14 Vectors, Lines and Planes

Assessment statements

4.1 Vectors as displacements in the plane and in three dimensions.
Components of a vector; column representation.
$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$
Algebraic and geometric approaches to the following topics:
the sum and difference of two vectors; the zero vector; the vector $-\mathbf{v}$;
multiplication by a scalar, $k\mathbf{v}$;
magnitude of a vector, $|\mathbf{v}|$;
unit vectors; base vectors, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$;
position vectors $\overrightarrow{OA} = \mathbf{a}$;
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$.

4.2 The scalar product of two vectors.
Perpendicular vectors; parallel vectors.
The angle between two vectors.

4.3 Vector equation of a line in two and three dimensions: $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$.
The angle between two lines.

4.4 Coincident, parallel, intersecting and skew lines, distinguishing between these cases.
Points of intersection.

4.5 The vector product of two vectors, $\mathbf{v} \times \mathbf{w}$.
Properties of the vector product.
Geometric interpretation of $|\mathbf{v} \times \mathbf{w}|$.

4.6 Vector equation of a plane $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$.
Use of normal vector to obtain the form $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$.
Cartesian equation of a plane $ax + by + cz = d$.

4.7 Intersections of: a line with a plane; two planes; three planes.
Angle between: a line and a plane; two planes.

Introduction

You have seen vectors in the plane in Chapter 9. We will limit our discussion to mainly three-dimensional space in this chapter. If you need to refresh your knowledge of the plane case, refer to Chapter 9.

Because we live in a three-dimensional world, it is essential that we study objects in three dimensions. To that end, we consider in this section a three-dimensional coordinate system in which points are determined by ordered triples. We construct the coordinate system in the following manner: Choose three mutually perpendicular axes, as shown in Figure 14.1, to serve as our reference. The orientation of the system is *right-handed* in the sense that if you hold your right hand so that the fingers curl from the positive x-axis towards the positive y-axis, your thumb points along the z-axis (see below). Looking at it in a different perspective, if you are looking straight at the system, the yz-plane is the plane facing you, and the xz-plane is perpendicular to it and extending out of the page towards you, and the xy-plane is the bottom part of that picture (Figure 14.2). The xy-, xz- and yz-planes are called the **coordinate planes**. Points in space are assigned coordinates in the same manner as in the plane. So, the point P (left) is assigned the ordered triple (x, y, z) to indicate that it is x, y and z units from the yz-, xz- and xy-planes.

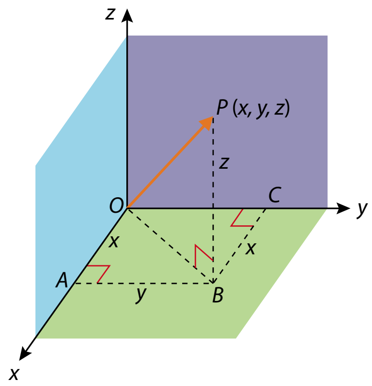
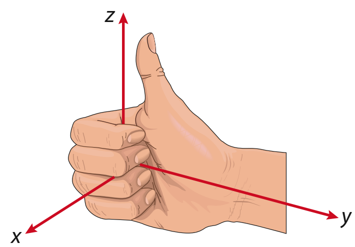


Figure 14.1

In this chapter, we will extend our study of vectors to space. The good news is that many of the rules you know from the plane also apply to vectors in space. So, we will only have to introduce a few new concepts. Some of the material will either be a repeat of what you have learned for two-dimensional space or an extension.



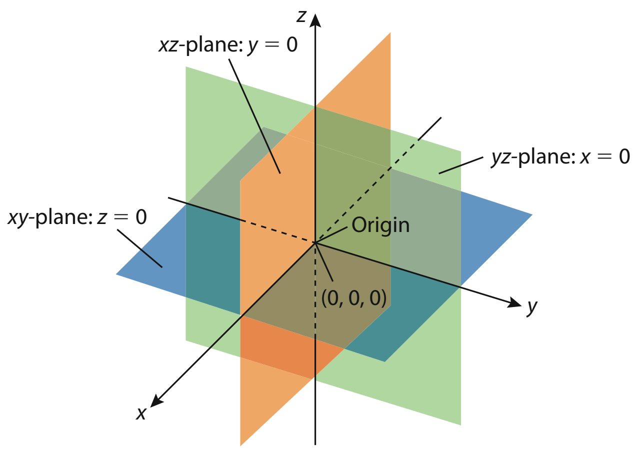


Figure 14.2 The coordinate planes divide space into 8 octants.

14.1 Vectors from a geometric viewpoint

Vectors can be represented geometrically by arrows in two- or three-dimensional space; the direction of the arrow specifies the direction of the vector, and the length of the arrow describes its magnitude. The first point on the arrow is called the **initial point** of the vector and the tip is called the **terminal point**. We shall denote vectors in lower-case boldface type, such as **v**, when using one letter to name the vector, and we will use $\overrightarrow{AB}$ to denote the vector from A to B. The handwritten notation will be the latter too.

If the initial point of a vector is at the origin, the vector is said to be in standard position. It is also called the **position vector** of point P. The terminal point will have coordinates of the form (x, y, z). We call these coordinates the **components** of $\mathbf{v}$ and we write $\mathbf{v} = (x, y, z)$ or $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

The length (magnitude) of a vector $\mathbf{v}$ is also known as its **modulus** or its **norm** and it is written as $|\mathbf{v}|$.

Look back at Figure 14.1. Using Pythagoras' theorem, we can show that the magnitude of a vector $\mathbf{v}$, $|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$.

Let $\overrightarrow{OP} = \mathbf{v}$, then

$|\mathbf{v}| = |\overrightarrow{OP}| = \sqrt{OB^2 + BP^2}$, since the triangle OBP is right-angled at B. Now, consider triangle OAB, which is right-angled at A:

$OB^2 = OA^2 + AB^2 = x^2 + y^2$, and, therefore,

$$|\mathbf{v}| = \sqrt{OB^2 + BP^2} = \sqrt{(x^2 + y^2) + z^2} = \sqrt{x^2 + y^2 + z^2}.$$

Two vectors like $\mathbf{v}$ and $\overrightarrow{AB}$ are equal (equivalent) if they have the same length (magnitude) and the same direction; we write $\mathbf{v} = \overrightarrow{AB}$. Geometrically, two vectors are equal if they are translations of one another as you see in Figures 14.3 and 14.4. Notice in Figure 14.4 that the four vectors are equal, even though they are in different positions.

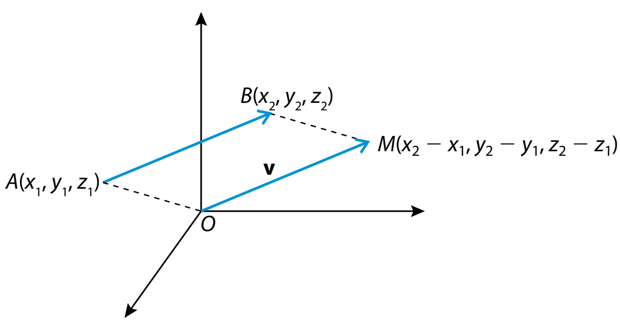


Figure 14.3

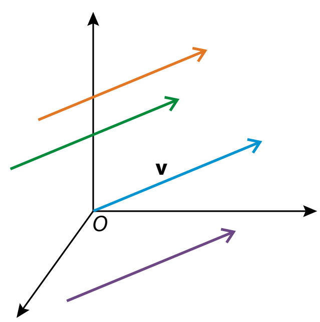


Figure 14.4

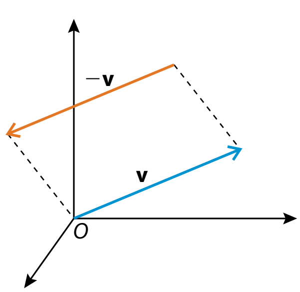


Figure 14.5

Because vectors are not affected by translation, the initial point of a vector $\mathbf{v}$ can be moved to any convenient position by making an appropriate translation.

Two vectors are said to be opposite if they have equal modulus but opposite direction (Figure 14.5).

If the initial and terminal points of a vector coincide, the vector has length zero; we call this the **zero vector** and denote it by $\mathbf{0}$.

The zero vector does not have a specific direction, so we will agree that it can be assigned any convenient direction in a specific problem.

Addition and subtraction of vectors

As you recall from Chapter 9, according to the **triangular rule**, if **u** and **v** are vectors, the sum **u** + **v** is the vector from the initial point of **u** to the terminal point of **v**, when the vectors are positioned so that the initial point of **v** is the terminal point of **u**, as shown in Figure 14.6.

Equivalently, **u** + **v** is also the diagonal of the parallelogram whose sides are **u** and **v**, as shown in Figure 14.7.

The difference of the two vectors **u** and **v** can be dealt with in the same manner. So, the vector **w** = **u** – **v** is a vector such that **u** = **v** + **w**.

In Figure 14.8, we can clearly see that the difference is along the diagonal joining the two terminal points of the vectors and in the direction from **v** to **u**.

If k is a real positive number, k**v** is a vector of magnitude $k|\mathbf{v}|$ and in the same direction as **v**. It follows that when k is negative, k**v** has magnitude $|k| \times |\mathbf{v}|$ and is in the opposite direction to **v** (Figure 14.9).

● **Hint:** When we discuss vectors, we will refer to real numbers as scalars.

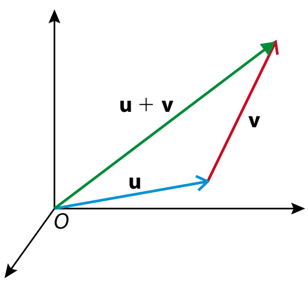


Figure 14.6

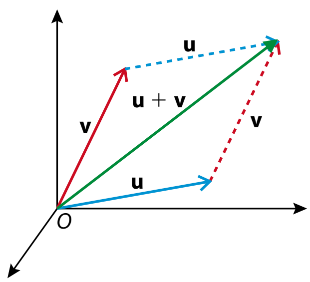


Figure 14.7

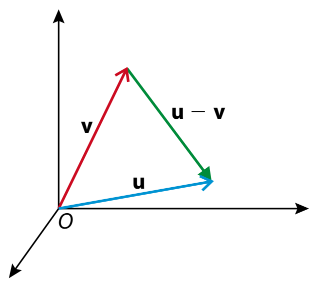


Figure 14.8

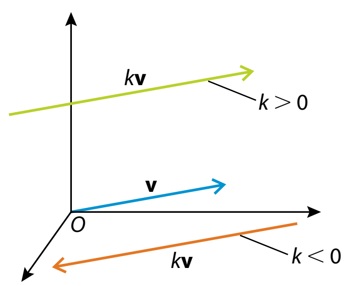


Figure 14.9

A result of the previous situation is the necessary and sufficient condition for two vectors to be parallel:

Two vectors are parallel if one of them is a scalar multiple of the other.
For example, the vector $(-3, 4, -2)$ is parallel to the vector $(4.5, -6, 3)$ since $(-3, 4, -2) = -\frac{2}{3}(4.5, -6, 3)$.

