9 Vectors

Assessment statements

4.1 Vectors as displacements in the plane.
Components of a vector; column representation.

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

Algebraic and geometric approaches to the following topics:
the sum and difference of two vectors; the zero vector; the vector $-\mathbf{v}$;
multiplication by a scalar, $k\mathbf{v}$;
magnitude of a vector, $|\mathbf{v}|$;
unit vectors; base vectors, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$;
position vectors $\overrightarrow{OA} = \mathbf{a}$;
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$.

4.2 The scalar product of two vectors.
Properties of the scalar product.
Perpendicular vectors; parallel vectors.
The angle between two vectors.

4.3 Representation of a line as $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.
The angle between two lines.
(See also Chapter 14.)

Introduction

Vectors are an essential tool in physics and a very significant part of mathematics. Historically, their primary application was to represent forces, and the operation called '**vector addition**' corresponds to the combining of various forces. Many other applications in physics and other fields have been found since. In this chapter, we will discuss what vectors are and how to add, subtract and multiply them by scalars; we will also examine why vectors are useful in everyday life and how they are used in real-life applications. Then we will discuss scalar products.

Control panel of a passenger jet cockpit.

9.1 Vectors as displacements in the plane

We can represent physical quantities like temperature, distance, area, speed, density, pressure and volume by a single number indicating magnitude or size. These are called **scalar quantities**. Other physical quantities possess the properties of magnitude and direction. We define the force needed to pull a truck up a 10° slope by its **magnitude** and **direction**. Force, displacement, velocity, acceleration, lift, drag, thrust and weight are quantities that cannot be described by a single number. These are called **vector quantities**. Distance and displacement, for example, have distinctly different meanings; so do speed and velocity. Speed is a scalar quantity that refers to 'how fast an object is moving'.

Velocity is a vector quantity that refers to 'the rate at which an object *changes its position*'. When evaluating the velocity of an object, we must keep track of direction. It would not be enough to say that an object has a velocity of 55 km/h; we must include direction information in order to fully describe the velocity of the object. For instance, you must describe the object's velocity as being 55 km/h east. This is one of the essential differences between speed and velocity. Speed is a **scalar** quantity and does not keep track of direction; velocity is a **vector** quantity and is direction-conscious.

Thus, an aeroplane moving westward with a speed of 600 km/h has a velocity of 600 km/h west. Note that speed has no direction (it is scalar) and velocity, at any instant, is simply the speed with a direction.

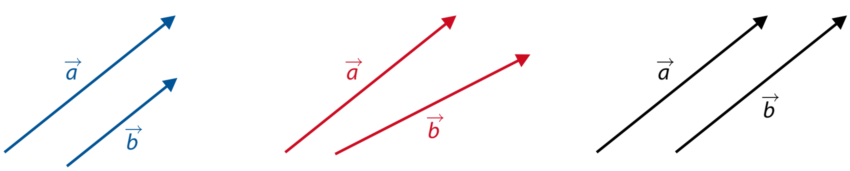
We represent vector quantities with **directed line segments** (Figure 9.1).

The directed line segment $\overrightarrow{AB}$ has **initial point** A and **terminal point** B. We use the notation $\overrightarrow{AB}$ to indicate that the line segment represents a vector quantity. We use $|\overrightarrow{AB}|$ to represent the **magnitude** of the directed line segment. The terms **size**, **length** or **norm** are also used. The direction of $\overrightarrow{AB}$ is from A to B. $\overrightarrow{BA}$ has the same length but the opposite direction to $\overrightarrow{AB}$ and hence cannot be equal to it.

Two directed line segments that have the same magnitude and direction are equivalent. For example, the directed line segments in Figure 9.2 are all equivalent.

We call the set of all directed line segments equivalent to a given directed line segment $\overrightarrow{AB}$ a **vector v**, and write $\mathbf{v} = \overrightarrow{AB}$. We denote vectors by lower-case, boldface letters such as **a**, **u**, and **v**.

We say that two vectors **a** and **b** are equal if their corresponding directed line segments are equivalent.



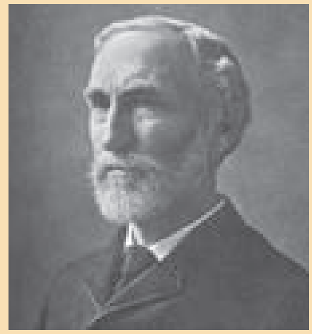
Vectors $\vec{a}$ and $\vec{b}$ have the same direction but different magnitudes $\Rightarrow \vec{a} \neq \vec{b}$.

Vectors $\vec{a}$ and $\vec{b}$ have equal magnitudes but different directions $\Rightarrow \vec{a} \neq \vec{b}$.

Vectors $\vec{a}$ and $\vec{b}$ have equal magnitudes and the same direction $\Rightarrow \vec{a} = \vec{b}$.

Figure 9.3

The notion of vector, as presented here, is due to the mathematician-physicist J. Williard Gibbs (1839–1903) of Yale University. His book *Vector Analysis* (1881) made these ideas accessible to a wide audience.



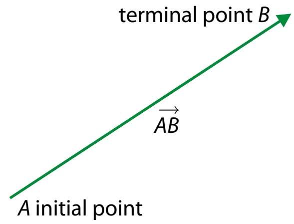


Figure 9.1

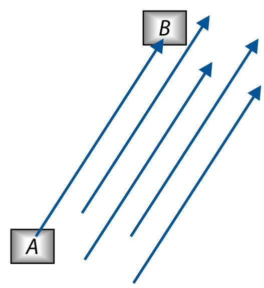


Figure 9.2

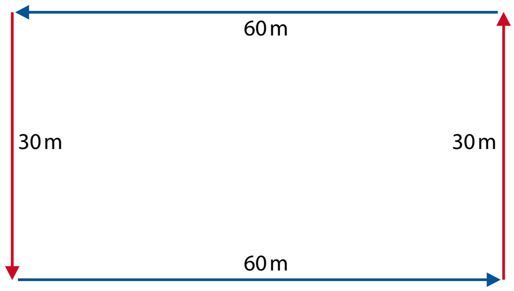
Hint: Note: When we handwrite vectors, we cannot use boldface, so the convention is to use the arrow notation.

Definition 1: Two vectors **u** and **v** are equal if they have the same magnitude and the same direction.

Definition 2: The negative of a vector **u**, denoted by $-$**u**, is a vector with the same magnitude but opposite direction.

Example 1

Marco walked around the park as shown in the diagram. What is Marco's displacement at the end of his walk?



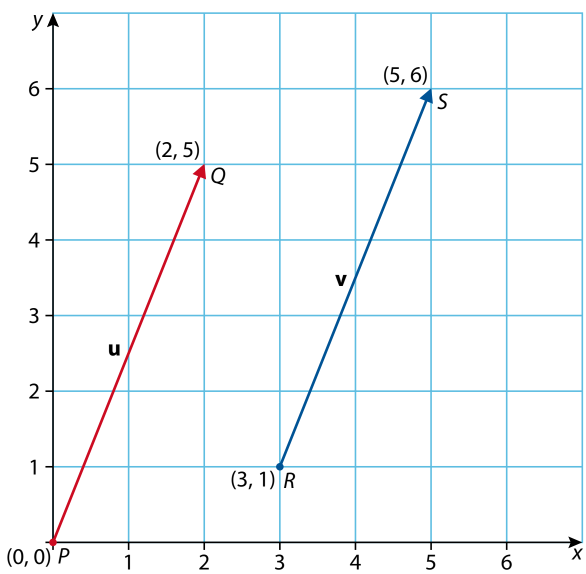
Solution

Even though he walked a total distance of 180 m, his displacement is zero since he returned to his original position. So, his displacement is **0**.

This is a displacement and hence direction is also important, not only magnitude. The 30 m south 'cancelled' the 30 m north, and the 60 m east is cancelled by the 60 m west.

Vectors can also be looked at as displacement/translation in the plane. Take, for example, the directed segments PQ and RS as representing the vectors **u** and **v**, respectively. The points $P(0, 0)$, $Q(2, 5)$, $R(3, 1)$ and $S(5, 6)$ are shown in Figure 9.4.

Figure 9.4



We can prove that these two vectors are equal.

The directed line segments representing the vectors have the same direction, since they both have a slope of $\frac{5}{2}$.

They also have the same magnitude, as:

$$|\overrightarrow{PQ}| = \sqrt{5^2 + 2^2} = \sqrt{29} \text{ and}$$
$$|\overrightarrow{RS}| = \sqrt{(5-3)^2 + (6-1)^2} = \sqrt{29}$$

Component form

The directed line segment with the origin as its initial point is the most convenient way of representing a vector. This representation of the vector is said to be in **standard position**. In Figure 9.4, **u** is in standard position. A vector in standard position can be uniquely represented by the coordinates of its terminal point (u_1, u_2). This is called the **component form of a vector u**, written as $\mathbf{u} = (u_1, u_2)$.

The coordinates u_1 and u_2 are the **components** of the vector **u**. In Figure 9.4, the components of the vector **u** are 2 and 5.

If the initial and terminal points of the vector are the same, the vector is a **zero vector** and is denoted by $\mathbf{0} = (0, 0)$.

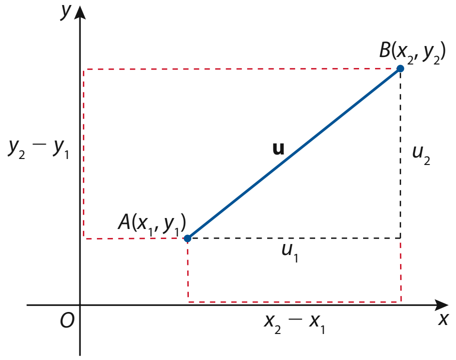
If **u** is a vector in the plane with initial point (0, 0) and terminal point (u_1, u_2), the **component form** of **u** is $\mathbf{u} = (u_1, u_2)$.
Note: The component form is also written as $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.

So, a vector in the plane is also an ordered pair (u_1, u_2) of real numbers. The numbers u_1 and u_2 are the components of **u**. The vector $\mathbf{u} = (u_1, u_2)$ is also called the **position vector** of the point (u_1, u_2).

If the vector **u** is not in standard position and is represented by a directed segment AB, then it can be written in its component form, observing the following fact:

$\mathbf{u} = (u_1, u_2) = (x_2 - x_1, y_2 - y_1)$, where $A(x_1, y_1)$ and $B(x_2, y_2)$ (Figure 9.5).

Figure 9.5



The length of vector **u** can be given using Pythagoras' theorem and/or the distance formula:

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 2

a) Find the components and the length of the vector between the points $P(-2, 3)$ and $Q(4, 7)$.

b) $\overrightarrow{RS}$ is another representation of the vector **u** where $R(7, -3)$. Find the coordinates of S.

Solution

a) $\overrightarrow{PQ} = (4 - (-2), 7 - 3) = (6, 4)$

$|\overrightarrow{PQ}| = \sqrt{36 + 16} = \sqrt{52} = 2\sqrt{13}$

b) Let S have coordinates (x, y). Therefore,

$\overrightarrow{RS} = (x - 7, y + 3)$.

But,

$\overrightarrow{RS} = \overrightarrow{PQ} \Rightarrow x - 7 = 6$ and $y + 3 = 4 \Rightarrow x = 13, y = 1$.

So, S has coordinates $(13, 1)$.

