\ln 12$ **b** $\ln 68$ **c** $\ln(1.4)$ **d** $\ln(0.7)$ **e** $\ln 500$

4 Explain why $\ln(-2)$ and $\ln 0$ cannot be found.

5 Use your calculator to write the following in the form e^k where k is correct to 4 decimal places:

a 6 **b** 60 **c** 6000 **d** 0.6 **e** 0.006

f 15 **g** 1500 **h** 1.5 **i** 0.15 **j** 0.000 15

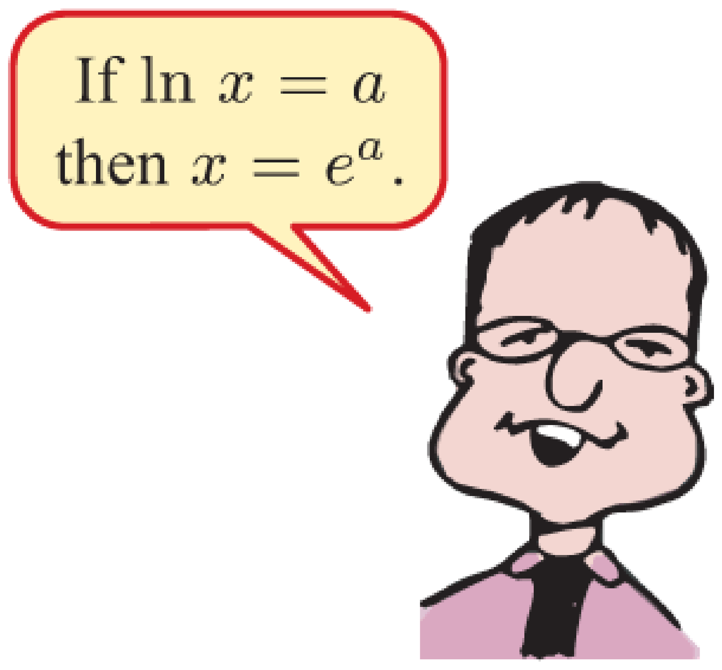
Example 13 Self Tutor

Find x if:

a $\ln x = 2.17$ **b** $\ln x = -0.384$

a $\ln x = 2.17$
$\therefore\ x = e^{2.17}$
$\therefore\ x \approx 8.76$

b $\ln x = -0.384$
$\therefore\ x = e^{-0.384}$
$\therefore\ x \approx 0.681$



6 Find x if:

a $\ln x = 3$ **b** $\ln x = 1$ **c** $\ln x = 0$ **d** $\ln x = -1$

e $\ln x = -5$ **f** $\ln x \approx 0.835$ **g** $\ln x \approx 2.145$ **h** $\ln x \approx -3.2971$

7 **a** Write in simplest form:

i $\ln(e^x)$ **ii** $e^{\ln x}$

b What does this tell us about the functions $y = e^x$ and $y = \ln x$?

Example 14 Self Tutor

Use the laws of logarithms to write as a single logarithm:

a $\ln 5 + \ln 3$ **b** $\ln 24 - \ln 8$ **c** $\ln 5 - 1$

a $\ln 5 + \ln 3$
$= \ln(5 \times 3)$
$= \ln 15$

b $\ln 24 - \ln 8$
$= \ln\left(\frac{24}{8}\right)$
$= \ln 3$

c $\ln 5 - 1$
$= \ln 5 - \ln(e^1)$
$= \ln\left(\frac{5}{e}\right)$

8 Write as a single logarithm or integer:

a $\ln 15 + \ln 3$ **b** $\ln 15 - \ln 3$ **c** $\ln 20 - \ln 5$

d $\ln 4 + \ln 6$ **e** $\ln 5 + \ln(0.2)$ **f** $\ln 2 + \ln 3 + \ln 5$

g $1 + \ln 4$ **h** $\ln 6 - 1$ **i** $\ln 5 + \ln 8 - \ln 2$

j $2 + \ln 4$ **k** $\ln 20 - 2$ **l** $\ln 12 - \ln 4 - \ln 3$

Example 15 **Self Tutor**

Use the laws of logarithms to simplify:

a $2\ln 7 - 3\ln 2$ **b** $2\ln 3 + 3$

a
$$\begin{aligned} &2\ln 7 - 3\ln 2 \\ &= \ln(7^2) - \ln(2^3) \\ &= \ln 49 - \ln 8 \\ &= \ln\left(\tfrac{49}{8}\right) \end{aligned}$$

b
$$\begin{aligned} &2\ln 3 + 3 \\ &= \ln(3^2) + \ln(e^3) \\ &= \ln 9 + \ln(e^3) \\ &= \ln(9e^3) \end{aligned}$$

9 Write in the form $\ln a$, $a \in \mathbb{R}$:

a $5\ln 3 + \ln 4$ **b** $3\ln 2 + 2\ln 5$ **c** $3\ln 2 - \ln 8$

d $3\ln 4 - 2\ln 2$ **e** $\frac{1}{3}\ln 8 + \ln 3$ **f** $\frac{1}{3}\ln\left(\frac{1}{27}\right)$

g $-\ln 2$ **h** $-\ln\left(\frac{1}{2}\right)$ **i** $-2\ln\left(\frac{1}{4}\right)$

j $4\ln 2 + 2$ **k** $\frac{1}{2}\ln 9 - 1$ **l** $-3\ln 2 + \frac{1}{2}$

10 Show that:

a $\ln 27 = 3\ln 3$ **b** $\ln\sqrt{3} = \frac{1}{2}\ln 3$ **c** $\ln\left(\frac{1}{16}\right) = -4\ln 2$

d $\ln\left(\frac{1}{6}\right) = -\ln 6$ **e** $\ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln 2$ **f** $\ln\left(\frac{e}{5}\right) = 1 - \ln 5$

g $\ln(6e) = \ln 6 + 1$ **h** $\ln\sqrt[3]{5} = \frac{1}{3}\ln 5$ **i** $\ln\left(\frac{1}{\sqrt[5]{2}}\right) = -\frac{1}{5}\ln 2$

j $\ln\left(\frac{e^2}{8}\right) = 2 - 3\ln 2$ **k** $\ln\left(\frac{\sqrt{3}}{e^4}\right) = \frac{1}{2}\ln 3 - 4$ **l** $\ln\left(\frac{1}{16 \times \sqrt[3]{e}}\right) = -4\ln 2 - \frac{1}{3}$

11 Find x and y given that $\ln\left(\frac{x}{y}\right) = 6$ and $\ln(x^3y^4) = 4$.

THEORY OF KNOWLEDGE

It is easy to take modern technology, such as the electronic calculator, for granted. Until electronic computers became affordable in the 1980s, a "calculator" was a *profession*, literally someone who would spend their time performing calculations by hand. They used mechanical calculators and techniques such as logarithms. They often worked in banks, but sometimes for astronomers and other scientists.

The logarithm was invented by **John Napier** (1550 - 1617) and first published in 1614 in a Latin book which translates as a *Description of the Wonderful Canon of Logarithms*. John Napier was the 8th Lord of Merchiston, which is now part of Edinburgh, Scotland. Napier wrote a number of other books on many subjects including religion and mathematics. One of his other inventions was a device for performing long multiplication which is now called "Napier's Bones". Other calculators, such as slide rules, used logarithms as part of their design. He also popularised the use of the decimal point in mathematical notation.

Logarithms were an extremely important development, and they had an immediate effect on the seventeenth century scientific community. **Johannes Kepler** used logarithms to assist with his calculations. This helped him develop his laws of planetary motion. Without logarithms these calculations would have taken many years. Kepler published a letter congratulating and acknowledging Napier. Kepler's laws gave **Sir Isaac Newton** important evidence to support his theory of universal gravitation. 200 years later, **Laplace** said that logarithms "by shortening the labours, doubled the life of the astronomer".

Johannes Kepler

1 Can anyone claim to have *invented* logarithms?

2 Can we consider the process of mathematical discovery as an *evolution* of ideas?

3 Has modern computing effectively doubled the life of a mathematician?

Many areas of mathematics have been developed over centuries as several mathematicians have worked in a particular area, or taken the knowledge from one area and applied it to another field. Sometimes the process is held up because a method for solving a particular class of problem has not yet been found. In other cases, pure mathematicians have published research papers on seemingly useless mathematical ideas, which have then become vital in applications much later.

In *Everybody Counts*: *A report to the nation on the future of Mathematical Education* by the National Academy of Sciences (National Academy Press, 1989), there is an excellent section on the Nature of Mathematics. It includes:

> "Even the most esoteric and abstract parts of mathematics - number theory and logic, for example - are now used routinely in applications (for example, in computer science and cryptography). Fifty years ago, the leading British mathematician G.H. Hardy could boast that number theory was the most pure and least useful part of mathematics. Today, Hardy's mathematics is studied as an essential prerequisite to many applications, including control of automated systems, data transmission from remote satellites, protection of financial records, and efficient algorithms for computation."

4 Should we only study the mathematics required to enter our chosen profession?

5 Why should we explore mathematics for its own sake, rather than to address the needs of science?

E LOGARITHMIC EQUATIONS

We can use the laws of logarithms to write equations in a different form. This can be particularly useful if an unknown appears as an exponent.

Since the logarithmic function is one-to-one, we can take the logarithm of both sides of an equation without changing the solution. However, we can only do this if both sides are positive.

Example 16 Self Tutor

Write as a logarithmic equation (in base 10):

a $y = a^2 b$ **b** $P = \frac{20}{\sqrt{n}}$

a
$$y = a^2 b$$
$$\therefore \log y = \log(a^2 b)$$
$$\therefore \log y = \log(a^2) + \log b$$
$$\therefore \log y = 2\log a + \log b$$

b
$$P = \left(\frac{20}{\sqrt{n}}\right)$$
$$\therefore \log P = \log\left(\frac{20}{n^{\frac{1}{2}}}\right)$$
$$\therefore \log P = \log 20 - \log(n^{\frac{1}{2}})$$
$$\therefore \log P = \log 20 - \tfrac{1}{2}\log n$$

Example 17 Self Tutor

Write without logarithms:

a $\log A = \log b + 2\log c$ **b** $\log_2 M = 3\log_2 a - 2$

a
$$\log A = \log b + 2\log c$$
$$\therefore \log A = \log b + \log(c^2)$$
$$\therefore \log A = \log(bc^2)$$
$$\therefore A = bc^2$$

b
$$\log_2 M = 3\log_2 a - 2$$
$$\therefore \log_2 M = \log_2(a^3) - \log_2(2^2)$$
$$\therefore \log_2 M = \log_2\left(\frac{a^3}{4}\right)$$
$$\therefore M = \frac{a^3}{4}$$

EXERCISE 3E

1 Write as a logarithmic equation (in base 10), with no powers, assuming all variables are positive:

a $y = 2^x$ **b** $y = 20b^3$ **c** $M = ad^4$ **d** $T = 5\sqrt{d}$

e $R = b\sqrt{l}$ **f** $Q = \frac{a}{b^n}$ **g** $y = ab^x$ **h** $F = \frac{20}{\sqrt{n}}$

i $L = \frac{ab}{c}$ **j** $N = \sqrt{\frac{a}{b}}$ **k** $S = 200 \times 2^t$ **l** $y = \frac{a^m}{b^n}$

2 Write without logarithms:

a $\log D = \log e + \log 2$ **b** $\log_a F = \log_a 5 - \log_a t$ **c** $\log P = \frac{1}{2}\log x$

d $\log_n M = 2\log_n b + \log_n c$ **e** $\log B = 3\log m - 2\log n$ **f** $\log N = -\frac{1}{3}\log p$

g $\log P = 3\log x + 1$ **h** $\log_a Q = 2 - \log_a x$

3 Write without logarithms:

a $\ln D = \ln x + 1$ **b** $\ln F = -\ln p + 2$ **c** $\ln P = \frac{1}{2}\ln x$

d $\ln M = 2\ln y + 3$ **e** $\ln B = 3\ln t - 1$ **f** $\ln N = -\frac{1}{3}\ln g$

g $\ln Q \approx 3\ln x + 2.159$ **h** $\ln D \approx 0.4\ln n - 0.6582$

4 **a** Write $y = 3 \times 2^x$ as a logarithmic equation in base 2.

b Hence write x in terms of y.

c Find the value of x when: **i** $y = 3$ **ii** $y = 12$ **iii** $y = 30$

5 Solve for x:

a $\log_3 27 + \log_3\left(\frac{1}{3}\right) = \log_3 x$ **b** $\log_5 x = \log_5 8 - \log_5(6 - x)$

c $\log_5 125 - \log_5 \sqrt{5} = \log_5 x$ **d** $\log_{20} x = 1 + \log_{20} 10$

e $\log x + \log(x+1) = \log 30$ **f** $\log(x+2) - \log(x-2) = \log 5$

g $\log 24 = \log 3 + x\log 2$ **h** $x\log_2 3 + \log_2 36 = 2$

6 Solve simultaneously for x and y:

a $\begin{cases} \log_x y = 2 \\ \log_{y-2} x = 1 \end{cases}$ **b** $\begin{cases} \log_x y = 3 \\ \log_{y+1}(x+1) = \frac{1}{2} \end{cases}$

7 Let $x = \log_2 7$.

a Write the equation without logarithms.

b Take the logarithm in base 10 of both sides of your equation from **a**. Hence show that $\log_2 7 = \dfrac{\log 7}{\log 2}$, and calculate this number.