Components provide a simple way to algebraically perform several operations on vectors. First, by definition, we know that two vectors are equal if they have the same length and the same magnitude. So, if we choose to draw the two equal vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ from the origin, their terminal points must coincide, and hence $u_1 = v_1$, $u_2 = v_2$ and $u_3 = v_3$. So, we showed that equal vectors have the same components. The converse is obviously true, i.e. if $u_1 = v_1$, $u_2 = v_2$ and $u_3 = v_3$, the two vectors are equal. The following results are also obvious from the simple geometry of similar figures:

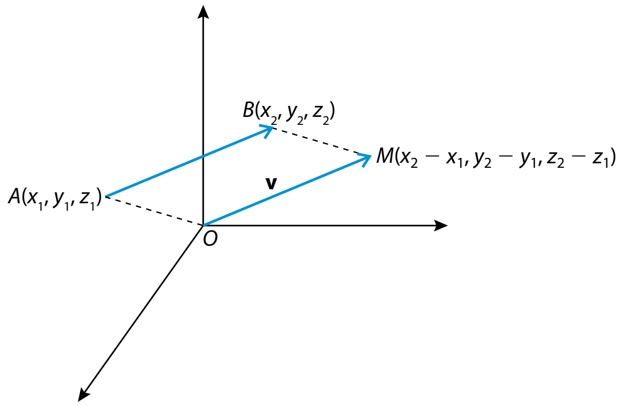
If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ and k is any real number, then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \text{ and } k\mathbf{u} = (ku_1, ku_2, ku_3).$$

If the initial point of the vector is not at the origin, the following theorem generalizes the previous notation to any position:

If $\overrightarrow{AB}$ is a vector with initial point $A(x_1, y_1, z_1)$ and terminal point $B(x_2, y_2, z_2)$, then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$, as you see in Figure 14.10.

Figure 14.10



As illustrated in Figure 14.10, either by applying the distance formula or by using the equality of vectors **v** and $\overrightarrow{AB}$,

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Additionally, the following results can follow easily from properties of real numbers: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$; $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$; $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$; and the other obvious relationships.

Example 1

Given the points $A(-2, 3, 5)$ and $B(1, 0, -4)$,

a) find the components of vector $\overrightarrow{AB}$

b) find the components of vector $\overrightarrow{BA}$

c) find the components of vector $3\overrightarrow{AB}$

d) find the components of vector $\overrightarrow{OA} + \overrightarrow{OB}$

e) calculate $|\overrightarrow{AB}|$ and $|\overrightarrow{BA}|$

f) calculate $|3\overrightarrow{AB}|$ and $|\overrightarrow{OA} + \overrightarrow{OB}|$.

Solution

a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$
$= (1 - (-2), 0 - 3, -4 - 5) = (3, -3, -9)$

b) Since $\overrightarrow{BA}$ is the opposite of $\overrightarrow{AB}$, then $\overrightarrow{BA} = (-3, 3, 9)$.

c) $3\overrightarrow{AB} = 3(3, -3, -9) = (9, -9, -27)$

d) $\overrightarrow{OA} + \overrightarrow{OB} = (-2 + 1, 3 + 0, 5 - 4) = (-1, 3, 1)$

e) $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{9 + 9 + 81} = 3\sqrt{11}$

$|\overrightarrow{BA}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{9 + 9 + 81} = 3\sqrt{11}$

f) $|3\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{81 + 81 + 729}$
$= \sqrt{891} = 9\sqrt{11}$

Obviously, $|3\overrightarrow{AB}| = 3|\overrightarrow{AB}|$!

$|\overrightarrow{OA} + \overrightarrow{OB}| = |(-1, 3, 1)| = \sqrt{1 + 9 + 1} = \sqrt{11}$

Notice that $|\overrightarrow{OA} + \overrightarrow{OB}| = \sqrt{11} \neq |\overrightarrow{OA}| + |\overrightarrow{OB}|$

$= \sqrt{4 + 9 + 25} + \sqrt{1 + 0 + 16} = \sqrt{38} + \sqrt{17}.$

In general, $|\lambda\mathbf{v}| = |\lambda|\,|\mathbf{v}|$, i.e. the magnitude of a multiple of a vector is equal to the absolute multiple of the magnitude of the vector. For example, $|-3\mathbf{v}| = 3|\mathbf{v}|$.

Example 2

Determine the relationship between the coordinates of point $M(x, y, z)$ so that the points M, $A(0, -1, 5)$ and $B(1, 2, 3)$ are collinear.

Solution

For the points to be collinear, it is enough to make $\overrightarrow{AM}$ parallel to $\overrightarrow{AB}$.
If the two vectors are parallel, then one of them is a scalar multiple of the other. Say $\overrightarrow{AM} = t\overrightarrow{AB}$.

$$\overrightarrow{AM} = (x, y + 1, z - 5) = t(1, 3, -2) = (t, 3t, -2t)$$

So, $x = t$, $y + 1 = 3t$, and $z - 5 = -2t$.

Unit vectors

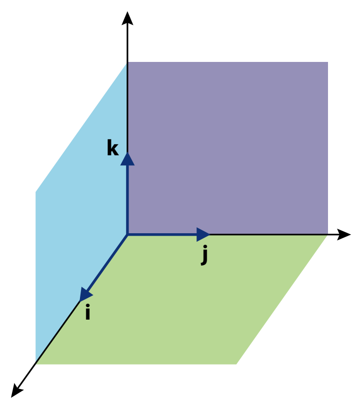


Figure 14.11

A vector of length 1 is called a **unit vector**. So, in two-dimensional space, the vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ are unit vectors along the x- and y-axes, and in three-dimensional space, the unit vectors along the axes are $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$. The vectors **i**, **j** and **k** are called the **base vectors** of the 3-space.

It follows immediately that each vector in 3-space can be expressed uniquely in terms of **i**, **j** and **k** as follows:

$$\mathbf{u} = (x, y, z) = (x, 0, 0) + (0, y, 0) + (0, 0, z)$$
$$= x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

• **Hint:** The terms '2-space' and '3-space' are short forms for two-dimensional space and three-dimensional space respectively.

So, in Example 1, $\overrightarrow{AB} = (3, -3, -9) = 3\mathbf{i} - 3\mathbf{j} - 9\mathbf{k}$.

Unit vectors can be found in any direction, not only in the direction of the axes. For example, if we want to find the unit vector in the same direction as **u**, we need to find a vector parallel to **u**, which has a magnitude of 1. Since **u** has a magnitude of $|\mathbf{u}|$, it is enough to multiply this vector by $1/|\mathbf{u}|$ to 'normalize' it. So, the unit vector **v** in the same direction as **u** is

$\mathbf{v} = \frac{1}{|\mathbf{u}|}\mathbf{u} = \frac{\mathbf{u}}{|\mathbf{u}|}$. This is a unit vector since its length is 1. This is why:

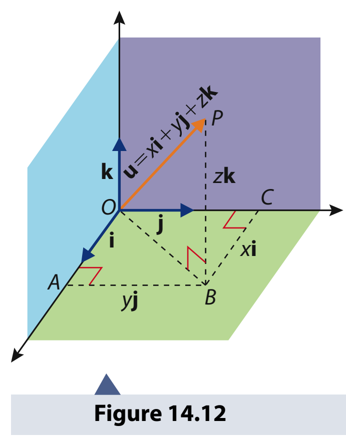


Figure 14.12

Recall that $|\mathbf{u}|$ is a real number (scalar), and so is $1/|\mathbf{u}|$.

Let $1/|\mathbf{u}| = k \Rightarrow \mathbf{v} = \frac{1}{|\mathbf{u}|}\mathbf{u} = k\mathbf{u} \Rightarrow |\mathbf{v}| = |k\mathbf{u}| = k|\mathbf{u}| = \frac{1}{|\mathbf{u}|}\cdot|\mathbf{u}| = 1.$

Example 3

Find a unit vector in the direction of $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

Solution

The length of the vector $\mathbf{v}$ is $\sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, so the unit vector is

$$\frac{1}{\sqrt{14}}(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = \frac{\mathbf{i}}{\sqrt{14}} - \frac{2\mathbf{j}}{\sqrt{14}} + \frac{3\mathbf{k}}{\sqrt{14}}.$$

To verify that this is a unit vector, we find its length:

$$\sqrt{\left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{2}{\sqrt{14}}\right)^2 + \left(\frac{3}{\sqrt{14}}\right)^2} = \sqrt{\frac{1}{14} + \frac{4}{14} + \frac{9}{14}} = 1$$

The unit vector plays another important role: it determines the direction of the given vector.

Recall from Chapter 9 that, in 2-space, we can write the vector in a form that gives us its direction (in terms of the angle it makes with the horizontal axis, called the direction angle) and its magnitude.

In the diagram below, θ is the angle with the horizontal axis.

The unit vector $\mathbf{v}$, in the same direction as $\mathbf{u}$, is:

$$\mathbf{v} = 1\cos\theta\,\mathbf{i} + 1\sin\theta\,\mathbf{j}$$

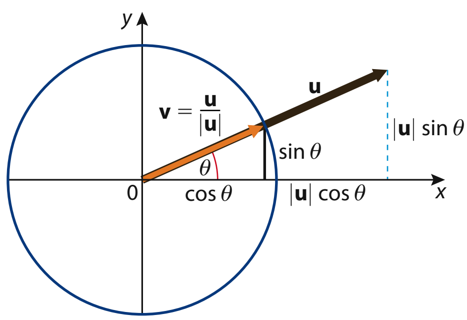
and from the results above,

$$\mathbf{v} = \frac{1}{|\mathbf{u}|}\mathbf{u} \Rightarrow$$

$$\mathbf{u} = |\mathbf{u}|\ (\mathbf{v})$$

$$= |\mathbf{u}|\cos\theta\,\mathbf{i} + |\mathbf{u}|\sin\theta\,\mathbf{j}$$

$$= |\mathbf{u}|(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}).$$



Example 4

Find the vector with magnitude 2 that makes an angle of 60° with the positive x-axis.

Solution

$$\mathbf{v} = |\mathbf{v}|\,(\cos 60°\,\mathbf{i} + \sin 60°\,\mathbf{j}) = 2\left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right) = \mathbf{i} + \sqrt{3}\mathbf{j}$$

Example 5

Find the direction and magnitude of the vector $\mathbf{v} = 2\sqrt{3}\mathbf{i} - 2\mathbf{j}$.

Solution

$$|\mathbf{v}| = \sqrt{(2\sqrt{3})^2 + 4} = 4$$

$$\cos\theta = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2},\ \sin\theta = \frac{-2}{4} = -\frac{1}{2} \Rightarrow \theta = -\frac{\pi}{6}$$

Example 6

a) Find the unit vector that has the same direction as $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$.

b) Find a vector of length 6 that is parallel to $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$.

Solution

a) The vector $\mathbf{v}$ has magnitude $|\mathbf{v}| = \sqrt{1 + 2^2 + (-2)^2} = 3$, so the unit vector $\boldsymbol{v}$ in the same direction as $\mathbf{v}$ is

$$\boldsymbol{v} = \frac{1}{3}\mathbf{v} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}.$$

b) Let $\mathbf{u}$ be the vector in question and $\boldsymbol{v}$ be the unit vector in the direction of $\mathbf{v}$.

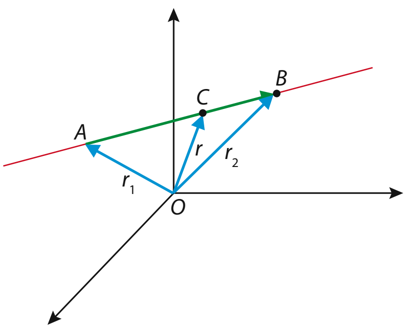
$$\mathbf{u} = 6 \cdot \boldsymbol{v} = 6 \times \frac{1}{\sqrt{14}}(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = \frac{6\mathbf{i}}{\sqrt{14}} - \frac{12\mathbf{j}}{\sqrt{14}} + \frac{18\mathbf{k}}{\sqrt{14}}$$

Example 7

Note: This problem introduces you to the vector equation of a line, as we will see in Section 14.4.

If $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of two points A and B in space, and λ is a real number, show that $\mathbf{r} = (1 - \lambda)\mathbf{r}_1 + \lambda\mathbf{r}_2$ is the position vector of a point C on the straight line joining A and B. Consider the cases where $\lambda = 0, 1, \frac{1}{2}, -1, 2$ and $\frac{2}{3}$.