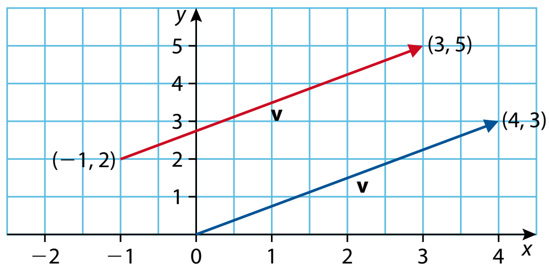
Example 3

The directed segment from $(-1, 2)$ to $(3, 5)$ represents a vector **v.** Find the length of vector **v**, draw the vector in standard position and find the opposite of the vector in component form.

Solution

The length of vector **v** can be found using the distance formula:

$$|\mathbf{v}| = \sqrt{(3 + 1)^2 + (5 - 2)^2} = 5$$



The opposite of this vector can be represented by $-\mathbf{v} = (-4, -3)$.

9.2 Vector operations

Two of the most basic and important operations are scalar multiplication and vector addition.

Scalar multiplication

In working with vectors, numbers are considered scalars. In this discussion, scalars will be limited to real numbers only. Geometrically, the product of a vector **u** and a scalar k, $\mathbf{v} = k\mathbf{u}$, is a vector that is $|k|$ times as long as **u.** If

k is positive, $\mathbf{v}$ has the same direction as $\mathbf{u}$, and when k is negative, $\mathbf{v}$ has the opposite direction to $\mathbf{u}$ (Figure 9.6).

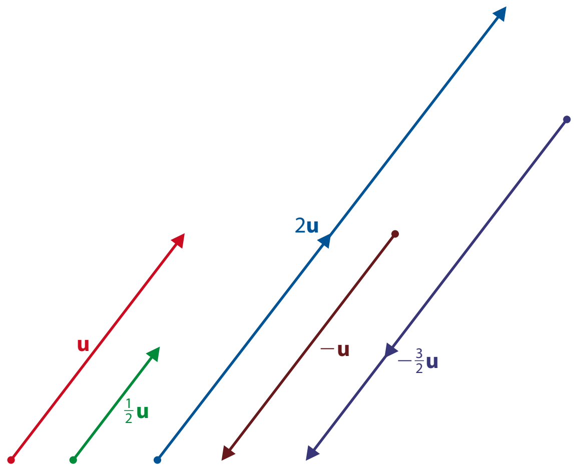


Figure 9.6

Consequence: It becomes clear from this discussion that for two vectors to be parallel, it is necessary and sufficient that one of them is a scalar multiple of the other. That is, if $\mathbf{v}$ and $\mathbf{u}$ are parallel, then $\mathbf{v} = k\mathbf{u}$; and vice versa, if $\mathbf{v} = k\mathbf{u}$, then $\mathbf{v}$ and $\mathbf{u}$ are parallel.

In terms of their components, the operation of scalar multiplication is straightforward.

If $\mathbf{u} = (u_1, u_2)$ then $\mathbf{v} = k\mathbf{u} = k(u_1, u_2) = (ku_1, ku_2)$.

Example 4

Find the magnitude of each vector.

a) $\mathbf{u} = (3, -4)$ b) $\mathbf{v} = (6, -8)$ c) $\mathbf{w} = (7, 0)$ d) $\mathbf{z} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Solution

a) $|\mathbf{u}| = \sqrt{3^2 + 4^2} = 5$

b) $|\mathbf{v}| = \sqrt{6^2 + (-8)^2} = 10$ Notice that $\mathbf{v} = 2\mathbf{u}$ and so $|\mathbf{v}| = 2|\mathbf{u}|$.

c) $|\mathbf{w}| = \sqrt{7^2 + 0^2} = 7$

d) $|\mathbf{z}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$ This is also called a unit vector as you will see later.

Vector addition

There are two equivalent ways of looking at the addition of vectors geometrically. One is the triangular method and the other is the parallelogram method.

Let $\mathbf{u}$ and $\mathbf{v}$ denote two vectors. Draw the vectors such that the terminal point of $\mathbf{u}$ and initial point of $\mathbf{v}$ coincide. The vector joining the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$ is the sum (resultant) of vectors $\mathbf{u}$ and $\mathbf{v}$ and is denoted by $\mathbf{u} + \mathbf{v}$ (Figure 9.7).

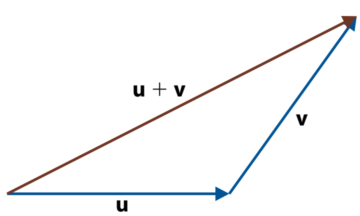


Figure 9.7

Another equivalent way of looking at the sum also gives us the grounds to say that vector addition is commutative.

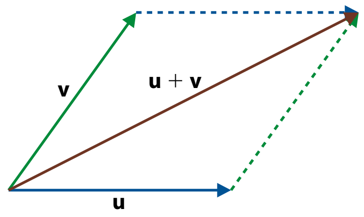


Figure 9.8

Let **u** and **v** denote two vectors. Draw the vectors such that the initial point of **u** and initial point of **v** coincide. The vector joining the common initial point of **u** and **v** to the opposite corner of the parallelogram, formed by the vectors as its adjacent sides, is the sum (resultant) of vectors **u** and **v** and is denoted by **u** + **v** (Figure 9.8).

The difference of two vectors is an extremely important rule that will be used later in the chapter.

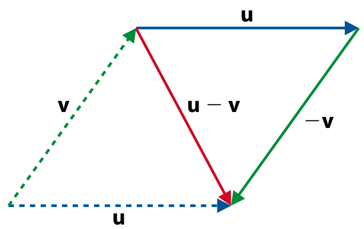


Figure 9.9

As Figure 9.9 shows, it is an extension of the addition rule. An easy way of looking at it is through a combination of the parallelogram rule and the triangle rule. We draw the vectors **u** and **v** in the usual way, then we draw −**v** starting at the terminal point of **u** and we add **u** + (−**v**) to get the difference **u** − **v**. As it turns out, the difference of the two vectors **u** and **v** is the diagonal of the parallelogram with its initial point the terminal of **v** and its terminal point the terminal point of **u**.

Example 5

Consider the vectors **u** = (2, −3) and **w** = (1, 3).

a) Write down the components of **v** = 2**u**.

b) Find |**u**| and |**v**| and compare them.

c) Draw the vectors **u**, **v**, **w**, 2**w**, **u** + **w**, **v** + 2**w**, **u** − **w**, **v** − 2**w**.

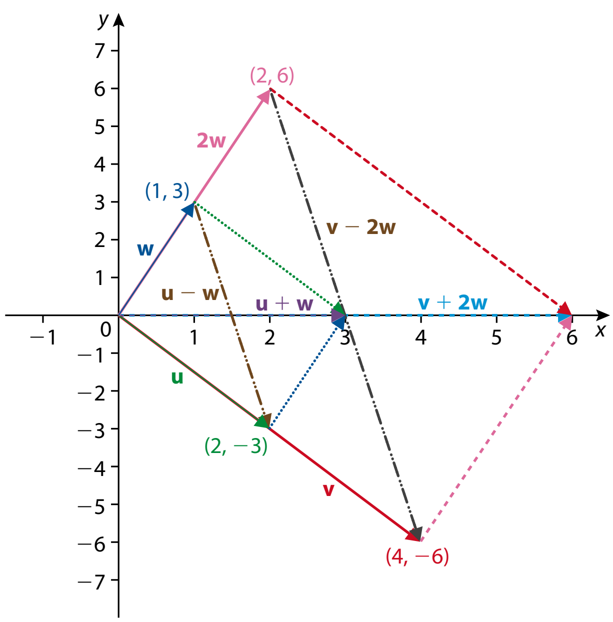
d) Comment on the results of c) above.

Solution

a) $\mathbf{v} = 2(2, -3) = (4, -6)$

b) $|\mathbf{u}| = \sqrt{4 + 9} = \sqrt{13}$, $|\mathbf{v}| = \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13}$. Clearly, $|\mathbf{v}| = 2|\mathbf{u}|$.

c)



d) We observe that **u** + **w** = (3, 0) which turns out to be (1 + 2, 3 − 3), the sum of the corresponding components. We observe the same for **v** + 2**w** = (6, 0)**,** which in turn is (2 + 4, 6 − 6).

We also observe that $\mathbf{v} + 2\mathbf{w} = 2\mathbf{u} + 2\mathbf{w} = 2(\mathbf{u} + \mathbf{w})$, and

$\mathbf{v} - 2\mathbf{w}$ is parallel to $\mathbf{u} - \mathbf{w}$ and is twice its length!

Can you draw more observations?

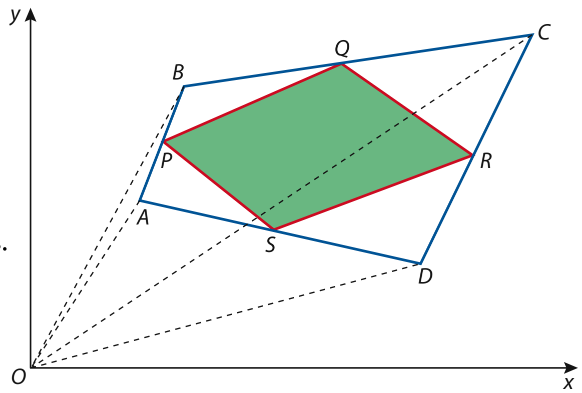
Example 6

ABCD is a quadrilateral with vertices that have position vectors **a**, **b**, **c**, and **d** respectively. *P*, *Q*, *R*, and *S* are the midpoints of the sides.

a) Express each of the following in terms of **a**, **b**, **c**, and **d**:

$\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{AP}$, and $\overrightarrow{OP}$

b) Prove that *PQRS* is a parallelogram using vector methods.



Solution

a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \mathbf{d} - \mathbf{c}$

$\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}(\mathbf{b} + \mathbf{a})$

b) One way of proving *PQRS* is a parallelogram is to show a pair of opposite sides parallel and congruent.

You can show that $\overrightarrow{OQ} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$, $\overrightarrow{OR} = \frac{1}{2}(\mathbf{d} + \mathbf{c})$, and $\overrightarrow{OS} = \frac{1}{2}(\mathbf{d} + \mathbf{a})$ as we did for $\overrightarrow{OP}$.

Now, $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \frac{1}{2}(\mathbf{b} + \mathbf{c}) - \frac{1}{2}(\mathbf{b} + \mathbf{a}) = \frac{1}{2}(\mathbf{c} - \mathbf{a})$, and

$\overrightarrow{SR} = \overrightarrow{OR} - \overrightarrow{OS} = \frac{1}{2}(\mathbf{d} + \mathbf{c}) - \frac{1}{2}(\mathbf{d} + \mathbf{a}) = \frac{1}{2}(\mathbf{c} - \mathbf{a})$.

Therefore, $\overrightarrow{PQ} = \overrightarrow{SR}$, and since they are opposite sides of the quadrilateral, so it is a parallelogram.

Base vectors in the coordinate plane

As you have seen before, vectors can also be represented in a coordinate system using their component form. This is a very useful tool that helps make many applications of vectors simple and easy. At the heart of the component approach to vectors we find the 'base' vectors **i** and **j**.

i is a vector of magnitude 1 with the direction of the positive *x*-axis and **j** is a vector of magnitude 1 with the direction of the positive *y*-axis. These vectors and any vector that has a magnitude of 1 are called **unit vectors**. Since vectors of same direction and length are equal, each vector **i** and **j** may be drawn at any point in the plane, but it is usually more convenient to draw them at the origin, as shown in Figure 9.10.