8 Consider the exponential equation $a^x = b$ where $a, b > 0$.

a Explain why $x = \log_a b$.

b Take the logarithm in base 10 of both sides of $a^x = b$.

c Hence show that $x = \log_a b = \dfrac{\log b}{\log a}$.

F THE CHANGE OF BASE RULE

In the previous Exercise you should have proven the base 10 case of the **change of base rule**:

$$\boldsymbol{\log_b a = \frac{\log_c a}{\log_c b}} \quad \text{for } a, b, c > 0 \text{ and } b, c \neq 1.$$

Proof: If $\log_b a = x$, then $b^x = a$

$\therefore \log_c b^x = \log_c a$ {taking logarithms in base c}

$\therefore x\log_c b = \log_c a$ {power law of logarithms}

$\therefore x = \dfrac{\log_c a}{\log_c b}$

$\therefore \log_b a = \dfrac{\log_c a}{\log_c b}$

We need the change of base rule to evaluate logarithms in bases other than 10 or e.

Example 18 **Self Tutor**

Find $\log_2 9$ by:

a changing to base 10 **b** changing to base e.

a $\log_2 9 = \dfrac{\log_{10} 9}{\log_{10} 2}$
≈ 3.17

b $\log_2 9 = \dfrac{\ln 9}{\ln 2}$
≈ 3.17

EXERCISE 3F

1 Use the change of base rule with base 10 to calculate:

a $\log_3 7$ **b** $\log_2 40$ **c** $\log_5 180$

d $\log_{\frac{1}{2}} 1250$ **e** $\log_3(0.067)$ **f** $\log_{0.4}(0.006\,984)$

Check your results using the change of base rule with base e.

2 Simplify $\log_m n \times \log_n(m^2)$.

3 Without using technology, show that $2^{\frac{4}{\log_5 4} + \frac{3}{\log_7 8}} = 175$.

Hint: Use the change of base rule with base 2.

4 Solve for x:

a $\log_4(x^3) + \log_2 \sqrt{x} = 8$ **b** $\log_{\frac{1}{9}} x = \log_9 5$

c $\log_{16}(x^5) = \log_{64} 125 - \log_4 \sqrt{x}$ **d** $\log_3(x^3) - 4\log_9 x - 5\log_{27} \sqrt{x} = \log_9 4$

e $\log_x 4 + \log_2 x = 3$

5 Given $x = \log_3(y^2)$, express $\log_y 81$ in terms of x.

6 Given $m = \log_4 3$, express $\log_2 24$ in terms of m.

7 **a** By evaluating each expression when $x = 9$, show that $\log_9(\log_3 x)$ and $\log_3(\log_9 x)$ are not always equal.

b Find the value of x such that $\log_9(\log_3 x) = \log_3(\log_9 x)$.

c Find, in terms of a, the solution to $\log_{a^2}(\log_a x) = \log_a(\log_{a^2} x)$.

d Show that when $\log_{a^k}(\log_a x) = \log_a(\log_{a^k} x)$, $k \neq 1$, both sides of the equation have the value $\log_a k^{\frac{1}{k-1}}$.

G SOLVING EXPONENTIAL EQUATIONS USING LOGARITHMS

In **Chapter 2** we found solutions to simple exponential equations where we could make equal bases and then equate exponents. However, it is not always easy to make the bases the same. In these situations we can use **logarithms**.

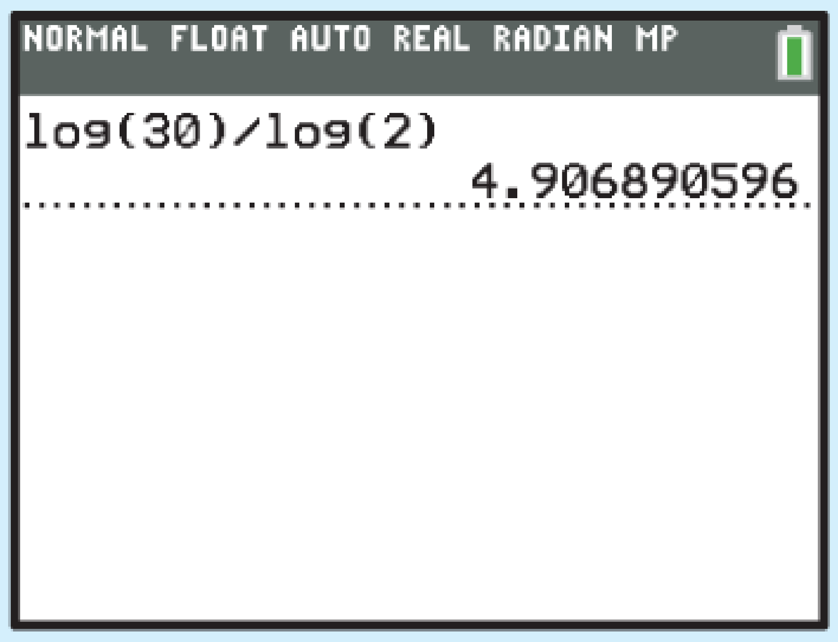
Example 19 — Self Tutor

a Solve the equation $2^x = 30$ exactly.

b Use your calculator to evaluate the solution correct to 2 decimal places.

a
$$2^x = 30$$
$$\therefore\ \log(2^x) = \log 30 \quad \{\text{taking the logarithm of each side}\}$$
$$\therefore\ x\log 2 = \log 30 \quad \{\log(b^m) = m\log b\}$$
$$\therefore\ x = \frac{\log 30}{\log 2}$$

b $\dfrac{\log 30}{\log 2} \approx 4.91$, so the solution is $x \approx 4.91$.



EXERCISE 3G

1 Consider the equation $3^x = 40$.

- **a** Explain why the solution to this equation lies between $x = 3$ and $x = 4$.
- **b** Find the solution exactly.
- **c** Use your calculator to evaluate the solution correct to 2 decimal places.

2 Solve for x: **i** exactly **ii** correct to 2 decimal places.

- **a** $2^x = 10$
- **b** $3^x = 20$
- **c** $4^x = 50$
- **d** $\left(\frac{1}{2}\right)^x = 0.0625$
- **e** $\left(\frac{3}{4}\right)^x = 0.1$
- **f** $10^x = 0.000\,015$

3 Solve for x, correct to 3 significant figures:

- **a** $5^x = 40$
- **b** $3^x = 2^{x+3}$
- **c** $2^{x+4} = 5^{2-x}$

Example 20 — Self Tutor

Find x exactly:

a $e^x = 30$ **b** $3e^{\frac{x}{2}} = 21$

a
$$e^x = 30$$
$$\therefore\ x = \ln 30$$

b
$$3e^{\frac{x}{2}} = 21$$
$$\therefore\ e^{\frac{x}{2}} = 7$$
$$\therefore\ \frac{x}{2} = \ln 7$$
$$\therefore\ x = 2\ln 7$$

4 Solve for x, giving an exact answer:

- **a** $e^x = 10$
- **b** $e^x = 1000$
- **c** $2e^x = 0.3$
- **d** $e^{\frac{x}{2}} = 5$
- **e** $e^{2x} = 18$
- **f** $e^{-\frac{x}{2}} = 1$

5 Solve for x, giving an exact answer:

a $3 \times 2^x = 75$ **b** $7 \times (1.5)^x = 20$ **c** $5 \times (0.8)^x = 3$

d $4 \times 2^{-x} = 0.12$ **e** $300 \times 5^{0.1x} = 1000$ **f** $32 \times e^{-0.25x} = 4$

6 Solve for x exactly:

a $25^x - 3 \times 5^x = 0$ **b** $8 \times 9^x - 3^x = 0$ **c** $2^x - 2 \times 4^x = 0$

7 Solve $3^{2x} = \frac{1}{16}$, writing your answer in terms of $\ln 2$ and $\ln 3$.

8 Solve $10^{2x} = 4^{x+3}$, writing your answer in terms of $\ln 2$ and $\ln 5$.

Example 21 **Self Tutor**

Find exactly the points of intersection of $y = e^x - 3$ and $y = 1 - 3e^{-x}$.
Check your solution using technology.

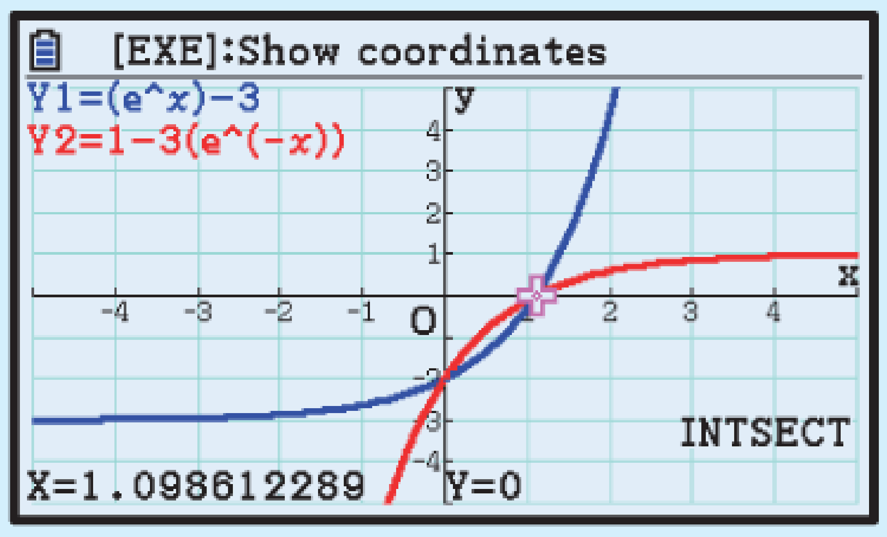
The functions meet where

$$e^x - 3 = 1 - 3e^{-x}$$
$$\therefore\ e^x - 4 + 3e^{-x} = 0$$
$$\therefore\ e^{2x} - 4e^x + 3 = 0 \qquad \{\text{multiplying each term by } e^x\}$$
$$\therefore\ (e^x - 1)(e^x - 3) = 0$$
$$\therefore\ e^x = 1 \text{ or } 3$$
$$\therefore\ x = \ln 1 \text{ or } \ln 3$$
$$\therefore\ x = 0 \text{ or } \ln 3$$

When $x = 0$, $y = e^0 - 3 = -2$
When $x = \ln 3$, $y = e^{\ln 3} - 3 = 0$

$\therefore$ the functions meet at $(0, -2)$ and at $(\ln 3, 0)$.