Solution



Rewrite the equation:

$$\mathbf{r} = (1 - \lambda)\mathbf{r}_1 + \lambda\mathbf{r}_2 = \mathbf{r}_1 + \lambda(\mathbf{r}_2 - \mathbf{r}_1)$$

Since $\mathbf{r}_2 - \mathbf{r}_1 = \overrightarrow{AB}$, then the position vector $\mathbf{r}$ of C, which is simply $\mathbf{r}_1 + \overrightarrow{AC}$ gives us $\mathbf{r} = \mathbf{r}_1 + \lambda\overrightarrow{AB} = \mathbf{r}_1 + \overrightarrow{AC}$, which in turn gives $\overrightarrow{AC} = \lambda\overrightarrow{AB}$. As you have seen before, this means that $\overrightarrow{AC}$ is parallel to $\overrightarrow{AB}$ and is a multiple of it.

If $\lambda = 0$, then $\mathbf{r} = \mathbf{r}_1$ and C is at A.

If $\lambda = 1$, then $\mathbf{r} = \mathbf{r}_1 + \overrightarrow{AB}$ and C is at B.

If $\lambda = \frac{1}{2}$, then $\mathbf{r} = \mathbf{r}_1 + \frac{1}{2}\overrightarrow{AB}$ and C is the midpoint of AB.

If $\lambda = -1$, then $\mathbf{r} = \mathbf{r}_1 + -\overrightarrow{AB}$ and A is the midpoint of CB.

If $\lambda = 2$, then $\mathbf{r} = \mathbf{r}_1 + 2\overrightarrow{AB}$ and B is the midpoint of AC.

If $\lambda = \frac{2}{3}$, then $\mathbf{r} = \mathbf{r}_1 + \frac{2}{3}\overrightarrow{AB}$ and C is $\frac{2}{3}$ the way between A and B.

Exercise 14.1

1 Write the vector $\overrightarrow{AB}$ in component form in each of the following cases.

a) $A\left(-\frac{3}{2}, -\frac{1}{2}, 1\right); B\left(1, -\frac{5}{2}, 1\right)$

b) $A\left(-2, -\sqrt{3}, -\frac{1}{2}\right); B\left(1, \sqrt{3}, -\frac{1}{2}\right)$

c) $A(2, -3, 5); B(1, -1, 3)$

d) $A(a, -a, 2a); B(-a, -2a, a)$

2 Given the coordinates of point P or Q and the components of $\overrightarrow{PQ}$, find the missing items.

a) $P\left(-\frac{3}{2}, -\frac{1}{2}, 1\right); \overrightarrow{PQ}\left(1, -\frac{5}{2}, 1\right)$

b) $\overrightarrow{PQ}\left(-\frac{3}{2}, -\frac{1}{2}, 1\right); Q\left(1, -\frac{5}{2}, 1\right)$

c) $P(a, -2a, 2a); \overrightarrow{PQ}(-a, -2a, a)$

3 Determine the relationship between the coordinates of point $M(x, y, z)$ so that the points M, A and B are collinear.

a) $A(0, 0, 5); B(1, 1, 0)$

b) $A(-1, 0, 1); B(3, 5, -2)$

c) $A(2, 3, 4); B(-2, -3, 5)$

4 Given the coordinates of the points A and B, find the symmetric image C of B with respect to A.

a) $A(3, -4, 0); B(-1, 0, 1)$

b) $A(-1, 3, 5); B\left(-1, \frac{1}{2}, \frac{1}{3}\right)$

c) $A(1, 2, -1); B(a, 2a, b)$

5 Given a triangle ABC and a point G such that $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = 0$, find the coordinates of G in each of the following cases.

a) $A(-1, -1, -1); B(-1, 2, -1); C(1, 2, 3)$

b) $A(2, -3, 1); B(1, -2, -5); C(0, 0, 1)$

c) $A(a, 2a, 3a); B(b, 2b, 3b); C(c, 2c, 3c)$

6 Determine the fourth vertex D of the parallelogram $ABCD$ having AB and BC as adjacent sides.

a) $A(\sqrt{3}, 2, -1); B(1, 3, 0); C(-\sqrt{3}, 2, -5)$

b) $A(\sqrt{2}, \sqrt{3}, \sqrt{5}); B(3\sqrt{2}, -\sqrt{3}, 5\sqrt{5}); C(-2\sqrt{2}, \sqrt{3}, -3\sqrt{5})$

c) $A\left(-\frac{1}{2}, \frac{1}{3}, 0\right); B\left(\frac{1}{2}, \frac{2}{3}, 5\right); C\left(\frac{7}{2}, -\frac{1}{3}, 1\right)$

7 Determine the values of m and n such that the vectors $\mathbf{v}(m - 2, m + n, -2m + n)$ and $\mathbf{w}(2, 4, -6)$ have the same direction.

8 Find a unit vector in the same direction as each vector.

a) $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

b) $\mathbf{v} = 6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

c) $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

9 Find a vector with the given magnitude and in the same direction as the given vector.

a) Magnitude 2, $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

b) Magnitude 4, $\mathbf{v} = 6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

c) Magnitude 5, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

10 Let $\mathbf{u} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + \mathbf{j}$. Find

a) $|\mathbf{u} + \mathbf{v}|$

b) $|\mathbf{u}| + |\mathbf{v}|$

c) $|-3\mathbf{u}| + |3\mathbf{v}|$

d) $\frac{1}{|\mathbf{u}|}\mathbf{u}$

e) $\left|\frac{1}{|\mathbf{u}|}\mathbf{u}\right|$

11 Find the terminal points for each vector.

a) $\mathbf{w} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, given the initial point $(-1, 2, -3)$

b) $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, given the initial point $(-2, 1, 4)$

12 Find vectors that satisfy the stated conditions:

a) opposite direction of $\mathbf{u} = (-3, 4)$ and third the magnitude of $\mathbf{u}$

b) length of 12 and same direction as $\mathbf{w} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

c) of the form $x\mathbf{i} + y\mathbf{j} - 2\mathbf{k}$ and parallel to $\mathbf{w} = \mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$

13 Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be the vectors from each vertex of a triangle to the midpoint of the opposite side. Find the value of $\mathbf{u} + \mathbf{v} + \mathbf{w}$.

14 Find the scalar t (or show that there is none) so that the vector $\mathbf{v} = t\mathbf{i} - 2t\mathbf{j} + 3t\mathbf{k}$ is a unit vector.

15 Find the scalar t (or show that there is none) so that the vector $\mathbf{v} = 2\mathbf{i} - 2t\mathbf{j} + 3t\mathbf{k}$ is a unit vector.

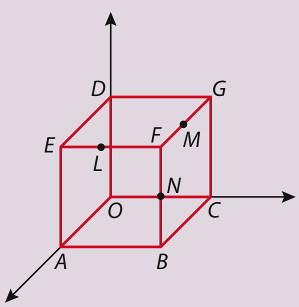
16 Find the scalar t (or show that there is none) so that the vector $\mathbf{v} = 0.5\mathbf{i} - t\mathbf{j} + 1.5t\mathbf{k}$ is a unit vector.

17 The diagram shows a cube of length 8 units.

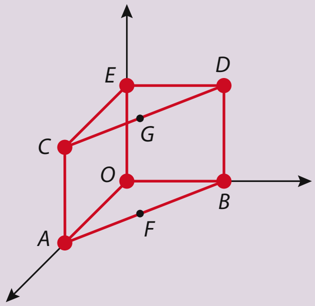
a) Find the position vectors of all the vertices.

b) L, M and N are the midpoints of the respective edges. Find the position vectors of L, M and N.

c) Show that $\overrightarrow{LM} + \overrightarrow{MN} + \overrightarrow{NL} = \vec{0}$.



18 A triangular prism is given with the lengths of the sides $OA = 8$, $OB = 10$ and $OE = 12$.



a) Find the position vectors of C and D.

b) F and G are the midpoints of the respective edges. Find their position vectors.

c) Find the vectors $\overrightarrow{AG}$ and $\overrightarrow{FD}$ and explain your results.

19 Find α such that $|\alpha\mathbf{i} + (\alpha - 1)\mathbf{j} + (\alpha + 1)\mathbf{k}| = 2$.

20 Let $\mathbf{a} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$, $\mathbf{d} = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$.

Find the scalars α, β and μ (or show that they cannot exist) such that $\mathbf{a} = \alpha\mathbf{b} + \beta\mathbf{c} + \mu\mathbf{d}$.

21 Repeat question 20 for $\mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$, $\mathbf{d} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

22 Repeat question 20 for $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{d} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$.

23 Let $\mathbf{u}$ and $\mathbf{v}$ be non-zero vectors such that $|\mathbf{u} - \mathbf{v}| = |\mathbf{u} + \mathbf{v}|$.

a) What can you conclude about the parallelogram with $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides?

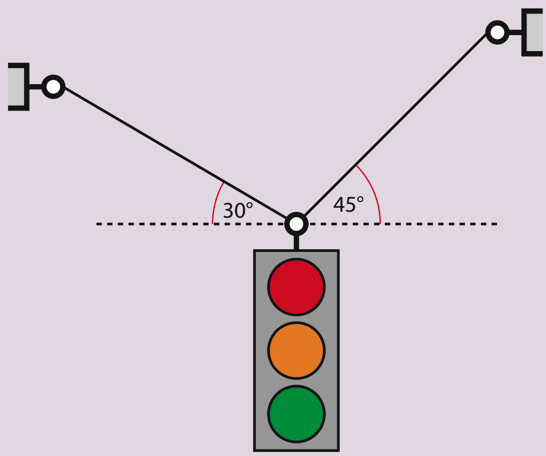
b) Show that if

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \text{ and } \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

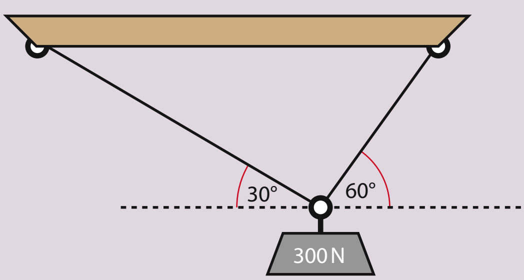
then

$$u_1v_1 + u_2v_2 + u_3v_3 = 0.$$

24 A 125 N traffic light is hanging from two flexible cables. The magnitude of the force that each cable applies to the 'eye ring' holding the lights is called the cable tension. Find the cable tensions if the light is in equilibrium.



25 Find the tension in the cables used to hold a weight of 300 N as shown in the diagram.



14.2 Scalar (dot) product

If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are two vectors, the dot product (scalar) is written as $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Result 1: $\mathbf{u}^2 = \mathbf{u} \cdot \mathbf{u} = u_1 \cdot u_1 + u_2 \cdot u_2 + u_3 \cdot u_3 = u_1^2 + u_2^2 + u_3^2 = |\mathbf{u}|^2$

From this definition, we can deduce another geometric 'definition' of the dot product:

$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$, where θ is the angle between the two vectors.

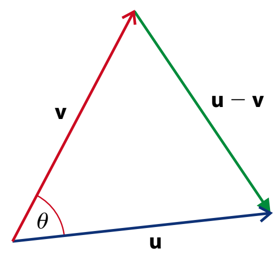


Figure 14.13

Proof:

Let **u** and **v** be drawn from the same point, as shown in Figure 14.13. Then

$$|\mathbf{u} - \mathbf{v}|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u}^2 + \mathbf{v}^2 - 2\mathbf{u} \cdot \mathbf{v}$$
$$= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\mathbf{u} \cdot \mathbf{v}.$$

Also, using the law of cosines,

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}| \cdot |\mathbf{v}| \cdot \cos \theta.$$

Conversely, using the law of cosines in the figure above gives

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}| \cdot |\mathbf{v}| \cdot \cos \theta,$$

which in turn will give

$$2|\mathbf{u}| \cdot |\mathbf{v}| \cdot \cos \theta = |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{u} - \mathbf{v}|^2$$
$$= (u_1^2 + u_2^2 + u_3^2) + (v_1^2 + v_2^2 + v_3^2) - [(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2]$$
$$= 2(u_1 v_1 + u_2 v_2 + u_3 v_3).$$

Thus, $|\mathbf{u}| \cdot |\mathbf{v}| \cdot \cos \theta = u_1 v_1 + u_2 v_2 + u_3 v_3$ and $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$.