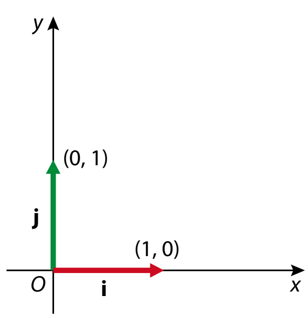


Figure 9.10

Now, the vector $k\mathbf{i}$ has magnitude k and is parallel to the vector **i**. Similarly, the vector $m\mathbf{j}$ has magnitude m and is parallel to **j**.

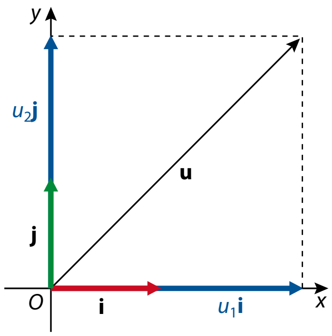


Figure 9.11

Consider the vector $\mathbf{u} = (u_1, u_2)$. This vector, in standard position, has an ***x*-component** u_1 and ***y*-component** u_2 (Figure 9.11).

Since the vector **u** is the diagonal of the parallelogram with adjacent sides $u_1\mathbf{i}$ and $u_2\mathbf{j}$, then it is the sum of the two vectors, i.e. $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$. It is customary to say that $u_1\mathbf{i}$ is the **horizontal component** and $u_2\mathbf{j}$ is the **vertical component** of **u**.

The previous discussion shows that it is always possible to express any vector in the plane as a linear combination of the unit vectors **i** and **j**.

This form of representation of vectors opens the door to a rich world of vector applications.

Vector addition and subtraction in component form

If vector **u** has components (u_1, u_2), then its component form is: $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$

Consider the two vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$.

(i) Vector sum $\mathbf{u} + \mathbf{v}$

$$\mathbf{u} + \mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j}) + (v_1\mathbf{i} + v_2\mathbf{j}) = (u_1\mathbf{i} + v_1\mathbf{i}) + (u_2\mathbf{j} + v_2\mathbf{j})$$
$$= (u_1 + v_1)\mathbf{i} + (u_2 + v_2)\mathbf{j}$$

For example, to add the two vectors $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} - 3\mathbf{j}$, it is enough to add the corresponding components:

$$\mathbf{u} + \mathbf{v} = (2 + 5)\mathbf{i} + (4 - 3)\mathbf{j} = 7\mathbf{i} + \mathbf{j}$$

(ii) Vector difference $\mathbf{u} - \mathbf{v}$

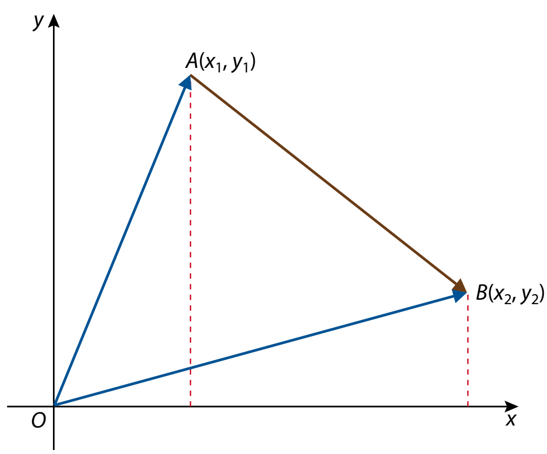
$$\mathbf{u} - \mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j}) - (v_1\mathbf{i} + v_2\mathbf{j}) = (u_1\mathbf{i} - v_1\mathbf{i}) + (u_2\mathbf{j} - v_2\mathbf{j})$$
$$= (u_1 - v_1)\mathbf{i} + (u_2 - v_2)\mathbf{j}$$

For example, to subtract the two vectors $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} - 3\mathbf{j}$, it is enough to subtract the corresponding components:

$$\mathbf{u} - \mathbf{v} = (2 - 5)\mathbf{i} + (4 + 3)\mathbf{j} = -3\mathbf{i} + 7\mathbf{j}$$

This interpretation of the difference gives us another way of finding the components of any vector in the plane, even if it is not in standard position (Figure 9.12).

Figure 9.12



Consider the vector $\overrightarrow{AB}$ where the position vectors of its endpoints are given by the vectors $\overrightarrow{OA} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\overrightarrow{OB} = x_2\mathbf{i} + y_2\mathbf{j}$.

As we have seen in section 9.1, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$.
This result was given in Section 9.1 as a definition.

- Many of the laws of ordinary algebra are also valid for vector algebra. These laws are:
  - Commutative law for addition: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
  - Associative law for addition: $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

    The verification of the associative law is shown in Figure 9.13.

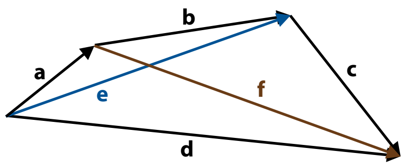


Figure 9.13

If we add **a** and **b** we get a vector **e**. And similarly, if **b** is added to **c**, we get **f**.

Now $\mathbf{d} = \mathbf{e} + \mathbf{c} = \mathbf{a} + \mathbf{f}$. Replacing **e** with $(\mathbf{a} + \mathbf{b})$ and **f** with $(\mathbf{b} + \mathbf{c})$, we get $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ and we see that the law is verified.

  - Commutative law for multiplication: $m\mathbf{a} = \mathbf{a}m$
  - Distributive law (1): $(m + n)\mathbf{a} = m\mathbf{a} + n\mathbf{a}$, where m and n are two different scalars.
  - Distributive law (2): $m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}$

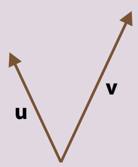
These laws allow the manipulation of vector quantities in much the same way as ordinary algebraic equations.

Two vectors **u** and **v** are parallel iff $\mathbf{v} = k\mathbf{u}$. This also means that in component form:

$$\frac{v_1}{u_1} = \frac{v_2}{u_2} = k$$

Exercise 9.1 and 9.2

1 Consider the vectors **u** and **v** given.
Sketch each indicated vector.
a) $2\mathbf{u}$
b) $-\mathbf{v}$
c) $\mathbf{u} + \mathbf{v}$
d) $2\mathbf{u} - \mathbf{v}$
e) $\mathbf{v} - 2\mathbf{u}$



For questions 2–5, consider the points A and B given and answer the following questions:
a) Find $|\overrightarrow{AB}|$.
b) Find the components of the vector $\mathbf{u} = \overrightarrow{AB}$ and sketch it in standard position.
c) Write the vector $\mathbf{v} = \frac{1}{|\overrightarrow{AB}|} \cdot \mathbf{u}$ in component form.
d) Find $|\mathbf{v}|$.
e) Sketch the vector **v** and compare it to **u**.

2 $A(3, 4)$ and $B(7, -1)$

3 $A(-2, 3)$ and $B(5, 1)$

4 $A(3, 5)$ and $B(0, 5)$

5 $A(2, -4)$ and $B(2, 1)$

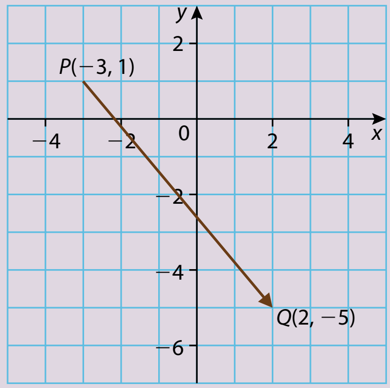
6 Consider the vector shown.

a) Write down the component representation of the vector.

b) Find the length of the vector.

c) Sketch the vector in standard position.

d) Find a vector equal to this one with initial point $(-1, 1)$.



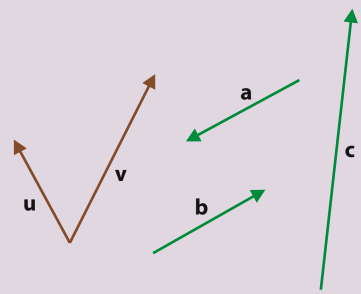
For questions 7–9, the initial point P and terminal point Q are given. Answer the same questions as in question 6.

7 $P(3, 2)$, $Q(7, 8)$

8 $P(2, 2)$, $Q(7, 7)$

9 $P(-6, -8)$, $Q(-2, -2)$

10 Which of the vectors **a**, **b**, or **c** in the figure shown right is equivalent to $\mathbf{u} - \mathbf{v}$?
Which is equivalent to $\mathbf{v} + \mathbf{u}$?



11 Find the terminal point of $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ if the initial point is $(-2, 1)$.

12 Find the initial point of $\mathbf{v} = (-3, 1)$ if the terminal point is $(5, 0)$.

13 Find the terminal point of $\mathbf{v} = (6, 7)$ if the initial point is $(-2, 1)$.

14 Find the initial point of $\mathbf{v} = 2\mathbf{i} + 7\mathbf{j}$ if the terminal point is $(-3, 2)$.

15 Consider the vectors $\mathbf{u} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{v} = -\mathbf{i} + 3\mathbf{j}$.

a) Find $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $2\mathbf{u} + 3\mathbf{v}$ and $2\mathbf{u} - 3\mathbf{v}$.

b) Find $|\mathbf{u} + \mathbf{v}|$, $|\mathbf{u} - \mathbf{v}|$, $|\mathbf{u}| + |\mathbf{v}|$ and $|\mathbf{u}| - |\mathbf{v}|$.

c) Find $|2\mathbf{u} + 3\mathbf{v}|$, $|2\mathbf{u} - 3\mathbf{v}|$, $2|\mathbf{u}| + 3|\mathbf{v}|$ and $2|\mathbf{u}| - 3|\mathbf{v}|$.

16 Let $\mathbf{u} = (1, 5)$ and $\mathbf{v} = (3, -4)$. Find the vector $\mathbf{x}$ such that $2\mathbf{u} - 3\mathbf{x} + \mathbf{v} = 5\mathbf{x} - 2\mathbf{v}$.

17 Find **u** and **v** if $\mathbf{u} - 2\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{u} + 3\mathbf{v} = \mathbf{i} + \mathbf{j}$.

18 Find the lengths of the diagonals of the parallelogram whose sides are the vectors $2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{i} + \mathbf{j}$.