GRAPHING PACKAGE



9 Solve for x:

a $e^{2x} = 2e^x$ **b** $e^x = e^{-x}$ **c** $e^{2x} - 5e^x + 6 = 0$

d $e^x + 2 = 3e^{-x}$ **e** $1 + 12e^{-x} = e^x$ **f** $e^x + e^{-x} = 3$

10 Find algebraically the point(s) of intersection of:

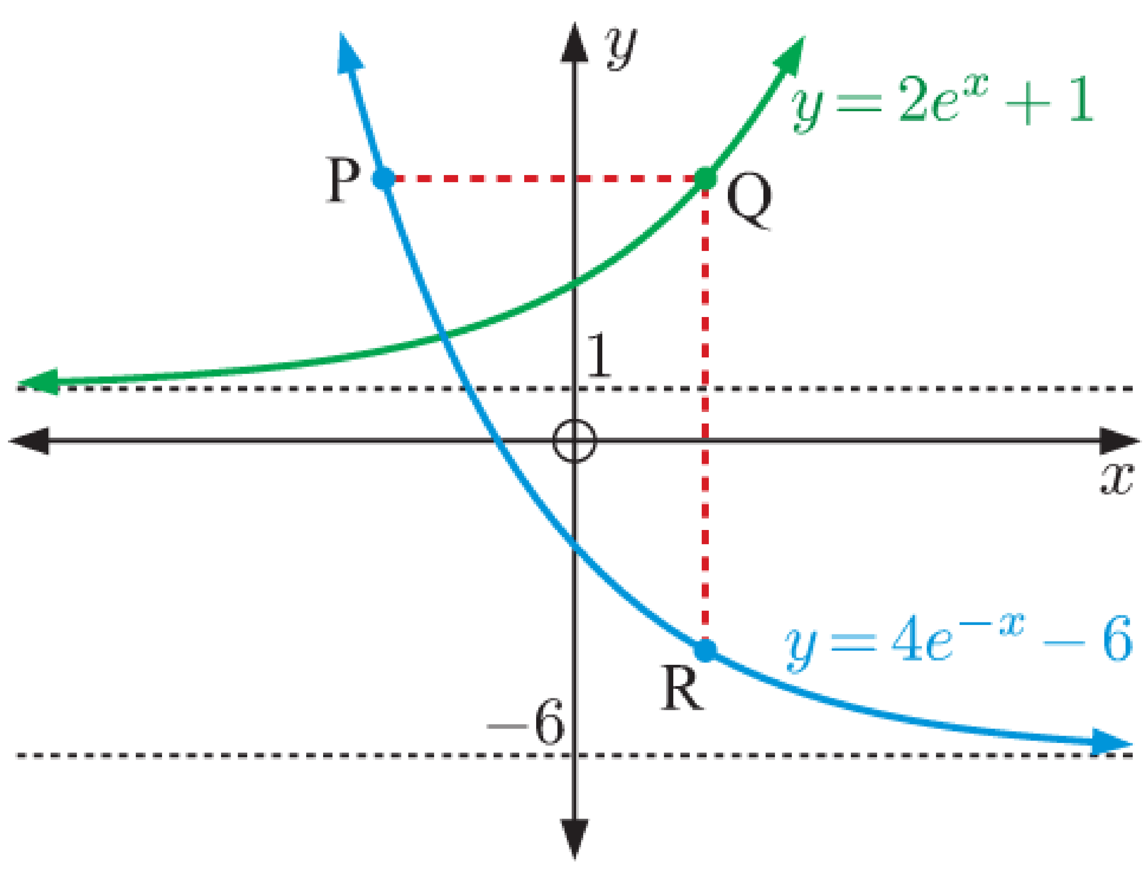
a $y = e^x$ and $y = e^{2x} - 6$ **b** $y = 2e^x + 1$ and $y = 7 - e^x$

c $y = 3 - e^x$ and $y = 5e^{-x} - 3$

Check your answers using technology.

11 **a** Find the exact coordinates of the intersection point of the graphs shown.

b Given that the vertical line segment [QR] has length 9 units, find the length of the horizontal line segment [PQ]. Give your answer in the form $\ln k$ units, where $k \in \mathbb{Q}$.



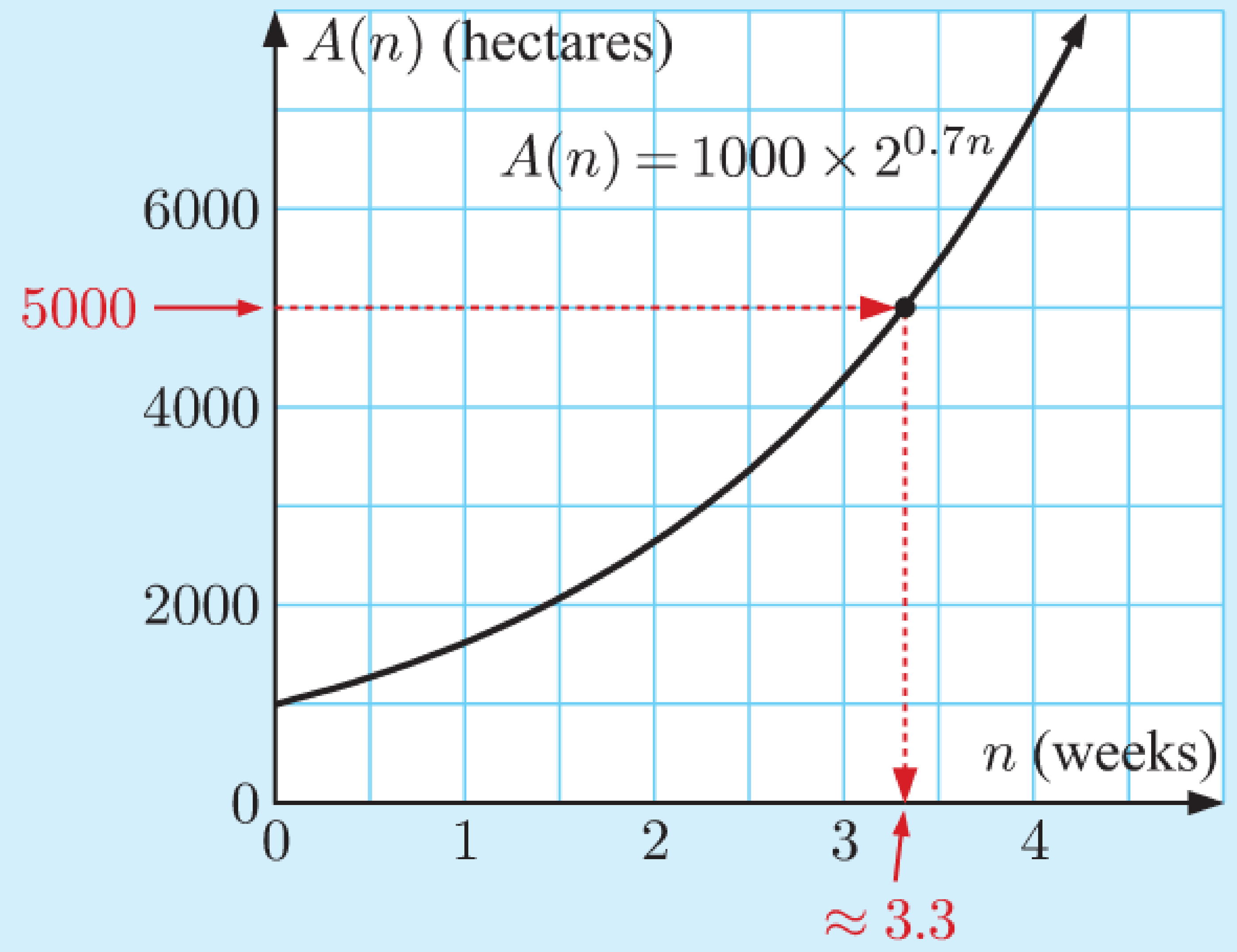
Example 22 **Self Tutor**

A farmer monitoring an insect plague finds that the area affected by the insects is given by $A(n) = 1000 \times 2^{0.7n}$ hectares, where n is the number of weeks after the initial observation.

a Use technology to help sketch the graph of $A(n)$. Hence estimate the time taken for the affected area to reach 5000 hectares.

b Check your answer to **a** using logarithms.

a From the graph, it appears that it will take about 3.3 weeks for the affected area to reach 5000 hectares.



b When $A(n) = 5000$,

$$1000 \times 2^{0.7n} = 5000$$
$$\therefore \; 2^{0.7n} = 5$$
$$\therefore \; \log(2^{0.7n}) = \log 5$$
$$\therefore \; 0.7n \log 2 = \log 5$$
$$\therefore \; n = \frac{\log 5}{0.7 \times \log 2} \approx 3.32$$

$\therefore$ it will take about 3 weeks and 2 days.

12 The population of turtles in an isolated colony is $P(t) = 852 \times (1.07)^t$, where t is the time in years after the colony was first recorded. How long will it take for the population to reach:

a 1000 turtles **b** 1500 turtles?

13 The weight of bacteria in a culture t hours after establishment is given by $W(t) = 20 \times 2^{0.15t}$ grams. Find, using logarithms, the time for the weight of the culture to reach:

a 30 grams **b** 100 grams.

14 A biologist is modelling an infestation of fire ants. He determines that the area affected by the ants is given by $A(n) = 2000 \times e^{0.57n}$ hectares, where n is the number of weeks after the initial observation.

a Use technology to help sketch the graph of $A(n)$.

b Hence estimate the time taken for the infested area to reach 10 000 hectares.

c Check your answer to **b** using logarithms.

15 A house is expected to increase in value at an average rate of 7.5% p.a. If the house is worth £360 000 now, when would you expect it to be worth £550 000?

16 Thabo has \$10 000 to invest in an account that pays 4.8% p.a. compounded annually. How long will it take for his investment to grow to \$15 000?

17 Dien invests \$15 000 at 8.4% p.a. compounded *monthly*. He will withdraw his money when it reaches \$25 000, at which time he plans to travel. The formula $t_n = t_0 \times r^n$ can be used to model the investment, where n is the time in months.

a Explain why $r = 1.007$.

b After how many months will Dien withdraw the money?

18 The mass M_t of radioactive substance remaining after t years is given by $M_t = 1000 \times e^{-0.04t}$ grams. Find the time taken for the mass to:

a halve **b** reach 25 grams **c** reach 1% of its original value.

19 The current I flowing in a transistor radio t seconds after it is switched off, is given by $I = I_0 \times 2^{-0.02t}$ amps. Show that it takes $\dfrac{50}{\log 2}$ seconds for the current to drop to 10% of its original value.

20 A sky diver jumps from an aeroplane. His speed of descent is given by $V(t) = 50(1 - e^{-0.2t})$ m s^{-1}, where t is the time in seconds.

a Show that it will take $5 \ln 5$ seconds for the sky diver's speed to reach 40 m s^{-1}.

b Write an expression for the time taken for his speed to reach v m s^{-1}.

21 Answer the **Opening Problem** on page **68**.

22 The weight of radioactive substance remaining after t years is given by $W = 1000 \times 2^{-0.04t}$ grams.

a Sketch the graph of W against t.

b Write a function for t in terms of W.

c Hence find the time required for the weight to reach:

i 20 grams **ii** 0.001 grams.

23 The temperature of a liquid t minutes after it is placed in a refrigerator, is given by $T = 4 + 96 \times e^{-0.03t}$ °C.

a Sketch the graph of T against t.

b Write a function for t in terms of T.