By comparing the two results, we can conclude that $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}| \cos \theta$.

From the geometric definition of the dot product, we can see that for vectors of a given magnitude, the dot product measures *the extent to which the vectors agree in direction*. As the difference in direction, from 0 to π *increases*, the dot product *decreases*:

If **u** and **v** have the same direction, then $\theta = 0$ and

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta = |\mathbf{u}||\mathbf{v}|.$$

This is the largest possible value for $\mathbf{u} \cdot \mathbf{v}$.

If **u** and **v** are at right angles, then $\theta = \frac{\pi}{2}$ and

$$\mathbf{u} \cdot \mathbf{v} = 0.$$

If **u** and **v** have opposite directions, then $\theta = \pi$ and

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \pi = -|\mathbf{u}||\mathbf{v}|.$$

This is the least possible value for $\mathbf{u} \cdot \mathbf{v}$.

The scalar product can be used, among other things, to find angles between vectors:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta \Leftrightarrow \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{\mathbf{u}}{|\mathbf{u}|} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = u \cdot v$, where u and v are unit vectors in the direction of **u** and **v** respectively. That is, the cosine of the angle between two vectors is the dot product of the corresponding unit vectors.

Example 8

Find the angle between the vectors $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = -3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$.

Solution

From the previous results, we have

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\cdot|\mathbf{v}|} = \frac{-3 - 12 + 4}{\sqrt{1 + 4 + 4}\sqrt{9 + 36 + 4}} = \frac{-11}{21}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-11}{21}\right) \approx 2.12 \text{ radians}$$

Result 2: A direct conclusion of the previous definitions is that if two vectors are *perpendicular*, the dot product is *zero*.

This is so because when the two vectors are perpendicular the angle between them is $\pm 90°$ and, therefore,

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta = |\mathbf{u}||\mathbf{v}| \cos 90° = |\mathbf{u}||\mathbf{v}|\cdot 0 = 0.$$

The base vectors of the coordinate system are obviously perpendicular: $\mathbf{i} \cdot \mathbf{j} = (1, 0, 0) \cdot (0, 1, 0) = 0$, and similarly, $\mathbf{i} \cdot \mathbf{k} = 0$ and $\mathbf{j} \cdot \mathbf{k} = 0$.

Result 3: If two vectors **u** and **v** are parallel, then $\mathbf{u}\cdot\mathbf{v} = \pm|\mathbf{u}||\mathbf{v}|$.

Again, this is so because when the vectors are parallel the angle between them is either 0° or 180° and, therefore,

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta = |\mathbf{u}||\mathbf{v}| \cos 0° = |\mathbf{u}||\mathbf{v}|\cdot 1 = |\mathbf{u}||\mathbf{v}|, \text{ or}$$

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta = |\mathbf{u}||\mathbf{v}| \cos 180° = |\mathbf{u}||\mathbf{v}|\cdot(-1) = -|\mathbf{u}||\mathbf{v}|.$$

Example 9

Determine which, if any, of the following vectors are orthogonal.

$$\mathbf{u} = 7\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}, \mathbf{v} = -3\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}, \mathbf{w} = \mathbf{i} + \mathbf{k}$$

Solution

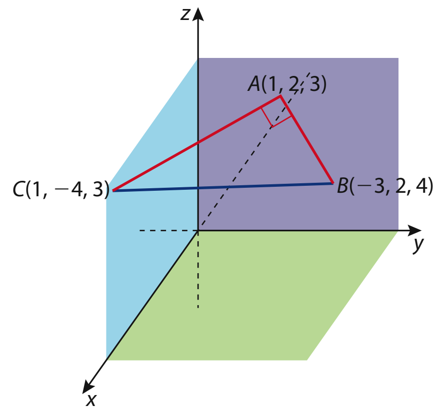
$\mathbf{u}\cdot\mathbf{v} = 7(-3) + 3 \times 5 + 2 \times 3 = 0$; orthogonal vectors

$\mathbf{u}\cdot\mathbf{w} = 7 \times 1 + 3 \times 0 + 2 \times 1 = 9$; not orthogonal

$\mathbf{v}\cdot\mathbf{w} = -3 \times 1 + 5 \times 0 + 3 \times 1 = 0$; orthogonal vectors

Example 10

$A(1, 2, 3)$, $B(-3, 2, 4)$ and $C(1, -4, 3)$ are the vertices of a triangle. Show that the triangle is right-angled and find its area.



Solution

$$\overrightarrow{AB} = (-3-1)\mathbf{i} + (2-2)\mathbf{j} + (4-3)\mathbf{k} = -4\mathbf{i} + \mathbf{k}$$

$$\overrightarrow{AC} = (1-1)\mathbf{i} + (-4-2)\mathbf{j} + (3-3)\mathbf{k} = -6\mathbf{j}$$

$$\overrightarrow{BC} = (1-(-3))\mathbf{i} + (-4-2)\mathbf{j} + (3-4)\mathbf{k} = 4\mathbf{i} - 6\mathbf{j} - \mathbf{k}$$

Since $\overrightarrow{AB}\cdot\overrightarrow{AC} = -4 \times 0 + 0 \times -6 + 1 \times 0 = 0$, the vectors are perpendicular. So the triangle is right-angled at A.

The area of this right triangle is half the product of the legs.

$$\text{Area} = \tfrac{1}{2}|\overrightarrow{AB}||\overrightarrow{AC}| = \tfrac{1}{2}\cdot\sqrt{(-4)^2 + 1}\cdot 6 = 3\sqrt{17}$$

Theorem

a) $|\mathbf{u}\cdot\mathbf{v}| \leqslant |\mathbf{u}|\,|\mathbf{v}|$

b) $|\mathbf{u}+\mathbf{v}| \leqslant |\mathbf{u}| + |\mathbf{v}|$

Proof

a) Since, $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$, then $|\mathbf{u}\cdot\mathbf{v}| = ||\mathbf{u}|\,|\mathbf{v}|\cos\theta| = |\mathbf{u}|\,|\mathbf{v}||\cos\theta|$, and as $|\cos\theta| \leqslant 1$, then $|\mathbf{u}\cdot\mathbf{v}| \leqslant |\mathbf{u}||\mathbf{v}|$.

b) $$\left.\begin{aligned}|\mathbf{u}+\mathbf{v}|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}\\ &= |\mathbf{u}|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2\end{aligned}\right\}\Leftarrow \mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$$

but, $\mathbf{u}\cdot\mathbf{v} \leqslant |\mathbf{u}\cdot\mathbf{v}|$, since $\mathbf{u}\cdot\mathbf{v}$ may also be negative while $|\mathbf{u}\cdot\mathbf{v}|$ is not.

Also, $|\mathbf{u}\cdot\mathbf{v}| \leqslant |\mathbf{u}|\,|\mathbf{v}|$, therefore

$$\begin{aligned}|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2 &\leqslant |\mathbf{u}|^2 + 2|\mathbf{u}\cdot\mathbf{v}| + |\mathbf{v}|^2\\ &\leqslant |\mathbf{u}|^2 + 2|\mathbf{u}||\mathbf{v}| + |\mathbf{v}|^2 = (|\mathbf{u}| + |\mathbf{v}|)^2.\end{aligned}$$

This then implies that $|\mathbf{u}+\mathbf{v}| \leqslant |\mathbf{u}| + |\mathbf{v}|$ (taking square roots).

Direction angles, direction cosines

Figure 14.14 shows a non-zero vector $\mathbf{v}$. The angles α, β and γ that the vector makes with the unit coordinate vectors are called the **direction angles** of $\mathbf{v}$, and $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the **direction cosines**.

Let $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Considering the right triangles OAP, OCP and ODP, the hypotenuse in each of these triangles is OP, i.e. $|\mathbf{v}|$. From your trigonometry chapters, you know that the side adjacent to an angle θ in a right triangle is related to it by

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \Leftrightarrow \text{adjacent} = \text{hypotenuse} \cdot \cos\theta, \text{ so in this case}$$

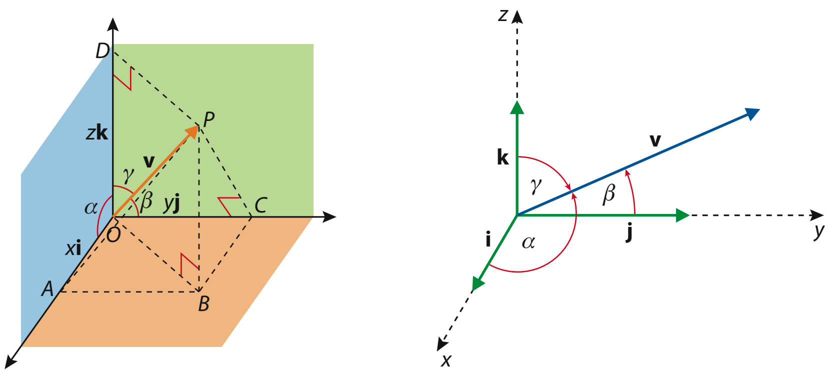
$x = |\mathbf{v}|\cos\alpha$, $y = |\mathbf{v}|\cos\beta$, $z = |\mathbf{v}|\cos\gamma$, and so

$\mathbf{v} = (|\mathbf{v}|\cos\alpha)\,\mathbf{i} + (|\mathbf{v}|\cos\beta)\,\mathbf{j} + (|\mathbf{v}|\cos\gamma)\,\mathbf{k} = |\mathbf{v}|(\cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k})$.

Taking the magnitude of both sides,

$$|\mathbf{v}| = |\mathbf{v}|\sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma}.$$

Figure 14.14



Therefore,
$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, i.e. the sum of the squares of the direction cosines is always 1.
For a unit vector, the expression will be of the form

$$\mathbf{u} = |\mathbf{u}|(\cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}) = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}\ (|\mathbf{u}| = 1).$$

This means that for a unit vector its x-, y- and z-coordinates are its direction cosines.

It is also important that you remember that $\cos\alpha = \frac{x}{|\mathbf{v}|}$, $\cos\beta = \frac{y}{|\mathbf{v}|}$, $\cos\gamma = \frac{z}{|\mathbf{v}|}$.

Example 11

Find the direction cosines of the vector $\mathbf{v} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, and then approximate the direction angles to the nearest degree.

Solution

$$|\mathbf{v}| = \sqrt{4^2 + (-2)^2 + 4^2} = 6 \Rightarrow \mathbf{v} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k},$$

$$\text{thus } \cos\alpha = \frac{2}{3}, \cos\beta = -\frac{1}{3}, \cos\gamma = \frac{2}{3}$$

From your GDC you will obtain:

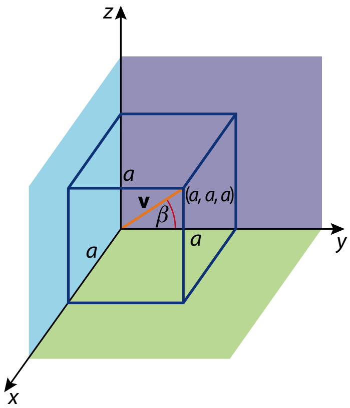
$$\alpha = \cos^{-1}\left(\frac{2}{3}\right) \approx 48°, \beta = \cos^{-1}\left(-\frac{1}{3}\right) \approx 109°, \gamma = \cos^{-1}\left(\frac{2}{3}\right) \approx 48°$$

Example 12

Find the angle that a main diagonal of a cube with side a makes with the adjacent edges.