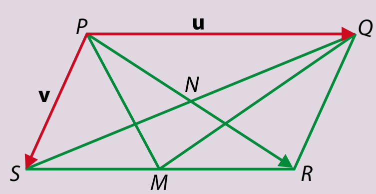
19 Vectors **u** and **v** form two sides of parallelogram $PQRS$, as shown. Express each of the following vectors in terms of **u** and **v**.

a) $\overrightarrow{PR}$

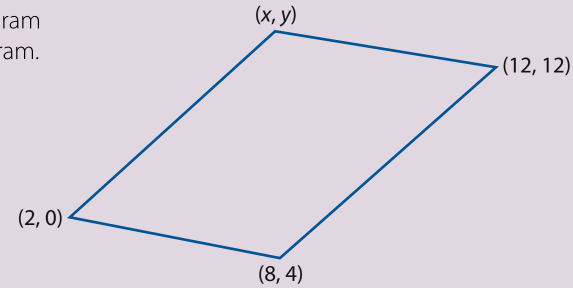
b) $\overrightarrow{PM}$, where M is the midpoint of $[RS]$

c) $\overrightarrow{QS}$

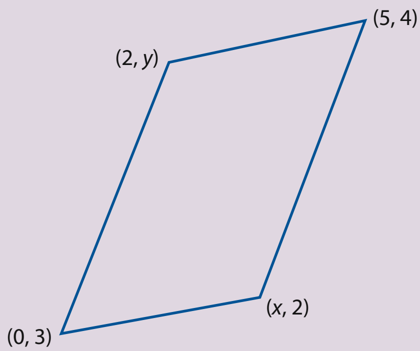
d) $\overrightarrow{QN}$



20 Find (x, y) so that the diagram at the right is a parallelogram.



21 Find x and y in the parallelogram shown right.



22 Find the scalars r and s such that

$\begin{pmatrix} 8 \\ 46 \end{pmatrix} = r\begin{pmatrix} 1 \\ 9 \end{pmatrix} + s\begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

Note: $\begin{pmatrix} 8 \\ 46 \end{pmatrix}$ is said to be written as a linear combination of $\begin{pmatrix} 1 \\ 9 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

23 Write (4, 7) as a linear combination of (2, 3) and (2, 1).

24 Write (5, −5) as a linear combination of (1, −1) and (−1, 1).

25 Write (−11, 0) as a linear combination of (2, 5) and (3, 2).

26 Let $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $\mathbf{v} = -\mathbf{i} + \mathbf{j}$. Show that, if $\mathbf{w}$ is any vector in the plane, then it can be written as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. (You can generalize the result to any two non-zero, non-parallel vectors $\mathbf{u}$ and $\mathbf{v}$.)

9.3 Unit vectors and direction angles

Consider the vector $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$. To find the magnitude of this vector, $|\mathbf{u}|$, we use the distance formula:

$$|\mathbf{u}| = \sqrt{3^2 + 4^2} = 5$$

If we divide the vector $\mathbf{u}$ by $|\mathbf{u}| = 5$, i.e. we multiply the vector $\mathbf{u}$ by the reciprocal of its magnitude, we get another vector that is parallel to $\mathbf{u}$, since they are scalar multiples of each other. The new vector is

$$\frac{\mathbf{u}}{5} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

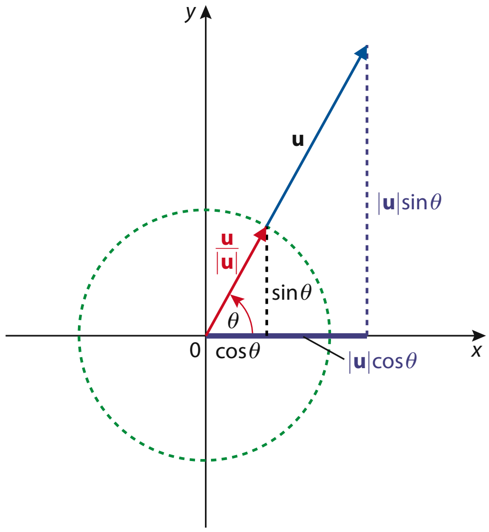
This vector is a unit vector in the same direction as $\mathbf{u}$, because

$$\left|\frac{\mathbf{u}}{5}\right| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1$$

Therefore, to find a unit vector in the same direction as a given vector, we divide that vector by its own magnitude.

This is tightly connected to the concept of the **direction angle** of a given vector. The **direction angle** of a vector (in standard position) is the angle it makes with the positive x-axis (Figure 9.14).

Figure 9.14



So, the vector **u** can be expressed in terms of the unit vector parallel to it in the following manner:

$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} = (|\mathbf{u}|\cos\theta)\mathbf{i} + (|\mathbf{u}|\sin\theta)\mathbf{j} = |\mathbf{u}|(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$ where $u_1 = |\mathbf{u}|\cos\theta$ and $u_2 = |\mathbf{u}|\sin\theta$. This fact implies two important tools that help us:

1. find the direction of a given vector
2. find vectors of any magnitude parallel to a given vector.

To find a unit vector parallel to a vector **u**, we simply find the vector $\frac{\mathbf{u}}{|\mathbf{u}|}$:

$$\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{\mathbf{u}}{\sqrt{u_1^2 + u_2^2}} = \left(\frac{u_1}{\sqrt{u_2^1 + u_2^2}}, \frac{u_2}{\sqrt{u_2^1 + u_2^2}}\right)$$

Applications of unit vectors and direction angles

Given a vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$, find the direction angle of this vector and another vector, whose magnitude is m, that is parallel to the vector **u.**

1. To help determine the direction angle, we observe the following:

   $u_1 = |\mathbf{u}|\cos\theta$ and $u_2 = |\mathbf{u}|\sin\theta$

   This implies that $\frac{u_2}{u_1} = \frac{|\mathbf{u}|\sin\theta}{|\mathbf{u}|\cos\theta} = \tan\theta.$

   So, $\tan^{-1}\theta$ is the reference angle for the direction angle in question. To know what the direction angle is, it is best to look at the numbers u_1 and u_2 in order to determine which quadrant the vector is in. The following example (Example 6) will clarify this point.

2. To find a vector of magnitude m parallel to **u**, we must first find the unit vector in the direction of **u** and then we multiply it by the scalar m.

   The unit vector in the direction of **u** is $\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{|\mathbf{u}|}(u_1\mathbf{i} + u_2\mathbf{j})$, and the vector of magnitude m in this direction will be

   $$m\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{m}{\sqrt{u_1^2 + u_2^2}}(u_1\mathbf{i} + u_2\mathbf{j}).$$

Example 7

Find the direction angle (to the nearest degree) of each vector, and find a vector of magnitude 7 that is parallel to each.

a) $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j}$

b) $\mathbf{v} = -3\mathbf{i} + 3\mathbf{j}$

c) $\mathbf{w} = 3\mathbf{i} - 4\mathbf{j}$

Solution

a) The direction angle for **u** is θ, as shown in Figure 9.15.

$$\tan\theta = \frac{2}{2} = 1 \Rightarrow \theta = 45°$$

A vector of magnitude 7 that is parallel to **u** is

$$7\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{7}{\sqrt{2^2 + 2^2}}(2\mathbf{i} + 2\mathbf{j}) = \frac{7}{2\sqrt{2}}(2\mathbf{i} + 2\mathbf{j}) = \frac{7}{\sqrt{2}}(\mathbf{i} + \mathbf{j}).$$

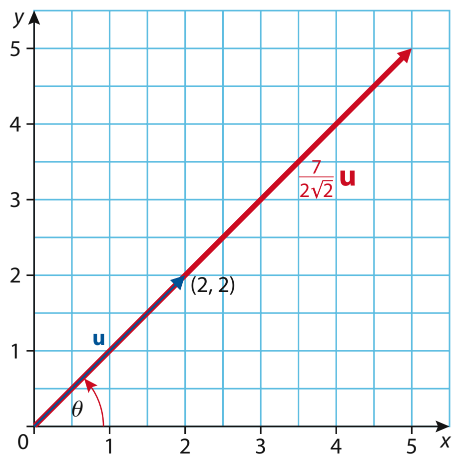


Figure 9.15

b) The direction angle for **v** is $180° - \theta$, as shown in Figure 9.16.

$$\tan\theta = \frac{-3}{3} = -1 \Rightarrow \theta = 180° - 45° = 135°$$

A vector of magnitude 7 that is parallel to **v** is

$$7\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{7}{\sqrt{3^2 + 3^2}}(-3\mathbf{i} + 3\mathbf{j}) = \frac{7}{3\sqrt{2}}(-3\mathbf{i} + 3\mathbf{j}) = \frac{7}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}).$$

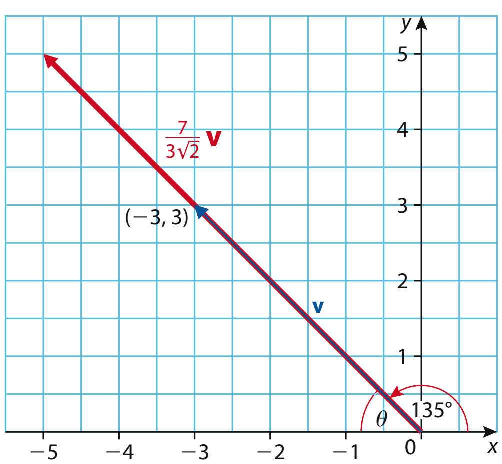


Figure 9.16

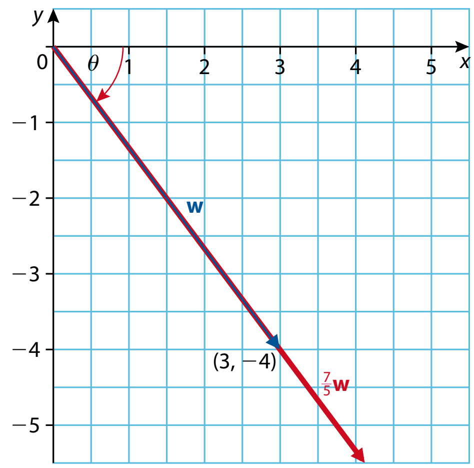
c) The direction angle for **w** is θ, as shown in Figure 9.17.

$\tan\theta = \frac{-4}{3} \Rightarrow \theta \approx -53°$

A vector of magnitude 7 that is parallel to **w** is

$$7\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{7}{\sqrt{3^2 + (-4)^2}}(3\mathbf{i} - 4\mathbf{j}) = \frac{7}{5}(3\mathbf{i} - 4\mathbf{j}).$$

Figure 9.17



Using vectors to model force, displacement and velocity

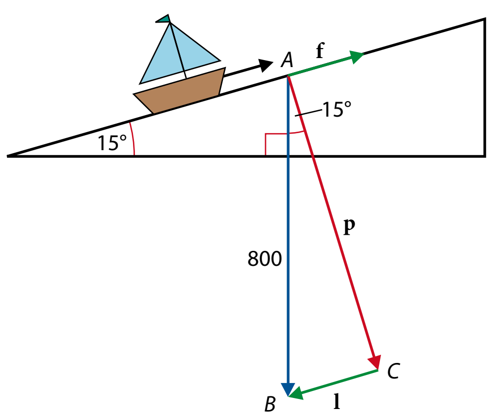
The force on an object can be represented by a vector. We can think of the force as a push or pull on an object such as a person pulling a box along a plane or the weight of a truck which is a downward pull of the Earth's gravity on the truck. If several forces act on an object, the **resultant** force experienced by the object is the vector sum of the forces.

Force

Example 8

What force is required to pull a boat of 800 N up a ramp inclined at 15° from the horizontal? Friction is ignored in this case.

Solution



The process of 'breaking-up' the vector into its components, as we did in the example, is called **resolving** the vector into its components. Notice that the process of resolving a vector is not unique. That is, you can resolve a vector into several pairs of directions.

The situation can be shown on a diagram. The weight is represented by the vector $\overrightarrow{AB}$. The weight of the boat has two components – one perpendicular to the ramp, which is the force responsible for keeping the boat on the ramp and preventing it from tumbling down (**p**). The other force is parallel to the ramp, and is the force responsible for pulling the boat down the ramp (**l**). Therefore, the force we need, **f**, must counter **l**.

In triangle *ABC*:

$$\sin \angle A = |\mathbf{l}|/800 \Rightarrow |\mathbf{l}| = 800 \sin \angle A = 800 \sin 15° = 207.06.$$

We need an upward force of 207.06 N along the ramp to move the boat.