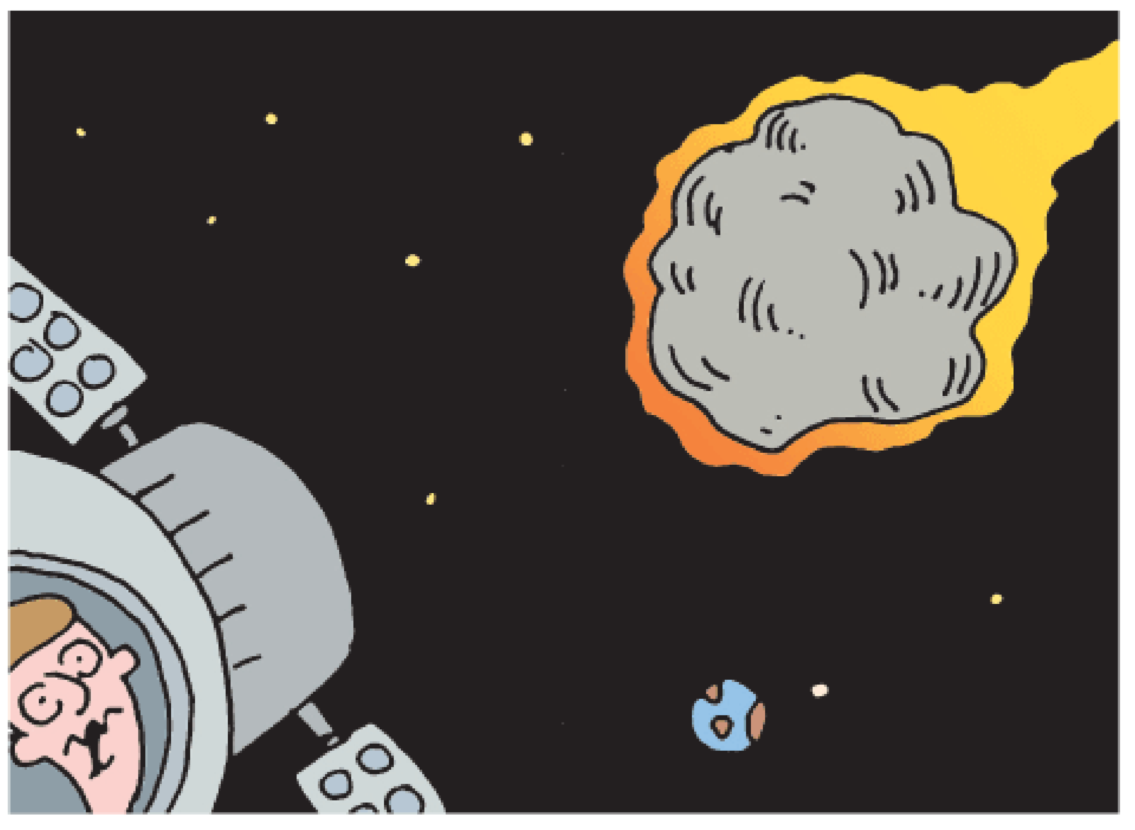
c Find the time required for the temperature to reach:

i 25°C **ii** 5°C.

24 A meteor hurtling through the atmosphere has speed of descent given by

$$V(t) = 650(4 + 2 \times e^{-0.1t}) \text{ m s}^{-1}$$

where t is the time in seconds after the meteor is sighted.



a Is the meteor's speed increasing or decreasing?

b Find the speed of the meteor:

i when it was first sighted

ii after 2 minutes.

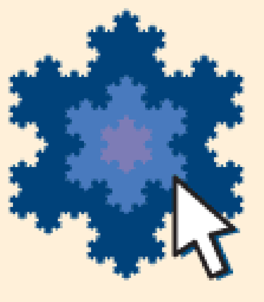
c How long will it take for the meteor's speed to reach 3000 m s^{-1}?

INVESTIGATION 2 THE "RULE OF 72"

The "rule of 72" is used to estimate the time a quantity takes to double in value, given the rate at which the quantity grows.

Click on the icon to view this Investigation.

RULE OF 72



H LOGARITHMIC FUNCTIONS

We have seen that $\log_a a^x = a^{\log_a x} = x$.

Letting $f(x) = \log_a x$ and $g(x) = a^x$, we have $f \circ g = g \circ f = x$.

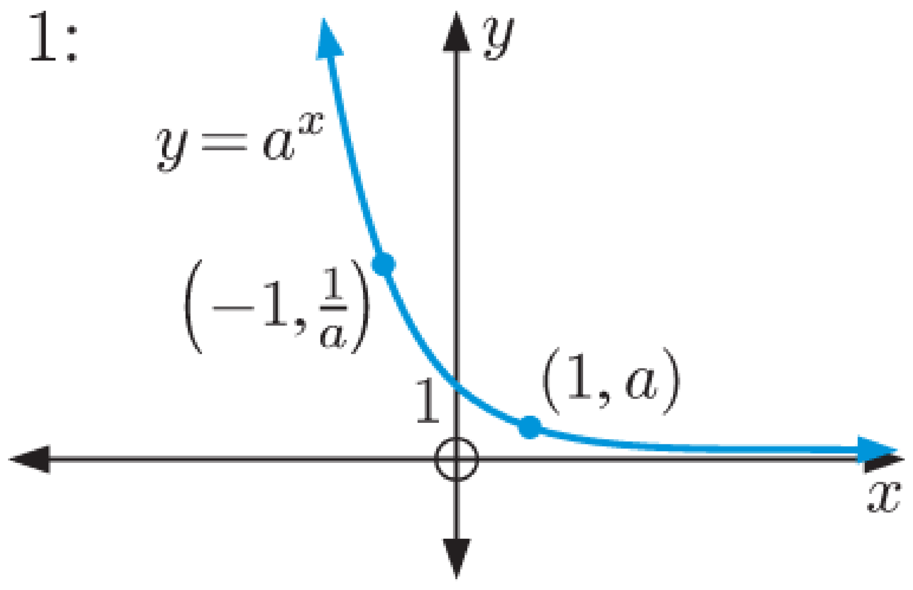
We can therefore say that the logarithmic function $\log_a x$ is the **inverse** of the exponential function a^x.

Algebraically, this has the effect that the logarithmic and exponential functions "undo" one another.

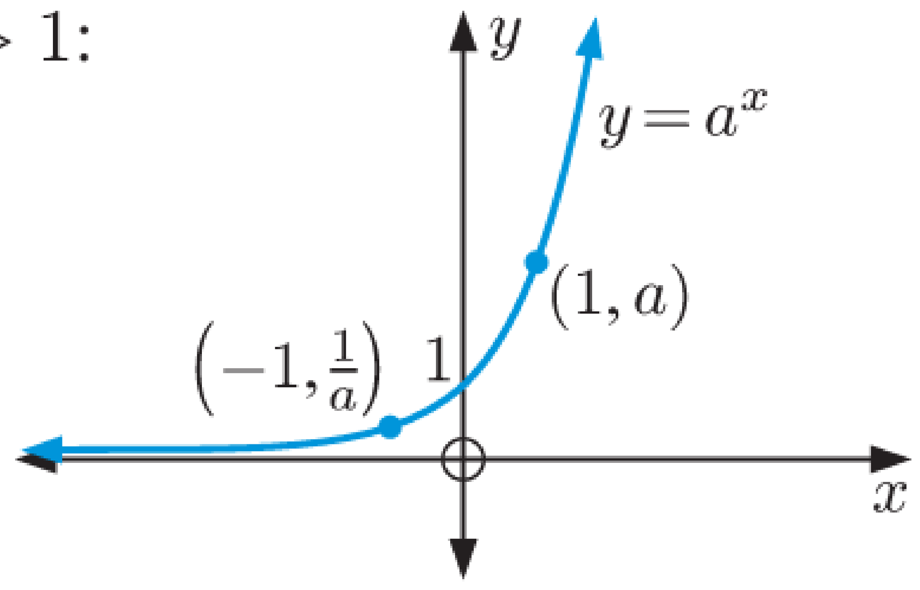
Geometrically, it means that the graph of $y = \log_a x$, $a > 0$, $a \neq 1$ is the *reflection* of the graph of $y = a^x$ in the line $y = x$.

We have seen previously the shape of the exponential function $y = a^x$ where $a > 0$, $a \neq 1$.

For $0 < a < 1$:



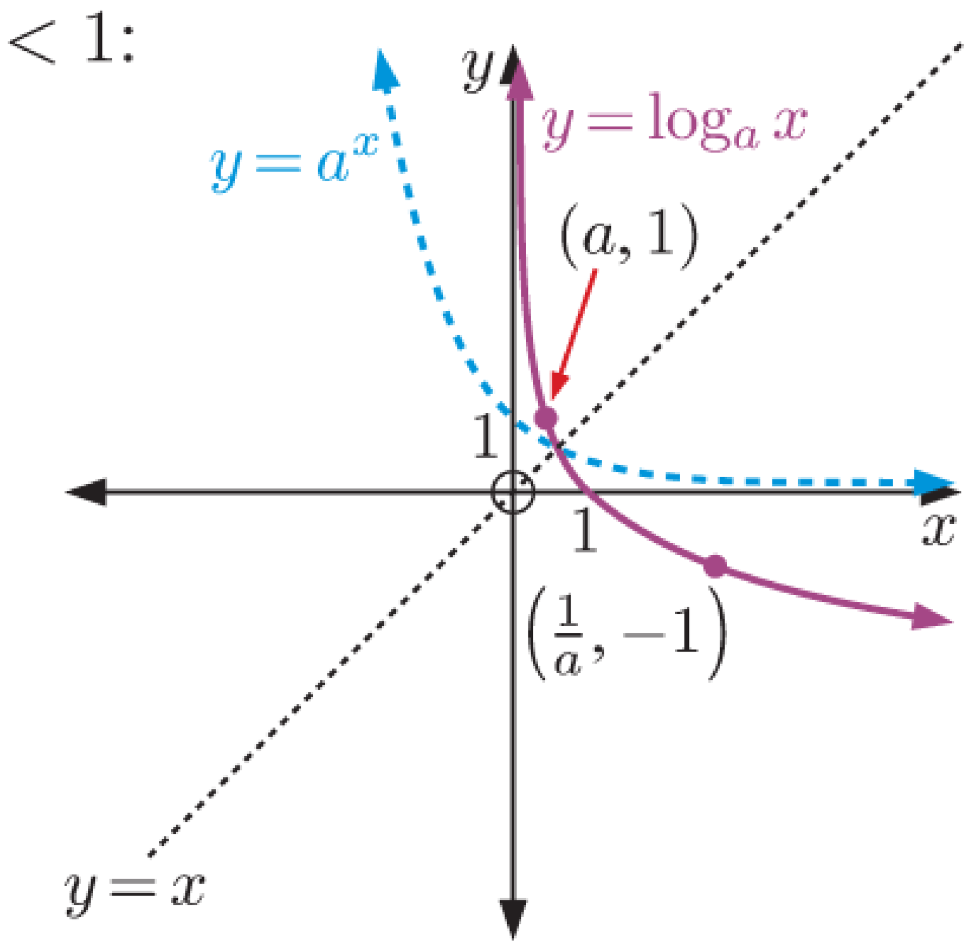
For $a > 1$:



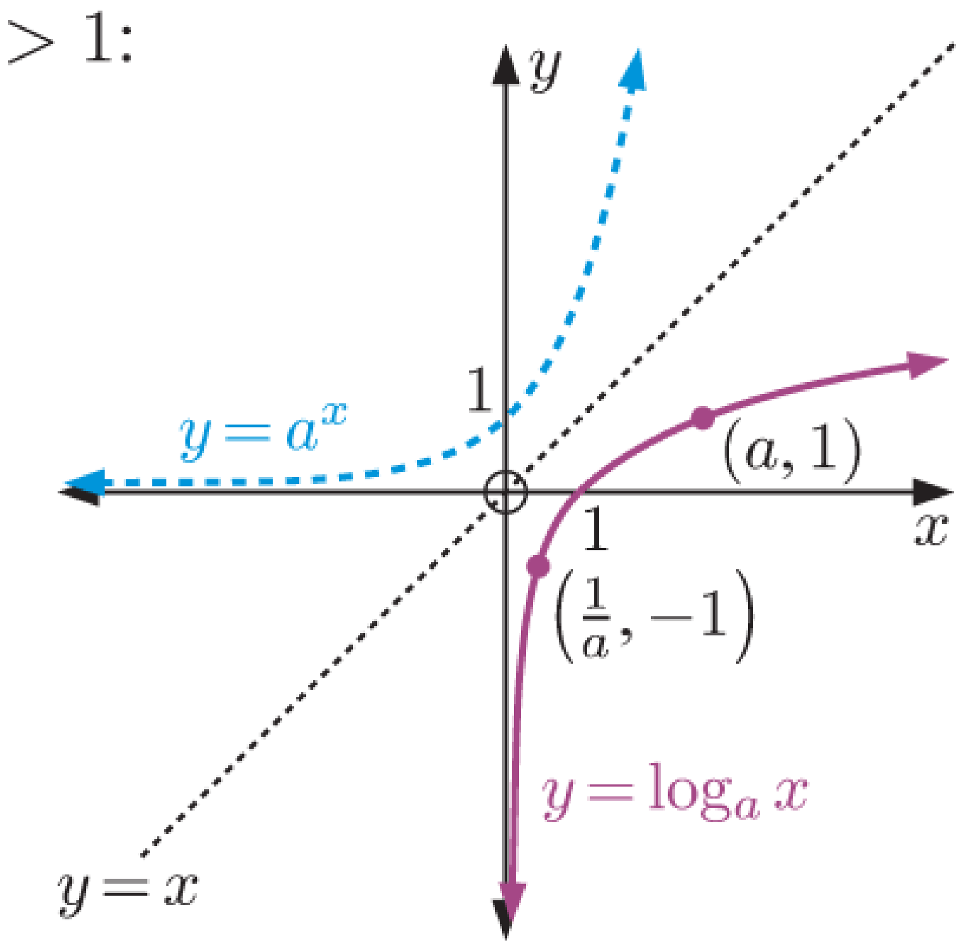
The horizontal asymptote for all of these functions is the x-axis $y = 0$.