Solution

We can place the cube in a coordinate system such that three of its adjacent edges lie on the coordinate axes as shown (right). The diagonal, represented by the vector $\mathbf{v}$ has a terminal point (a, a, a). Hence,

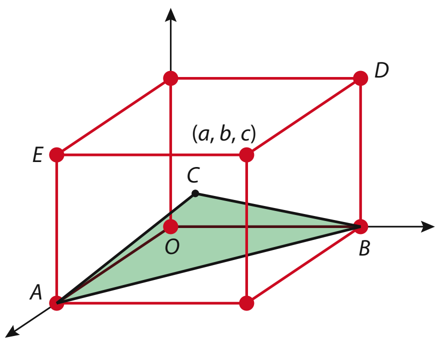


$|\mathbf{v}| = \sqrt{a^2 + a^2 + a^2} = a\sqrt{3}$.

Take angle β, for example:

$$\beta = \cos^{-1}\left(\frac{a}{a\sqrt{3}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.7°$$

Example 13



The point C is at the centre of the rectangular box whose edges have measures a, b and c. Find the measure of angle $A\hat{C}B$ in terms of a, b and c.

Solution

The point diagonally opposite to A is $D(0, b, c)$. So, $C\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$. Consequently,

$$\overrightarrow{CA} = \left(a - \frac{a}{2}, 0 - \frac{b}{2}, 0 - \frac{c}{2}\right) = \left(\frac{a}{2}, -\frac{b}{2}, -\frac{c}{2}\right) \text{ and } \overrightarrow{CB}$$
$$= \left(0 - \frac{a}{2}, b - \frac{b}{2}, 0 - \frac{c}{2}\right) = \left(-\frac{a}{2}, \frac{b}{2}, -\frac{c}{2}\right)$$

$$\cos A\hat{C}B = \frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|} = \frac{-\frac{a^2}{4} - \frac{b^2}{4} + \frac{c^2}{4}}{\sqrt{\frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4}}\sqrt{\frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4}}}$$
$$= \frac{c^2 - a^2 - b^2}{a^2 + b^2 + c^2}$$

Exercise 14.2

1 Find the dot product and the angle between the vectors.

a) $\mathbf{u} = (3, -2, 4)$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 6\mathbf{k}$

b) $\mathbf{u} = \begin{pmatrix} 2 \\ -6 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$

c) $\mathbf{u} = 3\mathbf{i} - \mathbf{j}$, $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j}$

d) $\mathbf{u} = \mathbf{i} - 3\mathbf{j}$, $\mathbf{v} = 5\mathbf{j} + 2\mathbf{k}$

e) $|\mathbf{u}| = 3$, $|\mathbf{v}| = 4$, the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{\pi}{3}$

f) $|\mathbf{u}| = 3$, $|\mathbf{v}| = 4$, the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{2\pi}{3}$

• **Hint:** Orthogonal means 'at right angles to each other'.

2 State whether the following vectors are orthogonal. If not orthogonal, is the angle acute?

a) $\mathbf{u} = \begin{pmatrix} 2 \\ -6 \\ 4 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$

b) $\mathbf{u} = 3\mathbf{i} - 7\mathbf{j}$, $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j}$

c) $\mathbf{u} = \mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$, $\mathbf{v} = 6\mathbf{j} + 3\mathbf{k}$

3 a) Show that the vectors $\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$ and $\mathbf{w} = y\mathbf{i} - x\mathbf{j}$ are both perpendicular to $\mathbf{u} = x\mathbf{i} + y\mathbf{j}$.

b) Find two unit vectors that are perpendicular to $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$. Plot the three vectors in the same coordinate system.

4 (i) Find the direction cosines of $\mathbf{v}$.

(ii) Show that they satisfy $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

(iii) Approximate the direction angles to the nearest degree.

a) $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$

b) $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

c) $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

d) $\mathbf{v} = 3\mathbf{i} - 4\mathbf{k}$

5 Find a unit vector with direction angles $\frac{\pi}{3}, \frac{\pi}{4}, \frac{2\pi}{3}$.

6 Find a vector with magnitude 3 and direction angles $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$.

7 Determine m so that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

a) $\mathbf{u} = (3, 5, 0)$; $\mathbf{v} = (m - 2, m + 3, 0)$

b) $\mathbf{u} = (2m, m - 1, m + 1)$; $\mathbf{v} = (m - 1, m, m - 1)$

8 Given the vectors $\mathbf{u} = (-3, 1, 2)$, $\mathbf{v} = (1, 2, 1)$, and $\mathbf{w} = \mathbf{u} + m\mathbf{v}$, determine the value of m so that the vectors $\mathbf{u}$ and $\mathbf{w}$ are orthogonal.

9 Given the vectors $\mathbf{u} = (-2, 5, 4)$ and $\mathbf{v} = (6, -3, 0)$, find, to the nearest degree, the measures of the angles between the following vectors.

a) $\mathbf{u}$ and $\mathbf{v}$

b) $\mathbf{u}$ and $\mathbf{u} + \mathbf{v}$

c) $\mathbf{v}$ and $\mathbf{u} + \mathbf{v}$

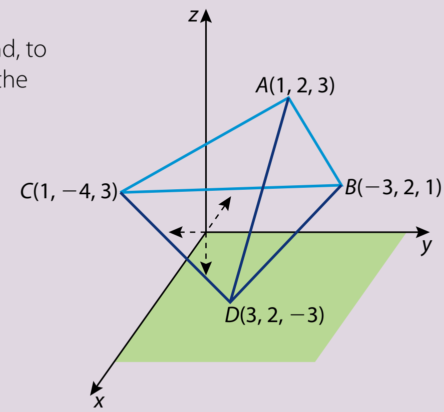
10 Consider the following three points: $A(1, 2, -3)$, $B(3, 5, -2)$ and $C(m, 1, -10m)$. Determine m so that

a) A, B and C are collinear

b) $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular.

11 Consider the triangle with vertices $A(4, -2, -1)$, $B(3, -5, -1)$ and $C(3, 1, 2)$. Find the vector equations of each of its medians and then find the coordinates of its centroid (i.e. where the medians meet).

12 Consider the tetrahedron $ABCD$ with vertices as shown in the diagram. Find, to the nearest degree, all the angles in the tetrahedron.



13 In question 12 above, use the angles you found to calculate the total surface area of the tetrahedron.

14 In question 12, what angles does $\overrightarrow{DC}$ make with each of the coordinate axes?

15 In question 12, find $(\overrightarrow{DA} - \overrightarrow{DB}) \cdot \overrightarrow{AC}$.

16 Find k such that the angle between the vectors $\begin{pmatrix} 3 \\ -k \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -3 \\ k \end{pmatrix}$ is $\frac{\pi}{3}$.

17 Find k such that the angle between the vectors $\begin{pmatrix} k \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ k \\ 1 \end{pmatrix}$ is $\frac{\pi}{3}$.

18 Find x and y such that $\begin{pmatrix} 2 \\ x \\ y \end{pmatrix}$ is perpendicular to both $\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$.

19 Consider the vectors $\begin{pmatrix} 1 - x \\ 2x - 2 \\ 3 + x \end{pmatrix}$ and $\begin{pmatrix} 2 - x \\ 1 + x \\ 1 + x \end{pmatrix}$. Find the value(s) of x such that the two vectors are parallel.

20 In triangle ABC, $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix}$.

Find the measure of $A\hat{B}C$.
Find $\overrightarrow{AC}$ and use it to find the measure of $B\hat{A}C$.

21 Find the value(s) of b such that the vectors are orthogonal.

a) $(b, 3, 2)$ and $(1, b, 1)$

b) $(4, -2, 7)$ and $(b^2, b, 0)$

c) $\begin{pmatrix} b \\ 11 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2b \\ -b \\ -5 \end{pmatrix}$

d) $\begin{pmatrix} 2 \\ 5 \\ 2b \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 4 \\ -b \end{pmatrix}$

22 If two vectors $\mathbf{p}$ and $\mathbf{q}$ are such that $|\mathbf{p}| = |\mathbf{q}|$, show that $\mathbf{p} + \mathbf{q}$ and $\mathbf{p} - \mathbf{q}$ are perpendicular. (This proves that the diagonals of a rhombus are perpendicular to each other!)

23 Shortly after take-off, a plane is rising at a rate of 300 m/min. It is heading at an angle of 45° north-west with an airspeed of 200 km/h. Find the components of its velocity vector. The x-axis is in the east direction, the y-axis north and the z-axis is the elevation.

24 For what value of t is the vector $2t\mathbf{i} + 4\mathbf{j} - (10 + t)\mathbf{k}$ perpendicular to the vector $\mathbf{i} + t\mathbf{j} + \mathbf{k}$?

25 For what value of t is the vector $t\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ perpendicular to the vector $\mathbf{i} + t\mathbf{j} + \mathbf{k}$?

26 For what value of t is the vector $4\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$ perpendicular to the vector $t^2\mathbf{i} + t\mathbf{j}$?

27 Find the angle between the diagonal of a cube and a diagonal of one of the faces. Consider all possible cases!

28 Show that the vector $|\mathbf{a}|\mathbf{b} + |\mathbf{b}|\mathbf{a}$ bisects the angle between the two vectors $\mathbf{a}$ and $\mathbf{b}$.

29 Let $\mathbf{u} = \mathbf{i} + m\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + n\mathbf{k}$. Compute all values of m and n for which $\mathbf{u} \perp \mathbf{v}$ and $|\mathbf{u}| = |\mathbf{v}|$.

30 Show that $\frac{\pi}{4}, \frac{\pi}{6}, \frac{2\pi}{3}$ cannot be the direction angles of a vector.

31 If a vector has direction angles $\alpha = \frac{\pi}{3}$ and $\beta = \frac{\pi}{4}$, find the third direction angle γ.

32 If a vector has all its direction angles equal, what is the measure of each angle?

33 If the direction angles of a vector $\mathbf{u}$ are α, β and γ, then what are the direction angles of $-\mathbf{u}$?

34 Find all possible values of a unit vector $\mathbf{u}$ that will be perpendicular to both $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

14.3 Vector (cross) product

In several applications of vectors there is a need to find a vector that is orthogonal to two given vectors. In this section we will discuss a new type of vector multiplication that can be used for this purpose.

If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are two vectors, then the vector (cross) product is written as $\mathbf{u} \times \mathbf{v}$ and is defined as

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k},$$

or, using the properties of determinants, we can observe that this definition is equivalent to

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Example 14

Given the vectors $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, find

a) $\mathbf{u} \times \mathbf{v}$ b) $\mathbf{v} \times \mathbf{u}$ c) $\mathbf{u} \times \mathbf{u}$

Solution

a) $$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 1 & 3 & -2 \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ 3 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -3 \\ 1 & 3 \end{vmatrix}\mathbf{k}$$

$$= 3\mathbf{i} + 5\mathbf{j} + 9\mathbf{k}.$$

You can also get the same result by simply evaluating the determinant using the short cut you learned in Chapter 6.

b) $$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ 2 & -3 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ -3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & -3 \end{vmatrix}\mathbf{k}$$

$$= -3\mathbf{i} - 5\mathbf{j} - 9\mathbf{k}.$$

Observe here that $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$!

c) $$\mathbf{u} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 2 & -3 & 1 \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ -3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -3 \\ 2 & -3 \end{vmatrix}\mathbf{k} = 0.$$

Determinants have many useful applications when we are dealing with vector products. Here are some of the properties which we state without proof.

1 If two rows of a determinant are proportional, then the value of that determinant is zero.

So, for example, if $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\mathbf{v} = ma\mathbf{i} + mb\mathbf{j} + mc\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ ma & mb & mc \end{vmatrix} = 0.$$

This result leads to an important property of vector products:

Two non-zero vectors are parallel if their cross product is zero.

2 If two rows of a determinant are interchanged, then its value is multiplied by (-1).

So, for instance, if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_2 \\ u_1 & u_2 & u_3 \end{vmatrix} = -(\mathbf{v} \times \mathbf{u}).$$

Properties

The following results are important in future work and are straightforward to prove. Most of the proofs are left as exercises.