Example 9

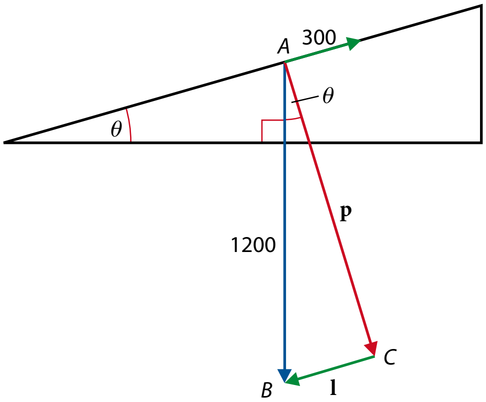
In many countries, it is a requirement that disabled people have access to all places without needing the help of others. Consider an office building whose entrance is 40 cm above ground level. Assuming, on average, that the weight of a person including the equipment used is 1200 N, answer the following questions:

a) At what angle should the ramp designed for disabled persons be set if, on average, the force that a person can apply using their hands is 300 N?

b) How long should the ramp be?

Solution

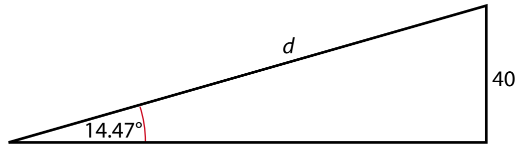
a)



As the diagram above shows, $|\mathbf{l}| = 300$, and

$$\sin \angle A = \frac{|\mathbf{l}|}{1200} = \frac{300}{1200} \Rightarrow \angle A = \sin^{-1} 0.25 \approx 14.47°.$$

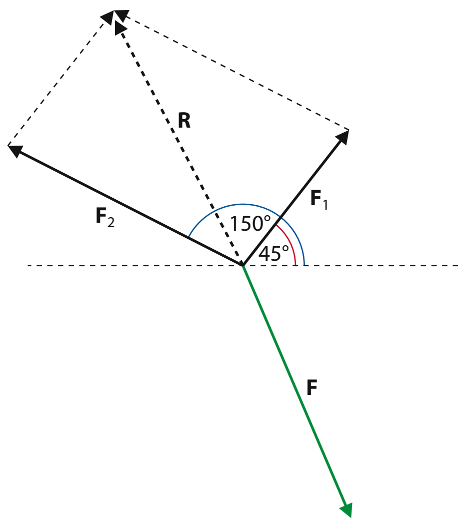
b) The length d of the ramp can be found using right triangle trigonometry:



$$\sin 14.47 = \frac{40}{d} \Rightarrow d = \frac{40}{\sin 14.47} \approx \frac{40}{0.25} = 160 \text{ cm}$$

Resultant force

Two forces $\mathbf{F}_1$ with magnitude 20 N and $\mathbf{F}_2$ with magnitude 40 N are acting on an object at equilibrium as shown in the diagram. Find the force $\mathbf{F}$ required to keep the object at equilibrium.



We will write the vectors for $\mathbf{F}_1$ and $\mathbf{F}_2$ in component form:

$$\mathbf{F}_1 = (20\cos 45°)\mathbf{i} + (20\sin 45°)\mathbf{j} = 10\sqrt{2}\mathbf{i} + 10\sqrt{2}\mathbf{j}$$
$$\mathbf{F}_1 = (40\cos 150°)\mathbf{i} + (40\sin 150°)\mathbf{j} = -20\sqrt{3}\mathbf{i} + 20\mathbf{j}$$

Now, the resultant force $\mathbf{R}$ is

$$\mathbf{R} = (10\sqrt{2}\mathbf{i} + 10\sqrt{2}\mathbf{j}) + (-20\sqrt{3}\mathbf{i} + 20\mathbf{j})$$
$$= (10\sqrt{2} - 20\sqrt{3})\mathbf{i} + (10\sqrt{2} + 20)\mathbf{j}$$

Finally, the force $\mathbf{F}$ required to keep the object at equilibrium is

$$\mathbf{F} = -\mathbf{R} = (-10\sqrt{2} + 20\sqrt{3})\mathbf{i} - (10\sqrt{2} + 20)\mathbf{j}$$

Vectors can be used to help tackle displacement situations. For example, an object at a position defined by the position vector (**a**, **b**) and a velocity vector (**c**, **d**) has a position vector (**a**, **b**) + *t*(**c**, **d**) after time *t*.

Displacement and velocity

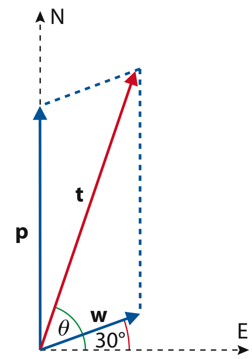
Note: In navigation, the convention is that the **course** or **bearing** of a moving object is the angle that its direction makes with the north direction measured clockwise. So, for example, a ship going east has a bearing of 90°.

The velocity of an object can be represented by a vector whose direction is the direction of motion and whose magnitude is the speed of the object.

When external forces interfere with the motion, such as wind, stream, and friction, then objects will move under the influence of the **resultant forces**.

Example 10

An aeroplane heads in a northerly direction with a speed of 450 km/h. The wind is blowing in the direction of N 60° E with a speed of 60 km/h.



a) Write down the component forms of the plane's air velocity and the wind velocity.

b) Find the true velocity of the plane.

c) Find the true speed and direction of the plane.

Solution

Let $\mathbf{p}$ be the vector for the plane's air velocity, $\mathbf{w}$ the wind's velocity, and $\mathbf{t}$ the true velocity.

a) $\mathbf{p} = 0\mathbf{i} + 450\mathbf{j}$

$$\mathbf{w} = (60\cos 30°)\mathbf{i} + (60\sin 30°)\mathbf{j} = 30\sqrt{3}\mathbf{i} + 30\mathbf{j}$$

b) The true velocity of the plane is the resultant of the two forces above, therefore

$$\mathbf{t} = \mathbf{p} + \mathbf{w} = (0\mathbf{i} + 450\mathbf{j}) + (30\sqrt{3}\mathbf{i} + 30\mathbf{j}) = 30\sqrt{3}\mathbf{i} + 480\mathbf{j}.$$

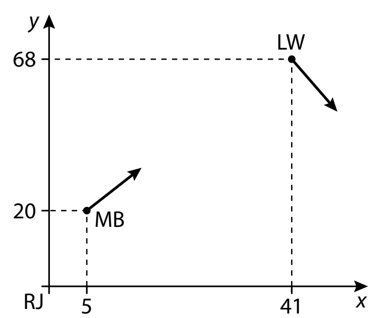
c) The true speed is given by the magnitude of $\mathbf{t}$,

$$|\mathbf{t}| = \sqrt{(30\sqrt{3})^2 + 480^2} \approx 482.8 \text{ km/h}.$$

The direction is determined by the angle θ that the true velocity makes with the horizontal. From our discussion earlier, this can be found by using the property that $\tan\theta = \frac{480}{30\sqrt{3}} \approx 9.24$, and so $\theta \approx 83.8°$. So, we can now give the true direction of the plane as N 6.2° E.

Example 11

The position vector of a ship (MB) from its starting position at a port RJ is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 20 \end{pmatrix} + t\begin{pmatrix} 12 \\ 16 \end{pmatrix}$. Distances are in kilometres and speeds are in km/h. t is time after 00 hour.



a) Find the position of the MB after 2 hours.

b) What is the speed of the MB?

c) Another ship (LW) is at sea in a location $\begin{pmatrix} 41 \\ 68 \end{pmatrix}$ relative to the same port. LW has stopped for some reason. Show that if LW does not start to move, the two ships will collide. Find the time of the potential collision.

d) To avoid collision, LW is ordered to leave its position and start moving at a velocity of $\begin{pmatrix} 15 \\ -36 \end{pmatrix}$ one hour after MB started. Find the position vector of LW.

e) How far apart are the two ships after two hours since the start of MB?

Solution

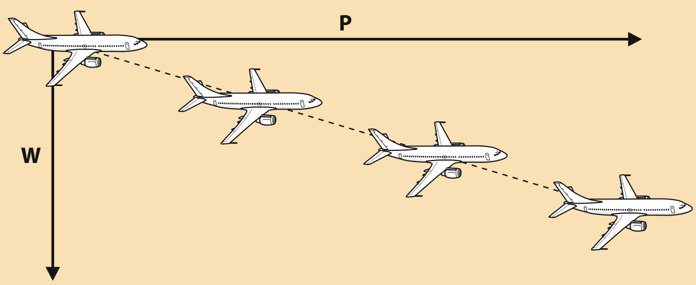
a) MB is at a position with vector $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 20 \end{pmatrix} + 2\begin{pmatrix} 12 \\ 16 \end{pmatrix} = \begin{pmatrix} 29 \\ 52 \end{pmatrix}$.

b) Since the velocity of the ship is $\begin{pmatrix} 12 \\ 16 \end{pmatrix}$, the speed is $\left|\begin{pmatrix} 12 \\ 16 \end{pmatrix}\right| = \sqrt{12^2 + 16^2}$ $= 20$ km/h.

c) The collision can happen if the position vectors of the two ships are equal: $\begin{pmatrix} 5 \\ 20 \end{pmatrix} + t\begin{pmatrix} 12 \\ 16 \end{pmatrix} = \begin{pmatrix} 41 \\ 68 \end{pmatrix} \Rightarrow 5 + 12t = 41$ and $20 + 16t = 68 \Rightarrow 12t = 36$ and $16t = 48 \Rightarrow t = 3$. After 3 hours, at 03:00, a collision could happen.

d) Since LW started one hour later, its position vector is
$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 41 \\ 68 \end{pmatrix} + (t - 1)\begin{pmatrix} 15 \\ -36 \end{pmatrix}, t \geqslant 1.$

e) MB is at $\begin{pmatrix} 29 \\ 52 \end{pmatrix}$ and LW is at $\begin{pmatrix} 41 \\ 68 \end{pmatrix} + (2 - 1)\begin{pmatrix} 15 \\ -36 \end{pmatrix} = \begin{pmatrix} 56 \\ 32 \end{pmatrix}$. The distance between them is $\sqrt{(56 - 29)^2 + (32 - 52)^2} = \sqrt{1129} = 33.6$ km.



When the wind is strong and is acting in a direction different from that of the airplane and if you watch the plane from the ground you will notice that the 'nose' of the plane is in a direction (air velocity) different from the motion of the plane's 'true' velocity.

Exercise 9.3

1 Find the direction angle for each vector.

a) $\mathbf{u} = (2, 0)$

b) $\mathbf{v} = (0, 3)$

c) $\mathbf{w} = (-3, 0)$

d) $\mathbf{u} + \mathbf{v}$

e) $\mathbf{v} + \mathbf{w}$

2 Find the magnitude and direction angle for each vector.

a) $\mathbf{u} = (3, 2)$ b) $\mathbf{v} = (-3, -2)$ c) $2\mathbf{u}$

d) $3\mathbf{v}$ e) $2\mathbf{u} + 3\mathbf{v}$ f) $2\mathbf{u} - 3\mathbf{v}$

3 Find the magnitude and direction angle for each vector.

a) $\mathbf{u} = (-4, 7)$ b) $\mathbf{v} = (2, 5)$ c) $3\mathbf{u}$

d) $-2\mathbf{v}$ e) $3\mathbf{u} + 2\mathbf{v}$ f) $\mathbf{u} - \mathbf{v}$

4 Write each of the following vectors in component form. θ is the angle that the vector makes with the positive horizontal axis.

a) $|\mathbf{u}| = 310, \theta = 62°$ b) $|\mathbf{u}| = 43.2, \theta = 19.6°$

c) $|\mathbf{u}| = 12, \theta = 135°$ d) $|\mathbf{u}| = 240, \theta = 300°$

5 Find the coordinates of a point D such that $\overrightarrow{AB} = 2\overrightarrow{CD}$ where $A(2, 1)$, $B(4, 7)$, and $C(-1, 1)$.