By reflecting these graphs in the line $y = x$, we obtain the graphs for $y = \log_a x$.

For $0 < a < 1$:



For $a > 1$:



The **vertical asymptote** of $y = \log_a x$ is the y-axis $x = 0$.

For $0 < a < 1$: as $x \to \infty$, $y \to -\infty$
as $x \to 0^+$, $y \to \infty$

For $a > 1$: as $x \to \infty$, $y \to \infty$
as $x \to 0^+$, $y \to -\infty$

PROPERTIES OF $y = \log_a x$

Since we can only find logarithms of positive numbers, the domain of $y = \log_a x$ is $\{x \mid x > 0\}$.

We can compare the functions $y = a^x$ and $y = \log_a x$ as follows:

Function	$y = a^x$	$y = \log_a x$
Domain	$x \in \mathbb{R}$	$x > 0$
Range	$y > 0$	$y \in \mathbb{R}$
Asymptote	horizontal $y = 0$	vertical $x = 0$

TRANSFORMATIONS OF LOGARITHMIC FUNCTIONS

LOGARITHMIC GRAPHS

Click on the icon to explore the graphs of functions of the form $y = p\ln(x-h)+k$.

Example 23 Self Tutor

Consider the function $f(x) = \log_2(x-1)+1$.

a State the transformation which maps $y = \log_2 x$ to $y = f(x)$.

b Find the domain and range of f.

c Find any asymptotes and axes intercepts.

d Sketch the graph of $y = f(x)$.

e Find the inverse function f^{-1}.

a $f(x)$ is a translation of $y = \log_2 x$ by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

b $x - 1 > 0$ when $x > 1$

So, the domain is $x > 1$ and the range is $y \in \mathbb{R}$.

c As $x \to 1^+$, $y \to -\infty$, so the vertical asymptote is $x = 1$.

As $x \to \infty$, $y \to \infty$, so there is no horizontal asymptote.

When $x = 0$, y is undefined, so there is no y-intercept.

When $y = 0$, $\log_2(x-1) = -1$

$\therefore\ x - 1 = 2^{-1}$

$\therefore\ x = \frac{3}{2}$ So, the x-intercept is $\frac{3}{2}$.

GRAPHICS CALCULATOR INSTRUCTIONS

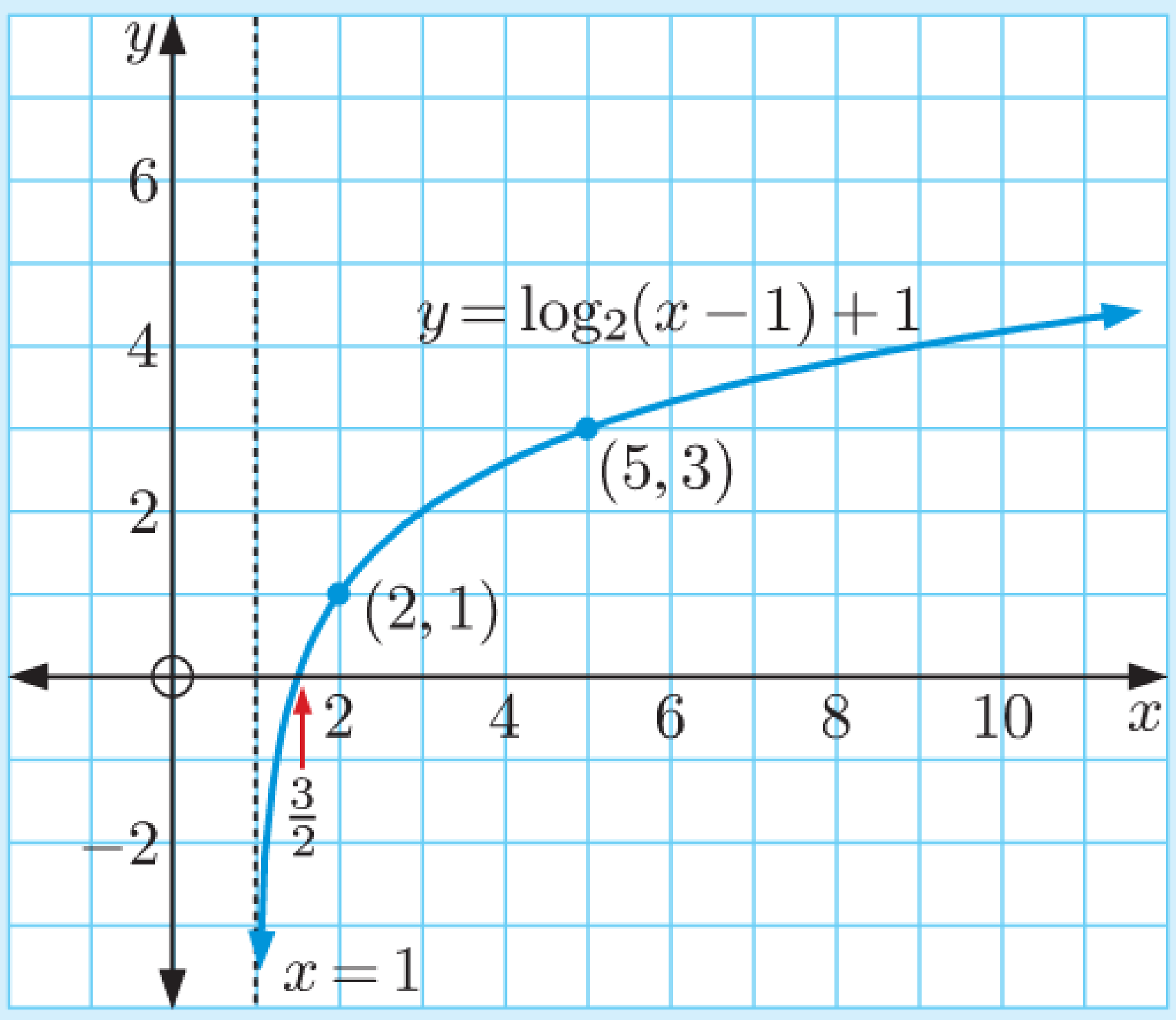
d When $x = 2$, $y = \log_2(2-1)+1 = 1$

When $x = 5$, $y = \log_2(5-1)+1$

$= \log_2 4 + 1$

$= 2 + 1$

$= 3$



e f is defined by $y = \log_2(x-1)+1$

$\therefore\ f^{-1}$ is defined by $x = \log_2(y-1)+1$

$\therefore\ x - 1 = \log_2(y-1)$

$\therefore\ y - 1 = 2^{x-1}$

$\therefore\ y = 2^{x-1}+1$

$\therefore\ f^{-1}(x) = 2^{x-1}+1$

EXERCISE 3H

1 For each of the following functions f:

i Find the domain and range.

ii Find any asymptotes and axes intercepts.

iii Sketch the graph of $y = f(x)$, showing all important features.

iv Solve $f(x) = -1$ algebraically and check the solution on your graph.

v Find the inverse function f^{-1}.

a $f : x \mapsto \log_2 x - 2$

b $f : x \mapsto \log_3(x+1)$

c $f : x \mapsto 1 - \log_3(x+1)$

d $f : x \mapsto \log_5(x-2) - 2$

e $f : x \mapsto 1 - \log_5(x-2)$

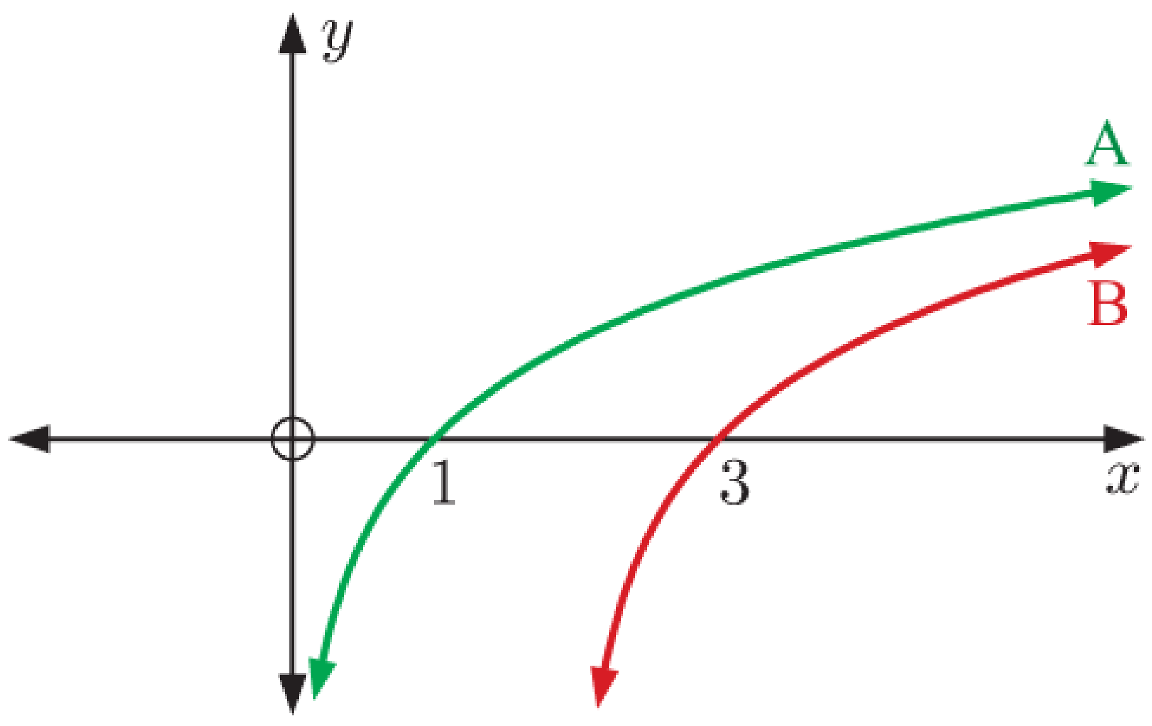
f $f : x \mapsto 1 - 2\log_2 x$

2 For each of the functions f:

- **i** State the transformation which maps $y = \ln x$ to $y = f(x)$.
- **ii** State the domain and range.
- **iii** Find any asymptotes and intercepts.
- **iv** Sketch the graph of $y = f(x)$, showing all important features.
- **v** Find the inverse function f^{-1}.