1 $\mathbf{u} \times (\mathbf{v} \pm \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \pm (\mathbf{u} \times \mathbf{w})$

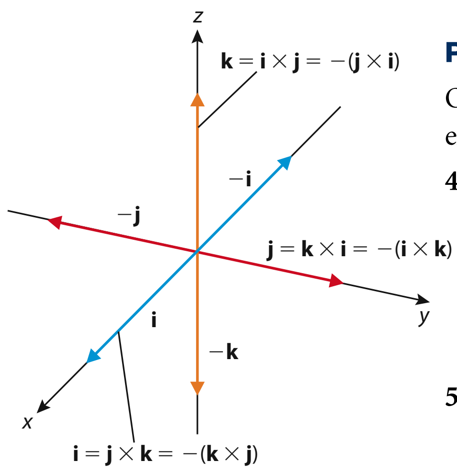
2 $\mathbf{u} \times 0 = 0$

3 $\mathbf{u} \times \mathbf{u} = 0$

4 $\mathbf{i} \times \mathbf{j} = \mathbf{k}$; $\mathbf{j} \times \mathbf{k} = \mathbf{i}$; $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

5 $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ (i.e. $\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{u}$.)

6 $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ (i.e. $\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{v}$.)



Proofs

Only properties 4 and 5 will be proved here, the rest will be left as an exercise.

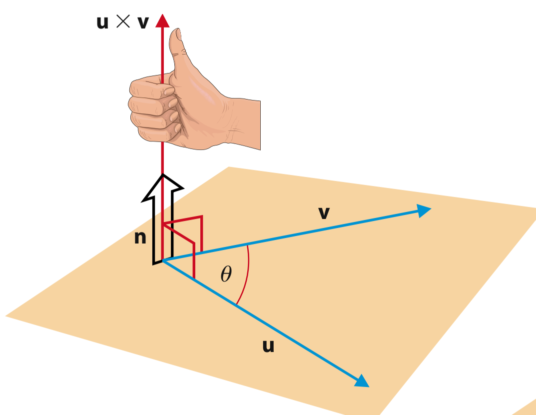
4 To prove the first result, we simply apply the definition.

$$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{k}, \text{ details are left as an exercise.}$$

5 $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = (u_1, u_2, u_3) \cdot \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$, so that

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= u_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - u_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + u_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\ &= u_1u_2v_3 - u_1u_3v_2 - u_2u_1v_3 + u_2u_3v_1 + u_3u_1v_2 - u_3u_2v_1 \\ &= 0 \end{aligned}$$

Properties 4 and 5 lead to an equivalent 'geometric' definition of the cross product:



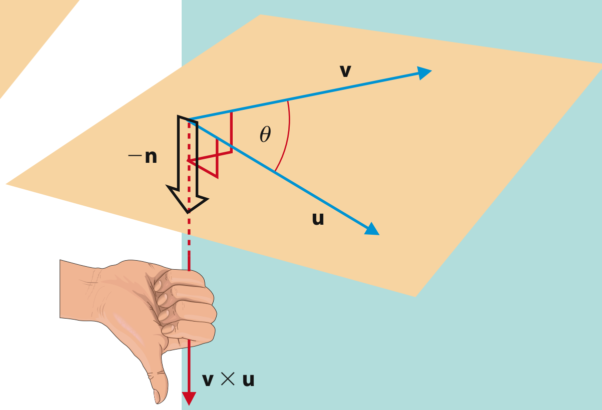
Definition of the cross product (Theorem):

($\mathbf{u} \times \mathbf{v}$) is a vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, obeying the right-hand rule shown right, and has the magnitude:

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta.$$

Note: As we mentioned before, please do not forget that

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}).$$



Proof

The algebraic manipulation required for the proof is tremendous, so we will keep the details away from this discussion:

$$\begin{aligned}
|\mathbf{u} \times \mathbf{v}| &= \sqrt{(u_2v_3 - u_3v_2)^2 + (u_1v_3 - u_3v_1)^2 + (u_1v_2 - u_2v_1)^2} \\
&= \sqrt{(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2} \\
&= \sqrt{|\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u}\cdot\mathbf{v})^2} = \sqrt{|\mathbf{u}|^2|\mathbf{v}|^2 - (|\mathbf{u}|\ |\mathbf{v}|\cos\theta)^2} \\
&= |\mathbf{u}||\mathbf{v}|\sqrt{1 - \cos^2\theta} = |\mathbf{u}||\mathbf{v}|\sin\theta
\end{aligned}$$

● **Hint:** The vector product gives you another method for finding the angle between two vectors.
$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$

$\Leftrightarrow \sin\theta = \dfrac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{u}||\mathbf{v}|}$

Example 15

Find a unit vector orthogonal to both vectors $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$.

Solution

$\mathbf{u} \times \mathbf{v}$ is orthogonal to both vectors.

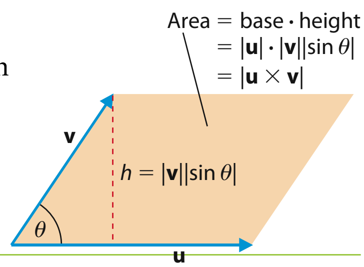
And $\mathbf{u} \times \mathbf{v} = 3\mathbf{i} + 5\mathbf{j} + 9\mathbf{k}$. A unit vector in the same direction as $\mathbf{u} \times \mathbf{v}$ will also be orthogonal to both vectors. Remembering that a unit vector is equal to the vector itself multiplied by the reciprocal of its magnitude, as we have seen in Chapter 9 and in Section 14.1, we find the magnitude of $\mathbf{u} \times \mathbf{v}$ first.

$$|\mathbf{u} \times \mathbf{v}| = \sqrt{9 + 25 + 81} = \sqrt{115},$$

and the required unit vector is therefore

$$\frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|} = \frac{3\mathbf{i} + 5\mathbf{j} + 9\mathbf{k}}{\sqrt{115}}.$$

Corollary: The last result leads to the conclusion that the magnitude of the cross product is the area of the parallelogram that has $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.



Example 16

Show that the quadrilateral $ABCD$ with its vertices at the following points is a parallelogram and find its area.
$A(3, 0, 2)$, $B(6, 2, 5)$, $C(1, 2, 2)$, $D(4, 4, 5)$

Solution

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}, \overrightarrow{BD} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}, \overrightarrow{CD} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$$

This implies that $\overrightarrow{AB} = \overrightarrow{CD}$ and $\overrightarrow{BD} = \overrightarrow{AC}$ which in turn means that the pairs of opposite sides of the quadrilateral are congruent and parallel. (You need only one pair.) Thus, $ABDC$ is a parallelogram with AB and BD as adjacent sides.

Furthermore, since

$$\overrightarrow{AB} \times \overrightarrow{BD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 3 \\ -2 & 2 & 0 \end{vmatrix} = -6\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

the area of the parallelogram is

$$|\overrightarrow{AB} \times \overrightarrow{BD}| = \sqrt{36 + 36 + 100} = \sqrt{172} = 2\sqrt{43}.$$

Example 17

Find the area of the triangle determined by the points $A(2, 2, 0)$, $B(-1, 0, 2)$ and $C(0, 4, 3)$.

Solution

The area of the triangle ABC is half the area of the parallelogram formed with AB and AC as its adjacent sides.

But $\overrightarrow{AB} = (-3, -2, 2)$ and $\overrightarrow{AC} = (-2, 2, 3)$, so $\overrightarrow{AB} \times \overrightarrow{AC} = (-10, 5, -10)$, and hence area of triangle $ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{15}{2}$.

The scalar triple product

(This product is very helpful in its geometric interpretation as it is of great help in finding the equation of a plane later in the chapter.)

If $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ are three vectors, then the scalar triple product is $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$. The component expression of this product can be found by applying the above definition:

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \mathbf{u} \cdot \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k} \right) \\
&= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \\
&= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}
\end{aligned}$$

Example 18

Calculate the scalar triple product of the vectors:

$$\mathbf{u} = 2\mathbf{i} - \mathbf{j} - 5\mathbf{k}, \mathbf{v} = 2\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}, \mathbf{w} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$$

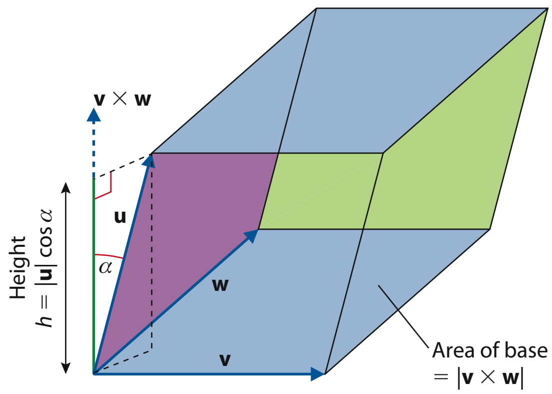
Solution

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & -1 & -5 \\ 2 & 5 & -5 \\ 1 & 4 & 3 \end{vmatrix} = 66$$

Geometric interpretation

$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ is the volume of the parallelepiped that has the three vectors as adjacent edges.

Proof



In the diagram above, $|\mathbf{v} \times \mathbf{w}|$ is the area of the parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$, which is the base of the parallelepiped. Also,

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\mathbf{u}||\mathbf{v} \times \mathbf{w}|\cos\alpha$$
$$= |\mathbf{v} \times \mathbf{w}||\mathbf{u}|\cos\alpha.$$

But, $|\mathbf{u}|\cos\alpha = h$, the height of the parallelepiped, and $|\mathbf{v} \times \mathbf{w}|$ is the area of the base; therefore, the triple product's absolute value is the volume of the parallelepiped.

A direct consequence of this theorem is that the volume of the parallelepiped is 0 if and only if the three vectors are coplanar. That is:

If $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ *and* $\mathbf{w} = (w_1, w_2, w_3)$ *are three vectors drawn from the same initial point, they lie in the same plane if:*

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0.$$

The parentheses in the scalar triple product is unnecessary, i.e. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$. Can you justify?

Example 19

Consider the three vectors

$$\mathbf{u} = 2\mathbf{i} + \mathbf{j} + m\mathbf{k},\ 3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \text{ and } \mathbf{w} = m\mathbf{i} + 2\mathbf{j} + \mathbf{k}.$$

a) Find the volume of the parallelepiped that has these vectors as sides.

b) Show that these vectors can never be on the same plane.

Solution

a) The volume of the parallelepiped is given by the absolute value of their scalar triple product:

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = \left\| \begin{matrix} 2 & 1 & m \\ 3 & 2 & 3 \\ m & 2 & 1 \end{matrix} \right\| = |-2m^2 + 9m - 11|$$

b) For the vectors to be coplanar, their scalar triple product must be zero. That is, $-2m^2 + 9m - 11 = 0$.

However, since this is a quadratic equation, it can have real roots if $b^2 - 4ac \geqslant 0$, but $b^2 - 4ac = 81 - 88 = -7 < 0$, and thus the equation does not admit any real roots and the three vectors can therefore never be coplanar.

Exercise 14.3

1 a) Find the cross product using the definition: $\mathbf{i} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$.

b) Compare your answer to $(\mathbf{i} \times \mathbf{i}) + (\mathbf{i} \times \mathbf{j}) + (\mathbf{i} \times \mathbf{k})$.

2 Repeat question 1 for $\mathbf{j} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$.

3 Repeat question 1 for $\mathbf{k} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$.

4 Use the definition of vector products to verify $\mathbf{u} \times (\mathbf{v} \pm \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \pm (\mathbf{u} \times \mathbf{w})$. (This is the distributive property of vector product over addition and subtraction.)