6 Find the unit vector in the same direction as **u** in each of the following cases.

a) $\mathbf{u} = (3, 4)$

b) $\mathbf{u} = 2\mathbf{i} - 5\mathbf{j}$

7 Find a unit vector in the plane making an angle θ with the positive x-axis where

a) $\theta = 150°$

b) $\theta = 315°$

8 Find a vector of magnitude 7 that is parallel to $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$.

9 Find a vector of magnitude 3 that is parallel to $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$.

10 Find a vector of magnitude 7 that is perpendicular to $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$.

11 Find a vector of magnitude 3 that is perpendicular to $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$.

12 A plane is flying on a bearing of 170° at a speed of 840 km/h. The wind is blowing in the direction N 120° E with a strength of 60 km/h.

a) Find the vector components of the plane's still-air velocity and the wind's velocity.

b) Determine the true velocity (ground) of the plane in component form.

c) Write down the true speed and direction of the plane.

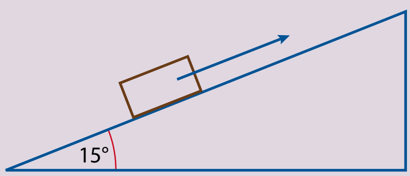
Note: In navigation, the convention is that the **course** or **bearing** of a moving object is the angle that its direction makes with the north direction measured clockwise. So, for example, a ship going east has a bearing of 090°.

13 A plane is flying on a compass heading of 340° at 520 km/h. The wind is blowing with the bearing 320° at 64 km/h.

a) Find the component form of the velocities of the plane and the wind.

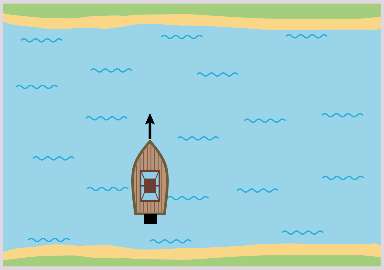
b) Find the actual ground speed and direction of the plane.

14

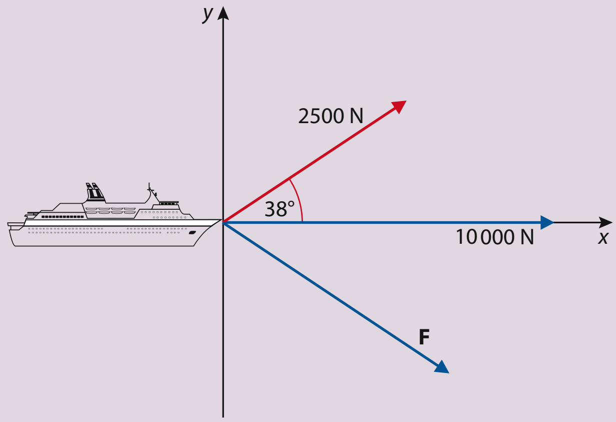


A box is being pulled up a 15° inclined plane. The force needed is 25 N. Find the horizontal and vertical components of the force vector and interpret each of them.

15 A motor boat with the power to steer across a river at 30 km/h is moving such that the bow is pointed in a northerly direction. The stream is moving eastward at 6 km/h. The river is 1 km wide. Where on the opposite side will the boat meet the land?



16 A force of 2500 N is applied at an angle of 38° to pull a 10 000 N ship in the direction given. What force **F** is needed to achieve this?

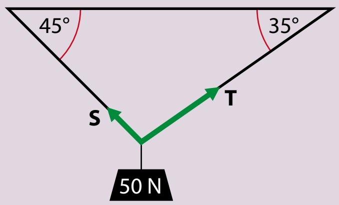


17 A boat is observed to have a bearing of 072°. The speed of the boat relative to still water is 40 km/h. Water is flowing directly south. The boat appears to be heading directly east.

a) Express the velocity of the boat with respect to the water in component form.

b) Find the speed of the water stream and the true speed of the boat.

18 A 50 N weight is suspended by two strings as shown. Find the tensions **T** and **S** in the strings.



19 A runner runs in a westerly direction on the deck of a cruise ship at 8 km/h. The cruise ship is moving north at a speed of 35 km/h. Find the velocity of the runner relative to the water.

20 The boat in question 15 wants to reach a point exactly north of the starting point. In which direction should the boat be steered in order to achieve this objective?

21 Forces **F** = (−10, 3), **G** = (−4, 1) and **H** = (4, −10) act on a point **P**. Find the additional force required to keep the system in equilibrium.

22 A wind is blowing due west at 60 km/h. A small plane with air speed of 300 km/h is trying to maintain a course due north. In what direction should the pilot steer the plane to keep the targeted course? How fast is the plane moving?

23 The points $P(2, 2)$, $Q(10, 2)$ and $R(12, 6)$ are three vertices of a parallelogram. Find the fourth vertex S if

a) P and R are vertices of the same diagonal

b) P and R are vertices of a common side.

24 Show, using vector operations, that the diagonals of a parallelogram intersect each other.

25 Show, using vector operations, that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and has half its length.

26 Prove that the midpoints of the sides of any quadrilateral are the vertices of a parallelogram.

27 An athlete is rowing a boat at a speed of 30 m per minute across a small river 150 m wide. The athlete keeps the boat heading perpendicular to the banks of the river.

a) How far down the river does the boat reach the opposite side if the river is flowing at a rate of 10 m/minute?

b) How long does the trip last?

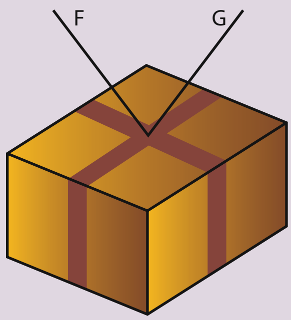
c) At what angle must the athlete steer the boat in order to reach a point directly opposite the starting point on the other side of the river? How long does the trip take?

28 A jet heads in the direction N 30° E at a speed of 400 km/h. The jet experiences a 20 km/h crosswind flowing due east. Find

a) the true velocity **p** of the jet,

b) the true speed and direction of the jet.

29 A box is carried by two strings F and G as shown right. The string F makes an angle of 45° with the horizontal while G makes an angle of 30°. The forces in F and G have a magnitude of 200 N each. The weight of the box is 300 N. What is the magnitude of the resultant force on the box and in which direction does it move?



9.4 Scalar product of two vectors

The multiplication of two vectors is not uniquely defined: in other words, it is unclear whether the product will be a vector or not. For this reason there are two types of vector multiplication:

The **scalar** or **dot product** of two vectors, which results in a scalar; and the **vector** or **cross product** of two vectors, which results in a vector.

In this chapter, we shall discuss only the scalar or dot product. We will discuss the vector product in Chapter 14.

The **scalar product of two vectors**, **a** and **b** denoted by $\mathbf{a} \cdot \mathbf{b}$, is defined as the product of the magnitudes of the vectors times the cosine of the angle between them:
$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$

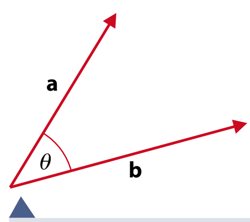


Figure 9.18

This is illustrated in Figure 9.18.

Note that the result of a dot product is a scalar, not a vector. The rules for scalar products are given in the following list:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$
$$0 \cdot \mathbf{a} = \mathbf{a} \cdot 0 = 0$$
$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$
$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$$
$$k(\mathbf{a} \cdot \mathbf{b}) = k\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot k\mathbf{b}, \text{ with } k \text{ any scalar.}$$

The first properties follow directly from the definition:

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$, and $\mathbf{b}\cdot\mathbf{a} = |\mathbf{b}|\,|\mathbf{a}|\cos\theta$, and, since multiplication of real numbers is commutative, it follows that $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ The third property will be proved later in this section. Proofs of the rest of the properties are left as exercises.

Using the definition, it is immediately clear that for two non-zero vectors **u** and **v**, if **u** and **v** are perpendicular, the dot product is zero. This is so, because
$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = |\mathbf{u}||\mathbf{v}|\cos 90° = |\mathbf{u}||\mathbf{v}| \times 0 = 0$.

The converse is also true: if $\mathbf{u}\cdot\mathbf{v} = 0$, the vectors are perpendicular,
$\mathbf{u}\cdot\mathbf{v} = 0 \Rightarrow |\mathbf{u}||\mathbf{v}|\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = 90°$.

Using the definition, it is also clear that for two non-zero vectors **u** and **v**, if **u** and **v** are parallel then the dot product is equal to $\pm|\mathbf{u}||\mathbf{v}|$. This is so, because
$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = |\mathbf{u}||\mathbf{v}|\cos 0° = |\mathbf{u}||\mathbf{v}| \times 1 = |\mathbf{u}||\mathbf{v}|$, or
$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = |\mathbf{u}||\mathbf{v}|\cos 180° = |\mathbf{u}||\mathbf{v}| \times (-1) = -|\mathbf{u}||\mathbf{v}|$.

The converse is also true: if $\mathbf{u}\cdot\mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$, the vectors are parallel, since
$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta \Rightarrow |\mathbf{u}||\mathbf{v}|\cos\theta = \pm|\mathbf{u}||\mathbf{v}| \Rightarrow \cos\theta = \pm 1 \Rightarrow \theta = 0°$ or $\theta = 180°$.

Another interpretation of the dot product

Projection

(*This subsection is optional – it is beyond the scope of the IB syllabus, but very helpful in clarifying the concept of dot products.*)

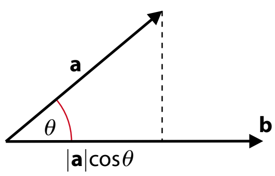


Figure 9.19

The quantity $|\mathbf{a}|\cos\theta$ is called the projection of the vector **a** on vector **b** (Figure 9.19). So, the dot product $\mathbf{b}\cdot\mathbf{a} = |\mathbf{b}||\mathbf{a}|\cos\theta = |\mathbf{b}|(|\mathbf{a}|\cos\theta)$
$= |\mathbf{b}| \times$ (the projection of **a** on **b**).

This fact is used in proving the third property on the list on page 419.

If we let B and C stand for the projections of **b** and **c** on **a**, we have
$\mathbf{a}(\mathbf{b} + \mathbf{c}) = |\mathbf{a}|(B + C) = |\mathbf{a}|B + |\mathbf{a}|C = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$.
This is called the **distributive property** of scalar products over vector addition. See Figure 9.20.

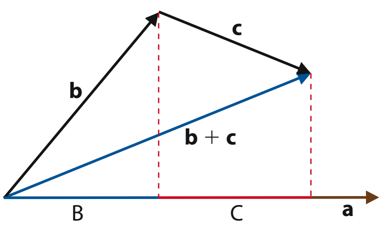


Figure 9.20

With this result, we can develop another definition for the dot product that is more useful in the calculation of this product.

Theorem

If vectors are expressed in component form, $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$, then $\mathbf{u}\cdot\mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j})\cdot(v_1\mathbf{i} + v_2\mathbf{j}) = u_1v_1 + u_2v_2$.