a $f(x) = \ln x - 4$ **b** $f(x) = \ln(x-1) + 2$ **c** $f(x) = 3\ln x - 1$

3 Consider the curves A and B. One of them is the graph of $y = \ln x$ and the other is the graph of $y = \ln(x-2)$.

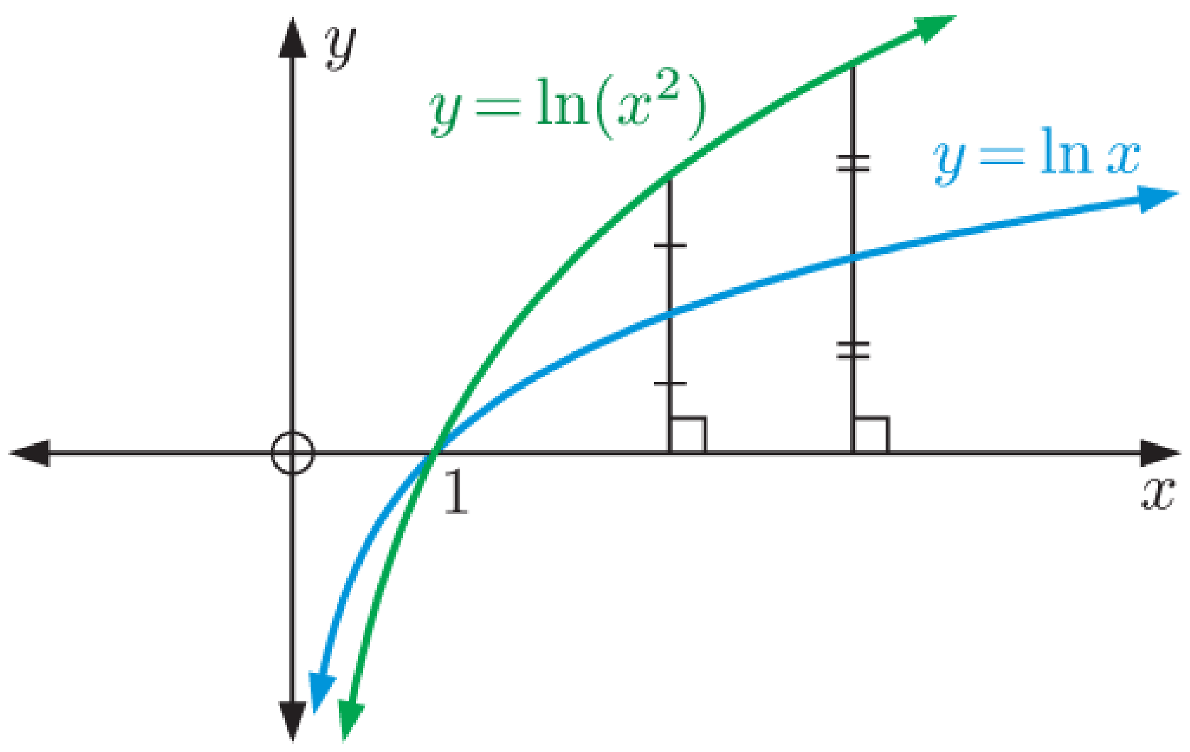


a Identify which curve is which, giving evidence for your answer.

b Copy the graphs onto a new set of axes, and then draw the graph of $y = \ln(x+2)$.

c Find the equation of the vertical asymptote for each graph.

4 Kelly said that in order to graph $y = \ln(x^2)$, $x > 0$, you could first graph $y = \ln x$ and then double the distance of each point on the graph from the x-axis. Is Kelly correct? Explain your answer.



5 Draw, on the same set of axes, the graphs of:

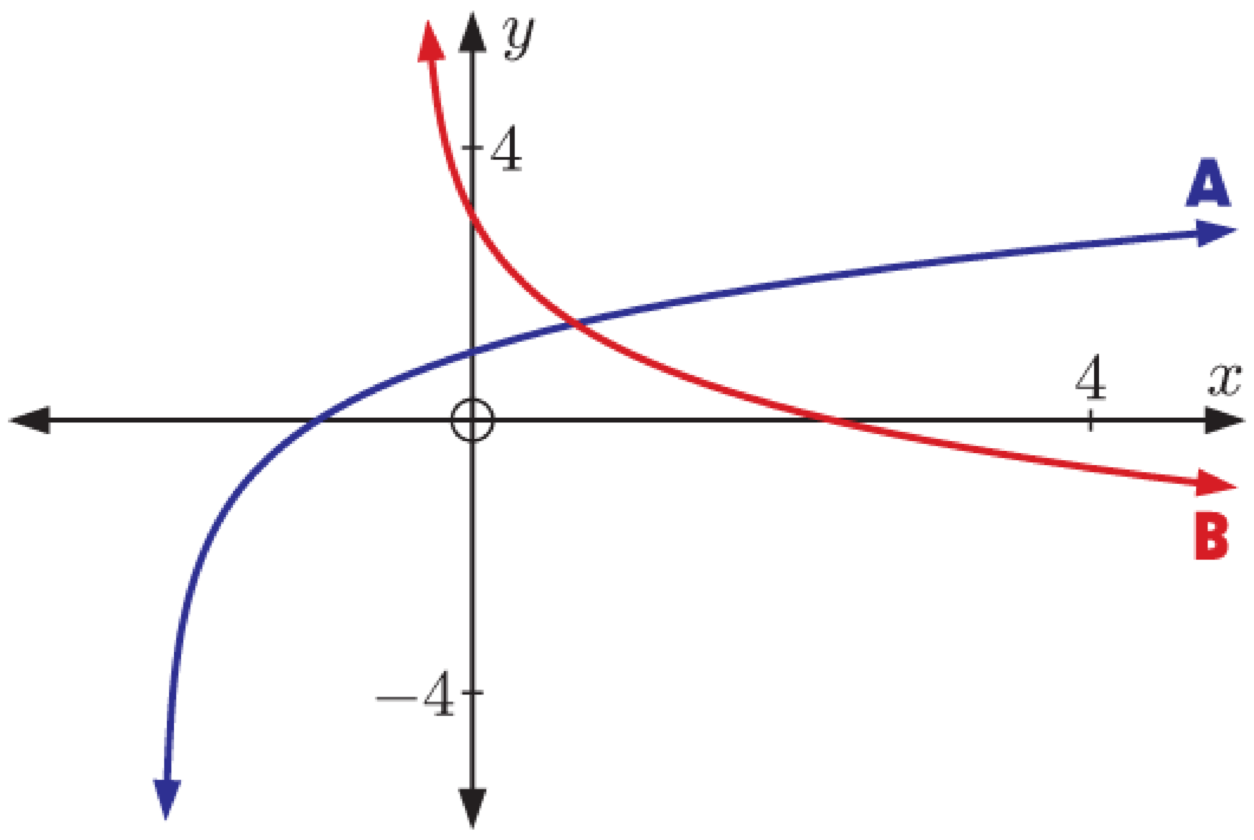
a $y = \ln x$ and $y = \ln(x^3)$ **b** $y = \ln x$ and $y = \ln\left(\frac{1}{x}\right)$

c $y = \ln x$ and $y = \ln(x+e)$ **d** $y = \ln x$ and $y = \ln(x-2) - 3$

e $y = 2\ln x$ and $y = \ln(x^2) + 2$

6 Describe a transformation which maps the graph of $y = \log_2 x$ to the graph of $y = \log_5 x$.

7 The logarithmic functions $y = 3 - \log_2(3x+1)$ and $y = \log_2(x+2)$ are graphed alongside.



a Identify which curve is which, giving evidence for your answer.

b Find the axes intercepts and asymptotes of each graph.

c Find the exact coordinates of the point where the graphs intersect.

8 For each of the following functions, find the inverse function f^{-1}, and state the domain and range of f^{-1}:

a $f(x) = 3^x$ **b** $f(x) = 2^{x+1}$ **c** $f(x) = e^{2x}$ **d** $f(x) = 5^x - 3$

9 Suppose $f(x) = be^x$ and $g(x) = \ln(bx)$. Find:

a $(f \circ g)(x)$

b $(g \circ f)(x)$

c the value of x, in terms of b, for which $(f \circ g)(x) = (g \circ f)(x)$.

10 Suppose $f(x) = e^{x+2}$ and $g(x) = \ln x - 3$.

a Find $(f \circ g)(x)$, and state its domain and range.

b Find $(g \circ f)(x)$, and state its domain and range.

11 Let $f(x) = \log_2(1 - 3x)$ and $g(x) = 2x + 1$.

a Find the domain and range of $f(x)$.

b Solve for x:

i $(f \circ g)(x) = 5$

ii $(g \circ f)(x) = 0$

c Find $f^{-1}(x)$, and state its domain and range.

12 Given $f : x \mapsto e^{2x}$ and $g : x \mapsto 2x - 1$, find:

a $(f^{-1} \circ g)(x)$

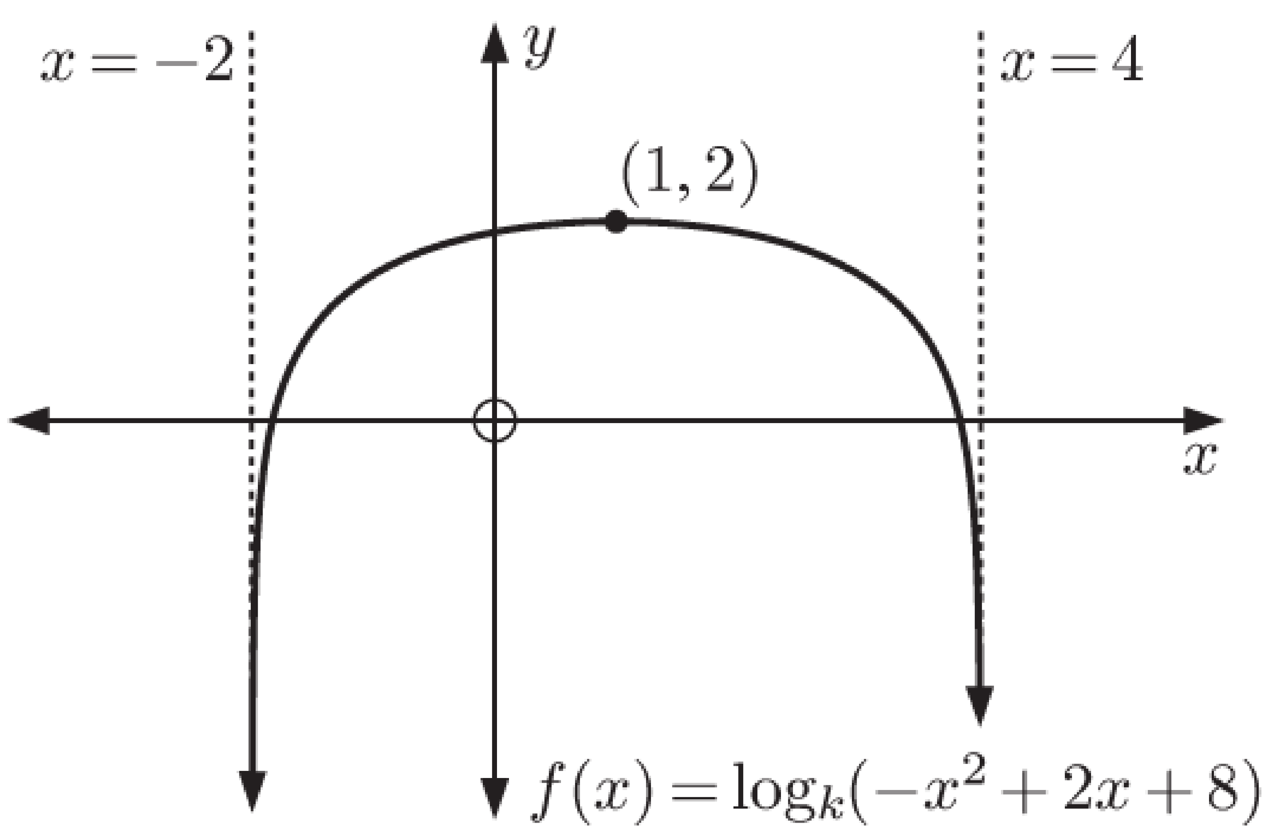
b $(g \circ f)^{-1}(x)$

13 The function $f(x) = \log_k(-x^2 + 2x + 8)$ is graphed alongside.

a Find k.

b Find the axes intercepts of $f(x)$.

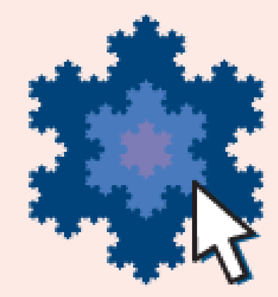
c Let $g(x) = f(x)$ on the restricted domain $-2 < x \leqslant 1$. Find $g^{-1}(x)$, and state its domain and range.



ACTIVITY

Click on the icon to obtain a card game for logarithmic functions.