In questions 5–8, find $\mathbf{u} \times \mathbf{v}$ and check that it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

5 $\mathbf{u} = (2, 3, -2)$, $\mathbf{v} = (-3, 2, 3)$

6 $\mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = -2\mathbf{j} + 2\mathbf{k}$

7 $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}$

8 $\mathbf{u} = 5\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + \mathbf{k}$

9 Consider the following vectors:

$$\mathbf{u} = 2\mathbf{i} + \mathbf{j} + m\mathbf{k},\ \mathbf{v} = 3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k},\ \mathbf{w} = m\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

Find

a) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

b) $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$

c) $\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})$

10 Consider the following vectors:

$$\mathbf{u} = (3, 0, 4),\ \mathbf{v} = (1, 2, 8),\ \mathbf{w} = (2, 5, 6)$$

Find

a) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$

b) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$

c) $(\mathbf{u} \times \mathbf{v}) \times (\mathbf{v} \times \mathbf{w})$

d) $(\mathbf{v} \times \mathbf{w}) \times (\mathbf{u} \times \mathbf{v})$

e) $(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

f) $(\mathbf{w} \cdot \mathbf{u})\mathbf{v} - (\mathbf{w} \cdot \mathbf{v})\mathbf{u}$

11 Find a unit vector that is orthogonal to both

$$\mathbf{u} = -6\mathbf{i} + 4\mathbf{j} + \mathbf{k} \text{ and } \mathbf{v} = 3\mathbf{i} + \mathbf{j} + 5\mathbf{k}.$$

12 Find a unit vector that is normal (perpendicular) to the plane determined by the points $A(1, -1, 2)$, $B(2, 0, -1)$ and $C(0, 2, 1)$.

In questions 13–14, find the area of the parallelogram that has **u** and **v** as adjacent sides.

13 $\mathbf{u} = 2\mathbf{i} + 3\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

14 $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 3\mathbf{j} - \mathbf{k}$

15 Verify that the points are the vertices of a parallelogram and find its area: $(2, -1, 1)$, $(5, 1, 4)$, $(0, 1, 1)$ and $(3, 3, 4)$.

16 Show that the points $P(1, -1, 2)$, $Q(2, 0, 1)$, $R(3, 2, 0)$ and $S(5, 4, -2)$ are coplanar.

17 For what value(s) of m are the following four points on the same plane? $A(m, 3, -2)$, $B(3, 4, m)$, $C(2, 0, -2)$ and $D(4, 8, 4)$.

In questions 18–19, find the area of the triangle with the given vertices.

18 $A(2, 6, -1)$, $B(1, 1, 1)$, $C(3, 5, 2)$

19 $A(3, 1, -2)$, $B(2, 5, 6)$, $C(6, 1, 8)$

In questions 20–22, find $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

20 $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

21 $\mathbf{u} = (2, -1, 3)$, $\mathbf{v} = (1, 4, 3)$, $\mathbf{w} = (-3, 2, -2)$

22 $\mathbf{u} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$

In questions 23–24, find the volume of the parallelepiped with **u**, **v** and **w** as adjacent edges.

23 $\mathbf{u} = (3, -5, 3)$, $\mathbf{v} = (1, 5, -1)$, $\mathbf{w} = (3, 2, -3)$

24 $\mathbf{u} = 4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = 5\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$, $\mathbf{w} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$

In questions 25–26, determine whether the three given vectors are coplanar.

25 $\mathbf{u} = (2, -1, 2)$, $\mathbf{v} = (4, 1, -1)$, $\mathbf{w} = (6, -3, 1)$

26 $\mathbf{u} = (4, -2, -1)$, $\mathbf{v} = (9, -6, -1)$, $\mathbf{w} = (6, -6, 1)$

In questions 27–28, find m such that the following vectors are coplanar; otherwise, show that it is not possible.

27 $\mathbf{u} = (1, m, 1)$, $\mathbf{v} = (3, 0, m)$, $\mathbf{w} = (5, -4, 0)$

28 $\mathbf{u} = (2, -3, 2m)$, $\mathbf{v} = (m, -3, 1)$, $\mathbf{w} = (1, 3, -2)$

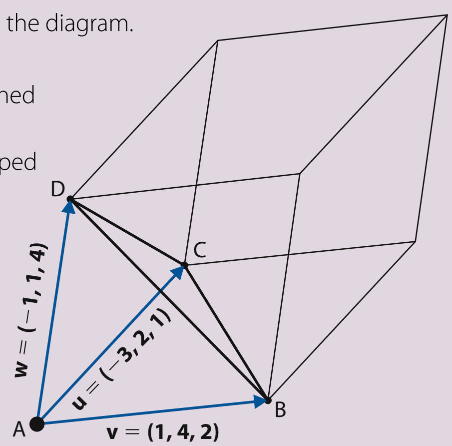
29 Consider the parallelepiped given in the diagram.

a) Find the volume.

b) Find the area of the face determined by **u** and **v**.

c) Find the height of the parallelepiped from vertex D to the base.

d) Find the angle that **w** makes with the plane determined by **u** and **v**.



30 a) From geometry, you know that the volume of a tetrahedron is $\frac{1}{3}$(base)(height). Use the results from the previous problem to find the volume of tetrahedron $ABCD$. Compare this volume to the volume of the parallelepiped and make a general conjecture.

b) Use the results you have from part a) to find the volume of the tetrahedron whose vertices are

$$A(0, 3, 1), B(3, 2, -2), C(2, 1, 2) \text{ and } D(4, -1, 4).$$

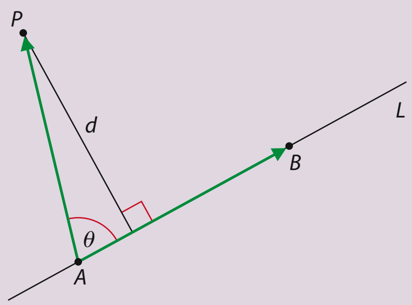
31 What can you conclude about the angle between two non-zero vectors **u** and **v** if $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u} \times \mathbf{v}|$?

32 Show that $|\mathbf{u} \times \mathbf{v}| = \sqrt{|\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2}$.

● **Hint:** Use right triangle trigonometry to find d in terms of θ first.

33 Use the diagram on the right to show that the distance from a point P in space to a line L through two points A and B can be expressed as

$$d = \frac{|\overrightarrow{AP} \times \overrightarrow{AB}|}{|\overrightarrow{AB}|}.$$



34 Use the result in the previous problem to find the distance from A to the line through the points B and C.

a) $A(-2, 2, 3), B(2, 2, 1), C(-1, 4, -3)$

b) $A(5, 4), B(3, 2), C(1, 3)$

c) $A(2, 0, 1), B(1, -2, 2), C(3, 0, 2)$

35 Express $(\mathbf{u} + \mathbf{v}) \times (\mathbf{v} - \mathbf{u})$ in terms of $(\mathbf{u} \times \mathbf{v})$.

36 Express $(2\mathbf{u} + 3\mathbf{v}) \times (4\mathbf{v} - 5\mathbf{u})$ in terms of $(\mathbf{u} \times \mathbf{v})$.

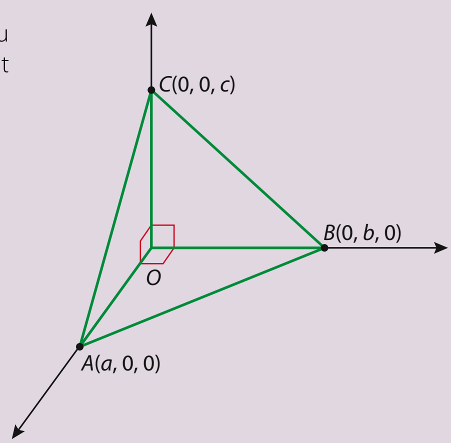
37 Express $(m\mathbf{u} + n\mathbf{v}) \times (p\mathbf{v} - q\mathbf{u})$ in terms of $(\mathbf{u} \times \mathbf{v})$, where m, n, p and q are scalars.

38 Refer to the diagram on the right. You are given a tetrahedron with vertex at the origin and base ABC.

a) Find the area of the base ABC. Call it o.

b) Find the area of each face of the tetrahedron and call them a, b and c.

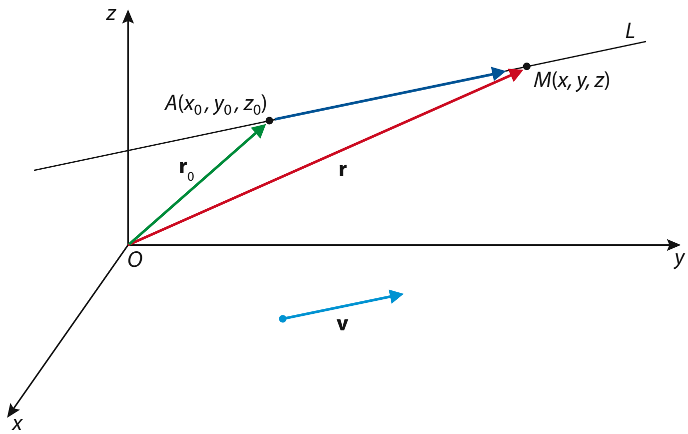
c) Show that $o^2 = a^2 + b^2 + c^2$. This is sometimes called the 3-D version of Pythagoras' theorem.



39 Find all vectors **v** such that $(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times \mathbf{v} = \mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$; otherwise, show that it is not possible.

40 Find all vectors **v** such that $(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times \mathbf{v} = \mathbf{i} + 5\mathbf{j}$; otherwise, show that it is not possible.

14.4 Lines in space



Similar to the plane, a straight line in space can be determined by any two points A and M that lie on it. Alternatively, the line can be determined by specifying a point on it and a direction given by a non-zero vector parallel to it. To investigate equations that describe lines in space, let us begin with a straight line L that passes through the point $A(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ as shown in the diagram. Now, if L is the line that passes through A and is parallel to the non-zero vector $\mathbf{v}$, then L consists of all the points $M(x, y, z)$ for which the vector $\overrightarrow{AM}$ is parallel to $\mathbf{v}$.

In Section 14.1 we established that:

Two vectors are parallel if one of them is a scalar multiple of the other. That is, $\mathbf{v}$ is parallel to $\mathbf{u}$ if and only if $\mathbf{v} = t\mathbf{u}$ for some real number t.

This means that for the point M to be on L, $\overrightarrow{AM}$ must be a scalar multiple of $\mathbf{v}$, i.e. $\overrightarrow{AM} = t\mathbf{v}$, where t is a scalar.

This equation can be written in coordinate form as

$$(x - x_0, y - y_0, z - z_0) = t(a, b, c) = (ta, tb, tc).$$

For two vectors to be equal, their components must be the same, then

$$x - x_0 = ta, y - y_0 = tb, z - z_0 = tc.$$

This leads to the result:

$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct.$$

The line that passes through the point $A(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = (a, b, c)$ has parametric equations:

$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct$$

Example 20

a) Find parametric equations of the line through $A(1, -2, 3)$ and parallel to $\mathbf{v} = 5\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$.

b) Find parametric equations of the line through the points $A(1, -2, 3)$ and $B(2, 4, -2)$.

Solution

a) From the previous theorem, $x = 1 + 5t$, $y = -2 + 4t$, $z = 3 - 6t$.

b) We need to find a vector parallel to the given line. The vector $\overrightarrow{AB}$ provides a good choice: $\overrightarrow{AB} = (1, 6, -5)$. So the equations are

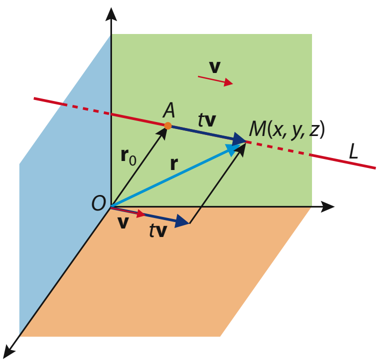
$$x = 1 + t,\ y = -2 + 6t,\ z = 3 - 5t.$$

Another set of equations could be

$$x = 2 + t,\ y = 4 + 6t,\ z = -2 - 5t.$$

Other sets are possible by considering any vector parallel to $\overrightarrow{AB}$.