Proof

$\mathbf{u}\cdot\mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j})\cdot(v_1\mathbf{i} + v_2\mathbf{j}) = u_1v_1\mathbf{i}^2 + u_1v_2\mathbf{ij} + u_2v_1\mathbf{ji} + u_2v_2\mathbf{j}^2$

However, $\mathbf{i}^2 = \mathbf{j}^2 = 1$ and $\mathbf{ij} = \mathbf{ji} = 0$. (Proof is left as an exercise for you.)

Therefore, $\mathbf{u}\cdot\mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j})\cdot(v_1\mathbf{i} + v_2\mathbf{j}) = u_1v_1 + u_2v_2$.

For example, to find the scalar product of the two vectors $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} - 3\mathbf{j}$, it is enough to add the products' corresponding components:

$$\mathbf{u}\cdot\mathbf{v} = 2 \times 5 + 4 \times (-3) = -2$$

If we start the definition of the scalar product as $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2$, we can deduce the other definition.

Start with the law of cosines which you learned in Chapter 8. Consider the diagram opposite and apply the law to finding *BC* in triangle *ABC*.

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}|\cos\theta$$

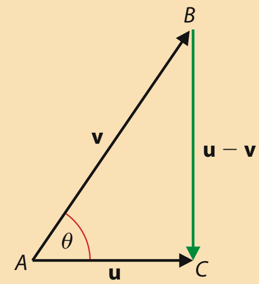
Using the fact that $\mathbf{u} \cdot \mathbf{u} = u_1u_1 + u_2u_2 = \mathbf{u}^2$,

$$\begin{aligned}|\mathbf{u} - \mathbf{v}|^2 &= (\mathbf{u} - \mathbf{v})^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})\\ &= u^2 - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + v^2 = u^2 - \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} + v^2\\ &= |\mathbf{u}|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2\end{aligned}$$

Now, comparing the two results

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 - 2(\mathbf{u} \cdot \mathbf{v}) + |\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}|\cos\theta$$

$$\Rightarrow -2(\mathbf{u} \cdot \mathbf{v}) = -2|\mathbf{u}|\,|\mathbf{v}|\cos\theta \Rightarrow \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$$



Example 12

Find the dot product of $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$.

Solution

$$\mathbf{u} \cdot \mathbf{v} = 2 \times 3 - 3 \times 2 = 0$$

What does this tell us about the two vectors?

The angle between two vectors

The basic definition of the scalar product offers us a method for finding the angle between two vectors.

Since $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$, then $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$.

Note: When the vectors **u** and **v** are given in component form, then the angle cosine can be directly calculated with

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{u_1v_1 + u_2v_2}{\sqrt{u_1^2 + u_2^2}\sqrt{v_1^2 + v_2^2}}$$

Example 13

Find the angle between the following two vectors:

$$\mathbf{v} = -3\mathbf{i} + 3\mathbf{j} \text{ and } \mathbf{w} = 2\mathbf{i} - 4\mathbf{j}$$

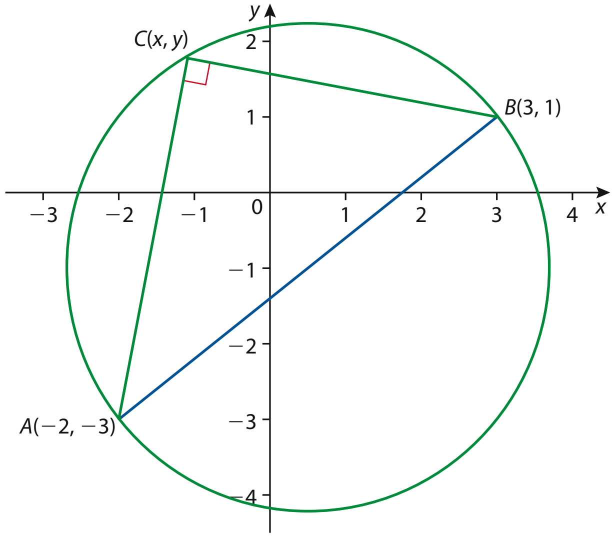
Solution

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|} = \frac{-3 \times 2 + 3 \times -4}{\sqrt{(-3)^2 + 3^2} \times \sqrt{2^2 + 4^2}} = \frac{-18}{\sqrt{18}\sqrt{20}} \Rightarrow \theta = 161.57°$$

Example 14

Consider the segment $[AB]$ with $A(-2, -3)$ and $B(3, 1)$. Use dot products to find the equation of the circle whose diameter is AB.

Solution



Consider any point $C(x, y)$ on the graph. Find the vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$. For the point C to be on the circle, the angle at C must be a right angle. Hence, the vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are perpendicular.

For perpendicular vectors, the dot product must be zero.

$$\overrightarrow{AC} = (x + 2, y + 3), \overrightarrow{BC} = (x - 3, y - 1)$$

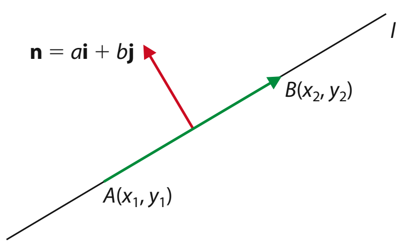
$$\overrightarrow{AC} \cdot \overrightarrow{BC} = 0 \Rightarrow (x + 2)(x - 3) + (y + 3)(y - 1) = 0$$

$$\Rightarrow x^2 - x + y^2 + 2y = 9$$

Example 15

Show that the vector $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$ is orthogonal (perpendicular) to the line l with equation $ax + by + c = 0$.

Solution



Consider two points A and B on the line with the coordinates as shown.

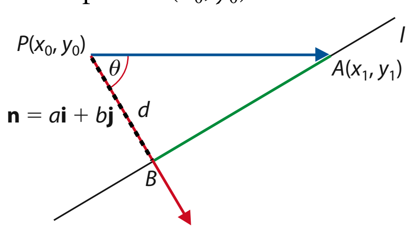
$\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1)$ and

$\mathbf{n} \cdot \overrightarrow{AB} = (a, b) \cdot (x_2 - x_1, y_2 - y_1) = (ax_2 + by_2) - (ax_1 + by_1)$, but A and B are on the line, so

$ax_2 + by_2 = -c$ and $ax_1 + by_1 = -c \Rightarrow \mathbf{n} \cdot \overrightarrow{AB} = -c + c = 0.$

Example 16

Find the distance from the point $P(x_0, y_0)$ to the line l with equation $ax + by + c = 0$.



Solution

The required distance, d, can be found using triangle PAB.

$$d = \left| |\overrightarrow{PA}| \cos\theta \right| = \left| |\overrightarrow{PA}| \frac{\overrightarrow{PA} \cdot \mathbf{n}}{|\overrightarrow{PA}||\mathbf{n}|} \right| = \left| \frac{\overrightarrow{PA} \cdot \mathbf{n}}{|\mathbf{n}|} \right|, \left(\frac{\overrightarrow{PA} \cdot \mathbf{n}}{|\mathbf{n}|} \text{ is called the component of } \overrightarrow{PA} \text{ along } \mathbf{n}. \right)$$

Now,

$$\overrightarrow{PA} = (x_1 - x_0, y_1 - y_0) \Rightarrow \overrightarrow{PA} \cdot \mathbf{n} = a(x_1 - x_0) + b(y_1 - y_0)$$

$$\Rightarrow \overrightarrow{PA} \cdot \mathbf{n} = ax_1 + by_1 - ax_0 - by_0 = -c - ax_0 - by_0$$

$$\text{Therefore, } d = \left| \frac{\overrightarrow{PA} \cdot \mathbf{n}}{|\mathbf{n}|} \right| = \left| \frac{-c - ax_0 - by_0}{\sqrt{a^2 + b^2}} \right| = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

So, for example, the distance from $A(2, -3)$ to the line with equation $5x + 3y = 2$ is

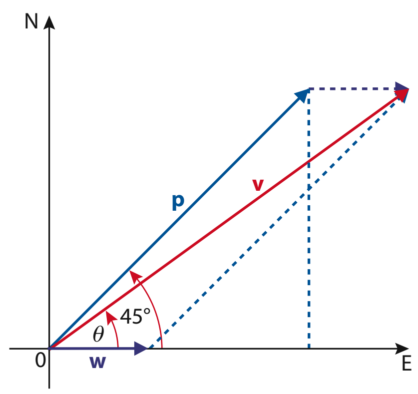
$$d = \frac{|5(2) + 3(-3) - 2|}{\sqrt{5^2 + 3^2}} = \frac{1}{\sqrt{34}} = \frac{\sqrt{34}}{34}.$$

Example 17

The instrument panel in a plane indicates that its airspeed (the speed of the plane relative to the surrounding air) is 200 km/h and that its compass heading (the direction in which the plane's nose is pointing) is N 45° E. There is a steady wind blowing from the west at 50 km/h. Because of the wind, the plane's *true* velocity is different from the panel reading. Find the true velocity of the plane. Also, find its true speed and direction.

Solution

A diagram can help clarify the situation.



The plane velocity $\mathbf{p}$ can be expressed in its component form:

$$x = |\mathbf{p}|\cos 45° = 200 \cos 45° = 100\sqrt{2},$$

$$y = |\mathbf{p}|\sin 45° = 200 \sin 45° = 100\sqrt{2},$$

so $\mathbf{p}$ can be written as $\mathbf{p} = (100\sqrt{2}, 100\sqrt{2})$.

The wind velocity $\mathbf{w}$ can also be expressed in component form:

$$\mathbf{w} = (50, 0)$$

So, the true velocity, $\mathbf{v} = (100\sqrt{2} + 50, 100\sqrt{2})$.

To find the true speed, we find the magnitude of the resultant found above:

$$|\mathbf{v}| = \sqrt{(100\sqrt{2} + 50)^2 + (100\sqrt{2})^2} \approx 238 \text{ km/h}$$

To find the true direction, we find θ and calculate the *heading* of the plane:

$$\tan\theta = \frac{100\sqrt{2}}{100\sqrt{2} + 50} \approx 0.739 \Rightarrow \theta \approx 36.5°,$$

so the true direction is N 53.5° E.

Exercise 9.4

1 Find (i) $\mathbf{u} \cdot \mathbf{v}$ and (ii) the angle between $\mathbf{u}$ and $\mathbf{v}$ to the nearest degree.

a) $\mathbf{u} = \mathbf{i} + \sqrt{3}\,\mathbf{j}$, $\mathbf{v} = \sqrt{3}\,\mathbf{i} - \mathbf{j}$

b) $\mathbf{u} = (2, 5)$, $\mathbf{v} = (4, 1)$

c) $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{v} = 4\mathbf{i} - \mathbf{j}$

d) $\mathbf{u} = 2\mathbf{j}$, $\mathbf{v} = -\mathbf{i} + \sqrt{3}\,\mathbf{j}$

e) $\mathbf{u} = (-3, 0)$, $\mathbf{v} = (0, 7)$

f) $\mathbf{u} = (3, 0)$, $\mathbf{v} = (\sqrt{3}, 1)$

g) $\mathbf{u} = -6\mathbf{j}$, $\mathbf{v} = -2\mathbf{i} + 2\sqrt{3}\mathbf{j}$

h) $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j}$, $\mathbf{v} = -4\mathbf{i} - 4\mathbf{j}$

2 Using the vectors $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j}$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = 4\mathbf{i} + 5\mathbf{j}$, find each of the indicated results.

a) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$

b) $\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

c) $\mathbf{u}(\mathbf{v} \cdot \mathbf{w})$

d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

e) $(\mathbf{u} \cdot \mathbf{v})(\mathbf{u} \cdot \mathbf{w})$

f) $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$

g) Looking at a)–d) write one paragraph to summarize what you learned!