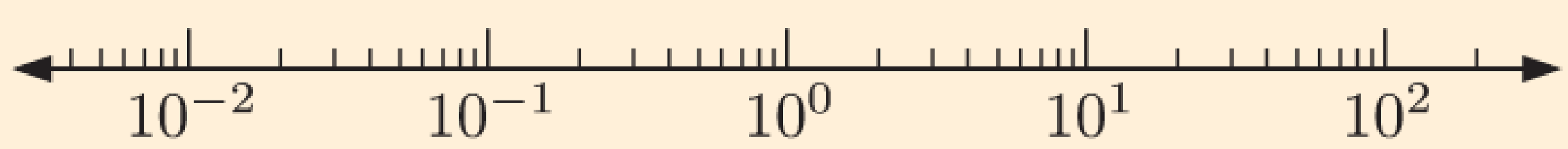
CARD GAME



INVESTIGATION 3 LOGARITHMIC SCALES

In a **logarithmic scale**, equally spaced major tick marks correspond to integer *powers* of a base number. We often call these **orders of magnitude**.

For example, in the logarithmic scale alongside, each major tick mark represents a power of 10.



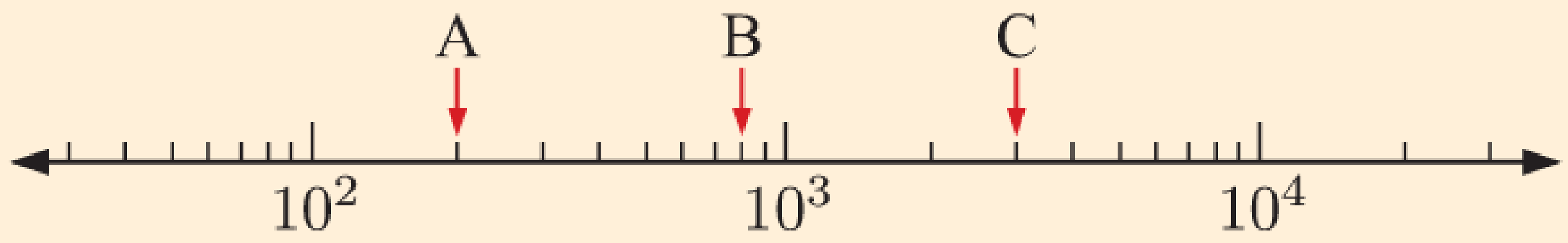
The minor tick marks correspond to integer *multiples* of each power of 10. So the minor tick marks between 10^1 and 10^2 represent 20, 30, 40, ..., and so on.

Logarithmic scales are useful when we want to represent both very large and very small numbers on the same number line. They allow us to compare real world quantities or events which are many orders of magnitude apart.

In this Investigation, we will explore the use of logarithmic scales in a variety of contexts.

What to do:

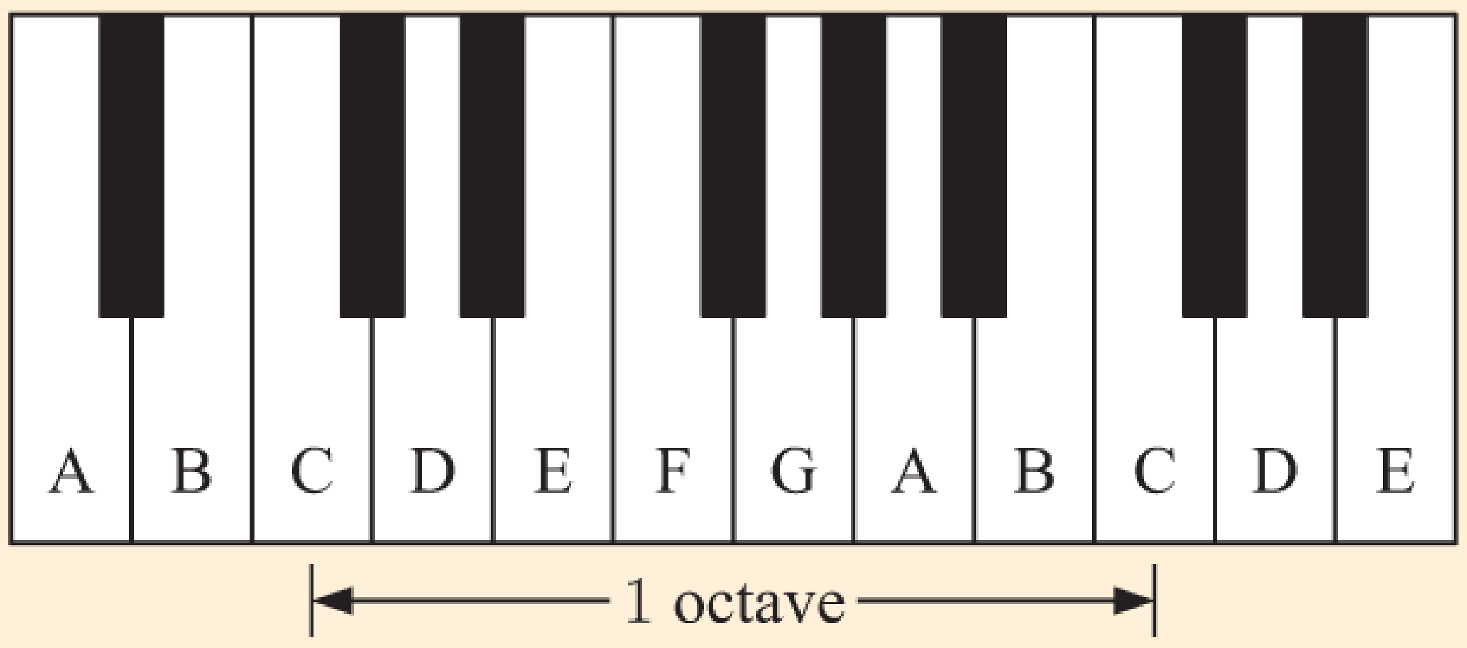
1 **a** For the logarithmic scale alongside, state the values of the points A, B, and C.



b Explain why the minor tick marks in a logarithmic scale are not equally spaced.

c Where is the value 0 on a logarithmic scale? Explain your answer.

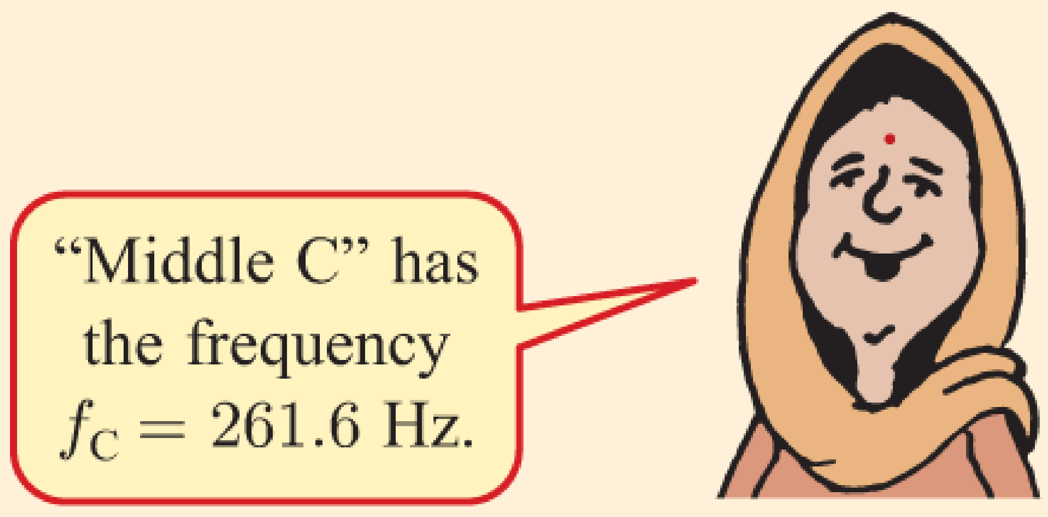
2 Musical notes are named according to the frequency of their sound waves. They are labelled with letters of the alphabet. A note which has *twice* the frequency of another is said to be one **octave** higher than it. So, one C is an octave below the next C.



a How many orders of magnitude apart are the frequencies of two notes separated by 3 octaves?

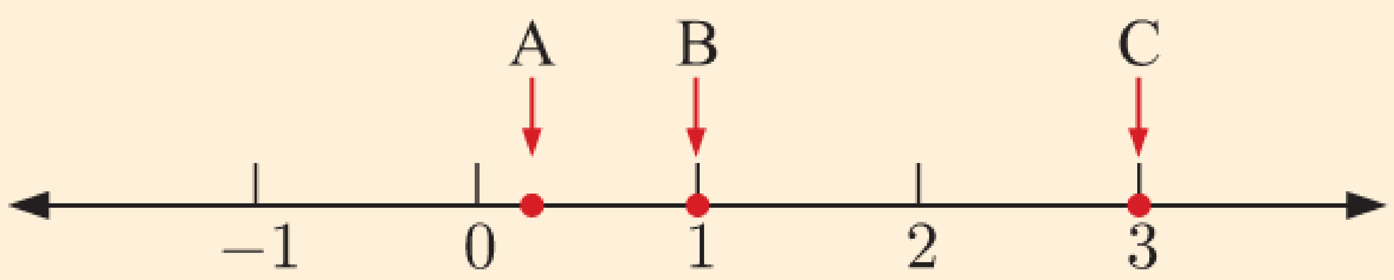
b Write an expression for the frequency of a musical note f, in terms of the number of octaves n *above* middle C.

c There are 12 different notes in an octave. They are equally spaced on the logarithmic scale. Find the ratio of frequencies between two adjacent notes.



3 In some situations, the logarithm is already applied to values placed on the number line. In these cases, the major tick marks represent the *exponents* rather than the numbers themselves.

For example, suppose the scale alongside is logarithmic with base 10. The major tick mark "2" represents the value 10^2, the major tick mark "3" represents the value 10^3, and so on.



a How many times larger is the value at C than the value at B?

b Estimate the position on the scale representing the value:

i 10 times smaller than A **ii** twice as large as B.

4 Earthquakes can range from microscopic tremors to huge natural disasters. The magnitude of earthquakes is measured on the **Richter scale** which relates to the energy released by the earthquake. For this logarithmic scale, the logarithm is part of the formula. It is calculated as $M = \log\left(\dfrac{I}{I_0}\right)$, where I is the earthquake intensity and I_0 is a reference intensity level.

a What does it mean for a tremor to have magnitude:

i 0 **ii** 1?

b Explain why an earthquake of magnitude 6 is *not* twice as intense as a magnitude 3 earthquake.

c Find the magnitude of an earthquake which has half the intensity of a magnitude 4 earthquake.

5 The *acidity* of a solution is determined by the concentration of hydronium ions (H_3O^+). The higher the concentration of H_3O^+, the more acidic it is. The opposite of acidic is *alkaline*.

a In extremely acidic solutions, the concentration of H_3O^+ is typically more than 10^{-3} units. In very alkaline solutions, it is usually less than 10^{-12} units.
Explain why a logarithmic scale would be useful in describing the acidity of a solution.

b In chemistry, the **pH** scale is used to measure acidity. The pH of a solution is given by $\text{pH} = -\log C$, where C is the concentration of H_3O^+. Find:

i the pH of a solution with H_3O^+ concentration 0.000 234 units

ii the H_3O^+ concentration in a solution with pH 7.

c Is it possible for a solution to have a negative pH? Explain what this means in terms of the concentration of H_3O^+.

6 Research the use of **decibels** in acoustics as a unit of measurement for loudness of sound. Compare the use of decibels to the scales in questions **4** and **5**.

REVIEW SET 3A

1 Find:

a $\log \sqrt{10}$ **b** $\log\left(\frac{1}{\sqrt[3]{10}}\right)$ **c** $\log(10^a \times 10^{b+1})$

2 Find:

a $\log_4 64$ **b** $\log_2 256$ **c** $\log_2(0.25)$ **d** $\log_{25} 5$

e $\log_8 1$ **f** $\log_{81} 3$ **g** $\log_9\left(\frac{1}{9}\right)$ **h** $\log_k \sqrt{k}$

3 Use your calculator to evaluate, correct to 3 decimal places:

a $\log 27$ **b** $\log(0.58)$ **c** $\log 400$ **d** $\ln 40$

4 If $y = \log_3 \sqrt{2-x}$, write x in terms of y.

5 Simplify:

a $4\ln 2 + 2\ln 3$ **b** $\frac{1}{2}\ln 9 - \ln 2$ **c** $2\ln 5 - 1$ **d** $\frac{1}{4}\ln 81$

6 Write as a single logarithm:

a $\log 16 + 2\log 3$ **b** $\log_2 16 - 2\log_2 3$ **c** $2 + \log_4 5$

7 Suppose $A = \log_5 2$ and $B = \log_5 3$. Write in terms of A and B:

a $\log_5 36$ **b** $\log_5 54$ **c** $\log_5\left(8\sqrt{3}\right)$

d $\log_5\left(\sqrt{6}\right)$ **e** $\log_5(20.25)$ **f** $\log_5\left(\frac{8}{9}\right)$

8 Write as a logarithmic equation:

a $M = ab^n$ **b** $T = \frac{5}{\sqrt{l}}$ **c** $G = \frac{a^2 b}{c}$

9 Solve for x:

a $3^x = 300$ **b** $30 \times 5^{1-x} = 0.15$ **c** $3^{x+2} = 2^{1-x}$

10 Solve exactly for x:

a $e^{2x} = 3e^x$ **b** $e^{2x} - 7e^x + 12 = 0$

11 Write without logarithms:

a $\ln P = 1.5 \ln Q + \ln T$ **b** $\ln M = 1.2 - 0.5 \ln N$

12 Solve for x:

a $3e^x - 5 = -2e^{-x}$ **b** $2\ln x - 3\ln\left(\frac{1}{x}\right) = 10$

13 Find x if:

a $\log_2 x = -3$ **b** $\log_5 x \approx 2.743$ **c** $\log_3 x \approx -3.145$

14 Solve for x: **i** exactly **ii** rounded to 2 decimal places.

a $2^x = 50$ **b** $7^x = 4$ **c** $(0.6)^x = 0.01$

15 Suppose $\log_a b = x$. Find, in terms of x, the value of $\log_a\left(\frac{1}{b}\right)$.

16 Write $\frac{8}{\log_5 9}$ in the form $a\log_3 b$.

17 Show that the solution to $16^x - 5 \times 8^x = 0$ is $x = \log_2 5$.

18 Solve for x, giving exact answers:

a $\ln x = 5$ **b** $3\ln x + 2 = 0$ **c** $e^x = 400$

d $e^{2x+1} = 11$ **e** $25e^{\frac{x}{2}} = 750$

19 Consider $f(x) = e^{3x-4} + 1$.

a Show that $f^{-1}(x) = \frac{\ln(x-1)+4}{3}$.

b Calculate $f^{-1}(8) - f^{-1}(3)$. Give your answer in the form $a\ln b$, where $a, b \in \mathbb{Q}^+$.

20 Solve simultaneously: $x = 16y$ and $\log_y x - \log_x y = \frac{8}{3}$.

21 Consider the function $g : x \mapsto \log_3(x+2) - 2$.

a State the transformation which maps $y = \log_3 x$ to $y = g(x)$.

b Find the domain and range.

c Find any asymptotes and axes intercepts for the graph of the function.

d Find the inverse function g^{-1}.

e Sketch the graphs of g, g^{-1}, and $y = x$ on the same set of axes.

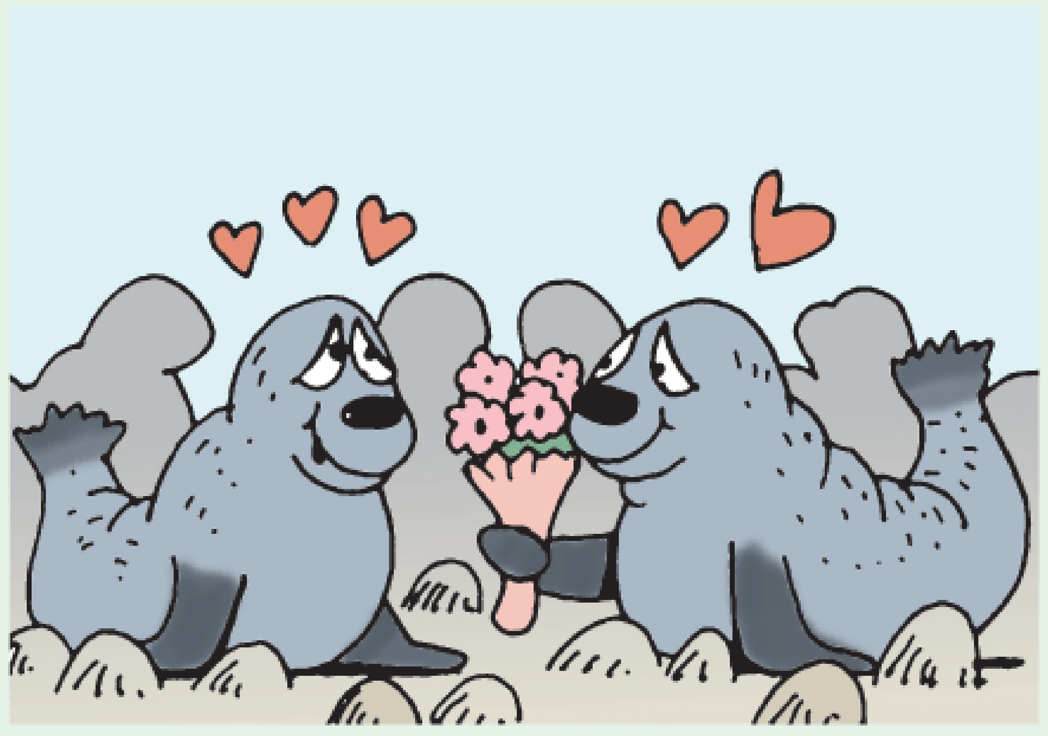
22 The weight of a radioactive isotope remaining after t weeks is given by $W_t = 8000 \times e^{-\frac{t}{20}}$ grams. Find the time for the weight to:

a halve **b** reach 1000 g **c** reach 0.1% of its original value.

23 A population of seals is given by $P(t) = 80 \times (1.15)^t$ where t is the time in years, $t \geqslant 0$.

a Find the time required for the population to double in size.

b Find the percentage increase in population during the first 4 years.



24 For each of the following functions:

i State the domain and range.
ii Find any asymptotes and axes intercepts.
iii Sketch the graph of the function, showing all important features.

a $f(x) = \log_2(x+4) - 1$ **b** $f(x) = \ln x + 2$

25 Draw, on the same set of axes, the graphs of:

a $y = \ln x$ and $y = \ln(x-3)$ **b** $y = \ln x$ and $y = \frac{1}{2}\ln x$

26 Consider $f(x) = e^x$ and $g(x) = \ln(x+4)$, $x > -4$. Find:

a $(f \circ g)(5)$ **b** $(g \circ f)(0)$

REVIEW SET 3B

1 Without using a calculator, find the base 10 logarithms of:

a $\sqrt{1000}$ **b** $\dfrac{10}{\sqrt[3]{10}}$ **c** $\dfrac{10^a}{10^{-b}}$

2 Find:

a $\log_2 128$ **b** $\log_3\left(\frac{1}{27}\right)$ **c** $\log_5\left(\frac{1}{\sqrt{5}}\right)$

3 Write in the form 10^x, giving x correct to 4 decimal places:

a 32 **b** 0.0013 **c** 8.963×10^{-5}

4 Find:

a $\ln(e\sqrt{e})$ **b** $\ln\left(\dfrac{1}{e^3}\right)$ **c** $\ln(e^{2x})$ **d** $\ln\left(\dfrac{e}{e^x}\right)$

5 Simplify:

a $\dfrac{\log_2 25}{\log_2 125}$ **b** $\dfrac{\log 64}{\log 32}$ **c** $\dfrac{\log_5 81}{\log_5 \sqrt{3}}$

6 Simplify:

a $e^{4\ln x}$ **b** $\ln(e^5)$ **c** $\ln(\sqrt{e})$

d $10^{\log x + \log 3}$ **e** $\ln\left(\dfrac{1}{e^x}\right)$ **f** $\dfrac{\log(x^2)}{\log_3 9}$

7 Write in the form e^x, where x is correct to 4 decimal places:

a 20 **b** 3000 **c** 0.075

8 Solve for x: **i** exactly **ii** rounded to 2 decimal places.

a $5^x = 7$ **b** $2^x = 0.1$

9 Write as a single logarithm:

a $\ln 60 - \ln 20$ **b** $\ln 4 + \ln 1$ **c** $\ln 200 - \ln 8 + \ln 5$

10 Solve for x, giving exact answers:

a $e^{2x} = 70$ **b** $3 \times (1.3)^x = 11$ **c** $5 \times 2^{0.3x} = 16$

11 What is the only value of x for which $\log x = \ln x$?

12 Show that the equation $\log_3(x-k)+\log_3(x+2)=1$ has a real solution for every real value of k.

13 Write as a logarithmic equation:

a $P=3\times b^x$ **b** $m=\dfrac{n^3}{p^2}$

14 Show that $\log_3 7\times 2\log_7 x = 2\log_3 x$.

15 Solve for x:

a $\log_2(x^2)+\log_8\sqrt{x}=3$ **b** $\log_{27}\left(\frac{1}{x}\right)+\log_3(x^4)=\log_3 10$

16 Write the following equations without logarithms:

a $\log T=2\log x-\log y$ **b** $\log_2 K=\log_2 n+\frac{1}{2}\log_2 t$

17 Write in the form $a\ln k$ where a and k are positive whole numbers and k is prime:

a $\ln 32$ **b** $\ln 125$ **c** $\ln 729$

18 Copy and complete:

	$y=\log_2 x$	$y=\ln(x+5)$
Domain		
Range		

19 **a** Factorise 4^x-2^x-20 in the form $(2^x+a)(2^x-b)$ where $a, b\in\mathbb{Z}^+$.

b Hence find the exact solution of $2^x(2^x-1)=20$.

c Suppose $p=\log_5 2$.

i Write the solution to **b** in terms of p.

ii Find the solution to $8^x=5^{1-x}$ in terms of p only.

20 Consider $g: x\mapsto 2e^x-5$.

a Find the inverse function g^{-1}.

b Sketch the graphs of g and g^{-1} on the same set of axes.

c State the domain and range of g and g^{-1}.

d State the asymptotes and intercepts of g and g^{-1}.

21 The temperature of a mug of water t minutes after it has been poured from a kettle is given by $T=60e^{-0.1t}+20$ °C.
Show that it will take $10\ln 3$ minutes for the temperature of the water to fall to 40°C.

22 The weight of a radioactive isotope after t years is given by $W(t)=2500\times 3^{-\frac{t}{3000}}$ grams.

a Find the initial weight of the isotope.

b Find the time taken for the isotope to reduce to 30% of its original weight.

c Find the percentage of weight lost after 1500 years.

23 Solve for x, giving an exact answer:

a $5^{\frac{x}{2}} = 9$ **b** $e^x = 30$ **c** $e^{1-3x} = 2$

24 Draw, on the same set of axes, the graphs of:

a $y = \ln x$ and $y = \ln(x+2)$ **b** $y = \ln x$ and $y = \ln(ex)$

25 *Hick's law* models the time taken for a person to make a selection from a number of possible options.

For a particular person, Hick's law determines that the time taken to choose between n equally probable choices is $T = 2\ln(n+1)$ seconds.

a Sketch the graph of T against n for $0 \leqslant n \leqslant 50$.

b How long will it take this person to choose between:

i 5 possible choices **ii** 15 possible choices?

c If the number of possible choices increases from 20 to 40, how much longer will the person take to make a selection?

26 Let $f(x) = \log_3 x - 2$ and $g(x) = 3 - \sqrt{x}$.

a Find the following functions, and state the domain and range of each function:

i $f^{-1}(x)$ **ii** $(f \circ g)(x)$ **iii** $(g \circ f)(x)$

b Solve for x:

i $(f \circ g)(x) = -2$ **ii** $(g \circ f)(x) = 0$

c Find $(g \circ f)^{-1}(x)$, and state its domain and range.