3 Determine whether $\mathbf{u}$ is orthogonal, parallel or neither to $\mathbf{v}$:

$$\mathbf{u} = \begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} -2 \\ \frac{1}{2} \end{pmatrix}$$

$$\mathbf{u} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 6 \\ -12 \end{pmatrix}$$

$$\mathbf{u} = \begin{pmatrix} 2\sqrt{3} \\ 2 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$$

● **Hint:** The work done by any force is defined as the product of the force multiplied by the distance it moves a certain object. In other words, it is the product of the force multiplied by the displacement of the object. As such, work is the dot product between the force and displacement $\mathbf{W} = \mathbf{F} \cdot \mathbf{D}$.

4 Find the work done by the force $\mathbf{F}$ in moving an object between points M and N.

a) $\mathbf{F} = 400\mathbf{i} - 50\mathbf{j}$, $M(2, 3)$, $N(12, 43)$

b) $\mathbf{F} = 30\mathbf{i} + 150\mathbf{j}$, $M(0, 30)$, $N(15, 70)$

c) $\mathbf{F} = \begin{pmatrix} 5 \\ 25 \end{pmatrix}$, $M(0, 0)$, $N(1, 6)$

5 Find the interior angles of the triangle ABC.

a) $A(1, 2)$, $B(3, 4)$, $C(2, 5)$

b) $A(3, 4)$, $B(-1, -7)$, $C(-8, -2)$

c) $A(3, -5)$, $B(1, -9)$, $C(-7, -9)$

6 Find a vector perpendicular to $\mathbf{u}$ in each case below. (Answers are not unique!)

a) $\mathbf{u} = (3, 5)$

b) $\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{3}{4}\mathbf{j}$

7 Use the dot product to find the equation of a circle whose diameter is $[AB]$.

a) $A(1, 2)$, $B(3, 4)$

b) $A(3, 4)$, $B(-1, -7)$

8 Decide whether the triangle ABC is right-angled using vector algebra:
$A(1, -3)$, $B(2, 0)$, $C(6, -2)$

9 Find t such that $\mathbf{a} = t\mathbf{i} - 3\mathbf{j}$ is perpendicular to $\mathbf{b} = 5\mathbf{i} + 7\mathbf{j}$.

10 For what value(s) of b are the vectors $(-6, b)$ and (b, b^2) perpendicular?

11 Find a unit vector that makes an angle of 60° with $\mathbf{u} = (3, 4)$.

12 Find t such that $\mathbf{a} = t\mathbf{i} - \mathbf{j}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j}$ make an angle of $\frac{3}{4}\pi$ radians.

13 Use the dot product to prove that the diagonals of a rhombus are perpendicular to each other.

14 Find the component of $\mathbf{u}$ along $\mathbf{v}$ if

a) $\mathbf{u} = (0, 7)$, $\mathbf{v} = (6, 8)$

b) $\mathbf{u} = \begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -2 \\ \frac{1}{2} \end{pmatrix}$

15 A young man pulls a sled horizontally by exerting a force of 16 N on the rope that is tied to its front end. The rope makes an angle of 45° with the horizontal. Find the work done in pulling the sled 55 m.

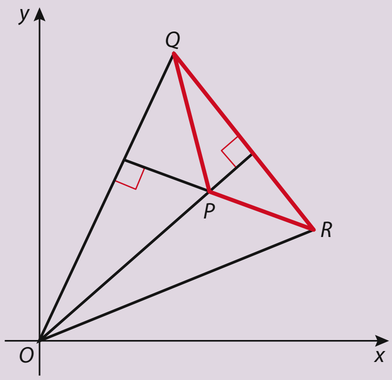
16 Find the distance from the point P to the line l in each case:

a) $P(0, 0)$, l: $3x - 4y + 5 = 0$

b) $P(2, 2)$, l: $3x - 2y = 2$

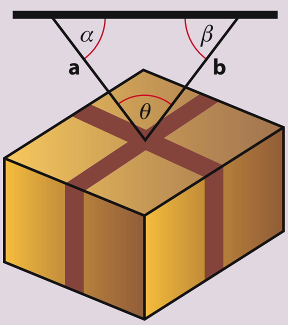
c) $P(1, 5)$, l: $5x - 3y = 11$

17 Given three points in the plane P, Q, and R such that $\overrightarrow{OP} \perp \overrightarrow{QR}$ and $\overrightarrow{OQ} \perp \overrightarrow{PR}$, use scalar product to show that $\overrightarrow{OR} \perp \overrightarrow{PQ}$.



18 Two vectors $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} x \\ 1 \end{pmatrix}$ have an angle of 30° between them. Find the possible values of x.

19 A weight of 1000 N is supported by two forces $\mathbf{a} = (-200, 400)$ and $\mathbf{b} = (200, 600)$. The weight is in equilibrium. Find the angles α, β, and θ.



20 Show that the vector $|\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a}$ bisects the angle between the two vectors $\mathbf{a}$ and $\mathbf{b}$.

Practice questions

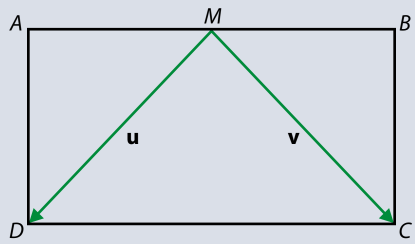
1 *ABCD* is a rectangle with *M* the midpoint of [*AB*]. **u** and **v** represent the vectors joining *M* to *D* and *C* respectively. Express each of the following vectors in terms of **u** and **v**.

a) $\overrightarrow{DC}$

b) $\overrightarrow{AM}$

c) $\overrightarrow{BC}$

d) $\overrightarrow{AC}$



2 Consider the vectors $\mathbf{u} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$.

a) Find the component form of the vector $\mathbf{w} = 2\mathbf{u} + \mathbf{v}$.

b) Find the vector **z** which has a magnitude of 6 units and same direction as **w**.

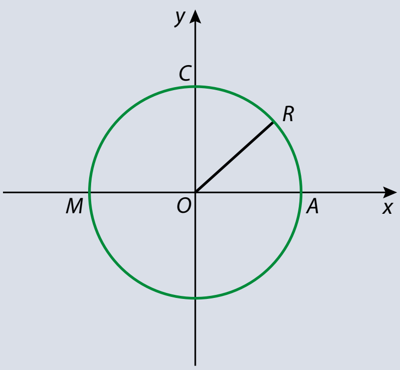
3 *M* and *A* are the ends of the diameter of a circle with centre at the origin. The radius of the circle is 15 cm and $\overrightarrow{OR} = \begin{pmatrix} 10 \\ 5\sqrt{5} \end{pmatrix}$.

a) Verify that *R* lies on the circle.

b) Find the vector $\overrightarrow{AR}$.

c) Find the cosine of $\angle OAR$.

d) Find the area of $\triangle MAR$.



4 Quadrilateral *MARC* has vertices with coordinates *M*(0, 0), *A*(6, 2), *R*(11, 4) and *C*(3, 8).

a) Find the vectors $\overrightarrow{MR}$ and $\overrightarrow{AC}$.

b) Find the angle between the diagonals of quadrilateral *MARC*.

c) Let the vector **u** be the vector joining the midpoints of [*MA*] and [*AR*], and **v** be the vector joining the midpoints of [*RC*] and [*CM*]. Compare **u** and **v** to $\overrightarrow{MR}$, and hence show that the quadrilateral connecting the midpoints of the sides of *MARC* form a parallelogram.

5 Vectors $\mathbf{u} = 5\mathbf{i} + 3\mathbf{j}$ and $\mathbf{v} = \mathbf{i} - 4\mathbf{j}$ are given. Find the scalars *m* and *n* such that $m(\mathbf{u} + \mathbf{v}) - 5\mathbf{i} + 7\mathbf{j} = n(\mathbf{u} - \mathbf{v})$.

6 Vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents a displacement in the eastern direction while vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represents a displacement north. Distances are in kilometres.

Two crews of workers are laying gas pipes in a north-south direction across the North Sea. Consider the base port where the crews leave to start work as the origin (0, 0).

At 07:00 the crews left the base port with their motor boats to two different locations. The crew called 'Marco' travel at a velocity of $\begin{pmatrix} 9 \\ 12 \end{pmatrix}$ and the crew called 'Tony' travel at a velocity of $\begin{pmatrix} 18 \\ -8 \end{pmatrix}$. Speeds are in km/h.

a) Find the speed of each boat.

b) Find the position vectors of each crew at 07:30.

c) Hence, or otherwise, find the distance between the vehicles at 07:30.

d) At 07:30 'Tony' stops and the crew begins laying pipes towards the north. 'Marco' continues travelling in the same direction at the same speed until it is exactly north of 'Tony'. At this point, 'Marco' stops and the crew then begins laying pipes towards the south. At what time does 'Marco' start work?

e) Each crew lays an average of 400 m of pipe in an hour. If they work non-stop until their lunch break at 12:30, what is the distance between them at this time?

f) How long would 'Marco' take to return to base port from its lunchtime position, assuming it travelled in a straight line and with the same average speed as on the morning journey? (Give your answer to the nearest minute.)

7 Triangle *TRI* is defined as follows:

$\overrightarrow{OT} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\overrightarrow{TR} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$, $\overrightarrow{TR} \cdot \overrightarrow{IR} = 0$, and $\overrightarrow{TI} = k\mathbf{j}$ where k is a scalar and $\mathbf{j}$ is the unit vector in the y-direction.

a) Draw an accurate diagram of $\triangle TRI$.

b) Write the vector $\overrightarrow{IR}$.

8 Vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents a displacement in the eastern direction while vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represents a displacement north. Distances are in kilometres.

The position vector of a plane for AUA airlines from its starting position in Vienna is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 25 \\ 40 \end{pmatrix} + t\begin{pmatrix} 360 \\ 480 \end{pmatrix}$. Speeds are in km/h and t is time after 00 hour.

a) Find the position of the AUA plane after 2 hours.

b) What is the speed of the plane?

c) A plane for LH airline started at the same time from a location $\begin{pmatrix} -155 \\ 1300 \end{pmatrix}$ relative to Vienna and moving with a velocity vector $\begin{pmatrix} 480 \\ -360 \end{pmatrix}$, flying at the same height as the AUA plane. Show that if the LH plane does not change route, the two planes will collide. Find the time of the potential collision.

d) To avoid collision, the LH plane is ordered to leave its position and start moving at a velocity of $\begin{pmatrix} 450 \\ -390 \end{pmatrix}$ one hour after it started. Find the position vector of the LH plane at that time.

e) How far apart are the two planes after two hours?

9 For what value(s) of n are the vectors $\begin{pmatrix} 3n \\ 2n+3 \end{pmatrix}$ and $\begin{pmatrix} 2n-1 \\ 4-2n \end{pmatrix}$ perpendicular. Otherwise, show that it is not possible.

10 Let α be the angle between the vectors **a** and **b**, where

$$\mathbf{a} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j},\ \mathbf{b} = (\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j} \text{ and } 0 < \theta < \frac{\pi}{4}.$$

Express α in terms of θ.

11 Given two non-zero vectors **a** and **b** such that $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$, find the value of $\mathbf{a} \cdot \mathbf{b}$.