Vector equation of a line



An alternative route to interpreting the equation

$$\overrightarrow{AM} = t\mathbf{v}$$

is to express it in terms of the position vectors $\mathbf{r}_0$ of the fixed point A, and $\mathbf{r}$, the position vector of M.

In Section 14.1, we discussed the difference of two vectors which can be of immediate use here.

$$\overrightarrow{AM} = \overrightarrow{OM} - \overrightarrow{OA}$$

$$\Rightarrow t\mathbf{v} = \mathbf{r} - \mathbf{r}_0$$

And hence we arrive at:

> The **vector equation** of the line
>
> $$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$
>
> where $\mathbf{r}$ is the position vector of any point on the line, while $\mathbf{r}_0$ is the position vector of a fixed point (A in this case) on the line and $\mathbf{v}$ is the vector parallel to the given line.

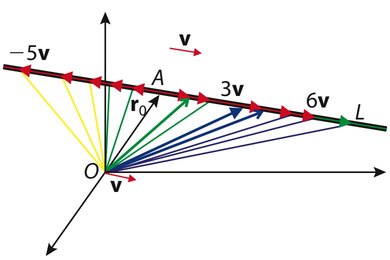


Figure 14.15 By observing Figure 14.15, you will notice, for example, that for each value of t you describe a point on the line. When $t > 0$, the points are in the same direction as $\mathbf{v}$. When $t < 0$, the points are in the opposite direction.

> The two approaches are very closely related. We can even say that the parametric equations are a detailed form of the vector equation!
>
> $$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$
> $$\Leftrightarrow (x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$$
> $$\Leftrightarrow (x, y, z) = (x_0, y_0, z_0) + (ta, tb, tc) = (x_0 + ta, y_0 + tb, z_0 + tc)$$
> $$\Leftrightarrow \begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

You can interpret vector equations in several ways. One of these has to do with displacement. That is, to reach point M from point O, you first arrive at A, and then go towards M along the line a multiple of $\mathbf{v}$, $t\mathbf{v}$.

Example 21

Find a vector equation of the line that contains $(-1, 3, 0)$ and is parallel to $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Solution

From the previous discussion,

$$\mathbf{r} = (-\mathbf{i} + 3\mathbf{j}) + t(3\mathbf{i} - 2\mathbf{j} + \mathbf{k}).$$

When $t = 0$, the equation gives the point $(-1, 3, 0)$. When $t = 1$, the equation yields

$\mathbf{r} = (-\mathbf{i} + 3\mathbf{j}) + (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, a point shifted by $1\mathbf{v}$ down the line. Similarly, when $t = 3$,

$\mathbf{r} = (-\mathbf{i} + 3\mathbf{j}) + 3(3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 8\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$, a point $3\mathbf{v}$ down the line, etc.

Alternatively, the equation can be written as

$$\mathbf{r} = (-1 + 3t)\mathbf{i} + (3 - 2t)\mathbf{j} + t\mathbf{k}.$$

This last form allows us to recognize the parametric equations of the line by simply reading the components of the vector on the right-hand side of the equation.

Example 22

Find a vector equation of the line passing through $A(2, 7)$ and $B(6, 2)$.

Solution

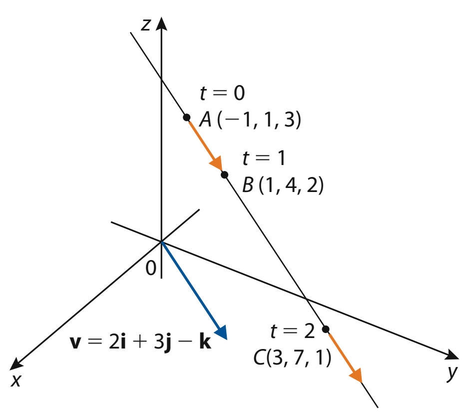
We let the vector $\overrightarrow{AB} = (6 - 2, 2 - 7) = (4, -5)$ be the vector giving the direction of the line, so

$$\mathbf{r} = (2, 7) + t(4, -5), \text{ or equivalently}$$

$$\mathbf{r} = 2\mathbf{i} + 7\mathbf{j} + t(4\mathbf{i} - 5\mathbf{j}).$$

Example 23

Find parametric equations for the line through $A(-1, 1, 3)$ and parallel to the vector $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.



Solution

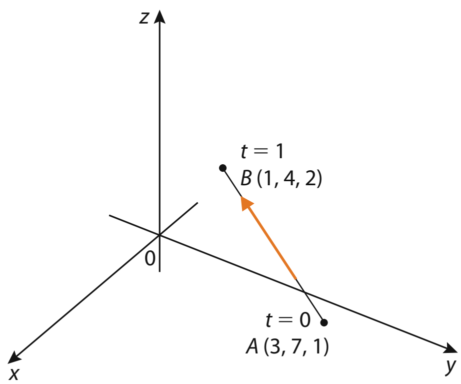
$$x = -1 + 2t, y = 1 + 3t, z = 3 - 1$$

If you select a few points with their parameter values you can see how the equation represents the line. For $t = 0$, as you expect, you are at point A; for $t = 1$, the point is B; and for $t = 2$, the point is C. The arrows show the direction of increasing values of t.

Line segments

Sometimes, we would like to 'parametrize' a line segment. That is, to write the equation so that it describes the points making up the segment. For example, to parametrize the line segment between $A(3, 7, 1)$ and $B(1, 4, 2)$, we first find the direction vector $AB = (-2, -3, 1)$, then we use point A as the fixed point on the line. Thus, the parametric equations are:

$$\begin{cases} x = 3 - 2t \\ y = 7 - 3t \\ z = 1 + t \end{cases}$$



Notice that when $t = 0$, the line starts at the point $A(3, 7, 1)$; when $t = 1$, the line is at $B(1, 4, 2)$. Therefore, to parametrize this segment we restrict the values of t to $0 \leqslant t \leqslant 1$. The new equations are then

$$x = 3 - 2t, y = 7 - 3t, z = 1 + t, 0 \leqslant t \leqslant 1.$$

In general, to parametrize a line segment AB so that we represent the points included between the endpoints only, we can use the vector equation

$$\mathbf{r}(t) = (1 - t)\overrightarrow{OA} + t\overrightarrow{OB}, 0 \leqslant t \leqslant 1.$$

In this parametrization, when $t = 0$, $\mathbf{r} = \overrightarrow{OA}$, and when $t = 1$, $\mathbf{r} = \overrightarrow{OB}$. This way $\mathbf{r}$ traces the segment AB from A to B for $0 \leqslant t \leqslant 1$.

Note: The parametrization with the vector equation can be expressed differently if we want to use parametric equation.

If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are the endpoints of the segment, then

$$\begin{aligned}\mathbf{r}(t) &= (1-t)\overrightarrow{OA} + t\overrightarrow{OB} = \overrightarrow{OA} - t\overrightarrow{OA} + t\overrightarrow{OB}\\ &= \overrightarrow{OA} + t(\overrightarrow{OB} - \overrightarrow{OA})\\ &= (x_1, y_1, z_1) + t(x_2 - x_1, y_2 - y_1, z_2 - z_1)\\ &= (x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1), z_1 + t(z_2 - z_1)).\end{aligned}$$

So the parametric equations are

$$x = x_1 + t(x_2 - x_1), y = y_1 + t(y_2 - y_1), z = z_1 + t(z_2 - z_1),$$
$$0 \leqslant t \leqslant 1.$$

Example 24

Parametrize the segment through $A(2, -1, 5)$ and $B(4, 3, 2)$. Use the equation to find the midpoint of the segment.

Solution

$$\begin{aligned}\mathbf{r}(t) &= (1-t)\overrightarrow{OA} + t\overrightarrow{OB}, 0 \leqslant t \leqslant 1\\ &= (1-t)(2, -1, 5) + t(4, 3, 2)\\ &= (2 + 2t, -1 + 4t, 5 - 3t)\end{aligned}$$

For the midpoint, $t = \frac{1}{2}$, and hence its coordinates are

$$\mathbf{r}\left(\tfrac{1}{2}\right) = \left(2 + 2\left(\tfrac{1}{2}\right), -1 + 4\left(\tfrac{1}{2}\right), 5 - 3\left(\tfrac{1}{2}\right)\right) = \left(3, 1, \tfrac{7}{2}\right).$$

Note: This method can be used to find points that divide the segment in any ratio: $\frac{2}{3}$ the way from A to B, etc.

Equivalently, the parametric equations can be used.

$$x = 2 + t(4 - 2), y = -1 + t(3 + 1), z = 5 + t(2 - 5)$$
$$x = 2 + 2t, y = -1 + 4t, z = 5 - 3t, 0 \leqslant t \leqslant 1$$

Symmetric (Cartesian) equations of lines

Another set of equations for a line is obtained by eliminating the parameter from the parametric equation.

If $a \neq 0$, $b \neq 0$ and $c \neq 0$, then the set of parametric equations can be re-arranged to yield the set of Cartesian (symmetric) equations:

$$\left.\begin{aligned} x - x_0 = ta &\Leftrightarrow \frac{x - x_0}{a} = t\\ y - y_0 = tb &\Leftrightarrow \frac{y - y_0}{b} = t\\ z - z_0 = tc &\Leftrightarrow \frac{z - z_0}{c} = t\end{aligned}\right\} \Leftrightarrow \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Notice that the coordinates (x_0, y_0, z_0) of the fixed point A on L appear in the numerators of the fractions, and that the components a, b and c of a direction vector appear in the denominators of these fractions.

Example 25

Find the Cartesian equations of the line through $A(3, -7, 4)$ and $B(1, -4, -1)$.

Solution

In order to use the Cartesian equation, we find the vector **v** parallel to the line. Since A and B are two points that lie on the line, the vector $\overrightarrow{AB}$ will suffice. Thus, we let

$$\mathbf{v} = \overrightarrow{AB} = (1 - 3)\mathbf{i} + (-4 + 7)\mathbf{j} + (-1 - 4)\mathbf{k} = -2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}.$$

If we use A as the fixed point, then the Cartesian equations are

$$\frac{x-3}{-2} = \frac{y+7}{3} = \frac{z-4}{-5}.$$

Similarly, if we use B as the fixed point, then

$$\frac{x-1}{-2} = \frac{y+4}{3} = \frac{z+1}{-5}.$$

Example 26

Let L be the line with Cartesian equations

$$\frac{x-2}{3} = \frac{y+1}{-2} = z - 4.$$

Find a set of parametric equations for L.

Solution

Since the numbers in the denominators are the components of a vector parallel to L, then

$$\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

The point $(2, -1, 4)$ lies on L.

Thus, a set of parametric equations of L is

$$x = 2 + 3t, y = -1 - 2t, z = 4 + t.$$

A vector equation would be

$$\mathbf{r} = (2, -1, 4) + t(3, -2, 1).$$

Note: If any of the components a, b or c is zero, then the Cartesian equations are written in a mixed form. For example, if $c = 0$, then we write

$$\frac{x-x_0}{a} = \frac{y-y_0}{b}, z = z_0.$$

For example, the Cartesian set of equations for a line parallel to $2\mathbf{i} - 3\mathbf{j}$ through the point $(2, 1, -3)$ is

$$\frac{x-2}{2} = \frac{y-1}{-3}, z = -3.$$

Intersecting, parallel and skew straight lines

In the plane, lines can coincide, intersect or be parallel. This is not necessarily so in space. In addition to the three cases above, there is the case of skew straight lines. Although these lines are not parallel, they do not intersect either. They lie in different planes.

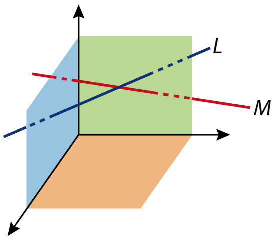


Figure 14.16a Two intersecting straight lines.

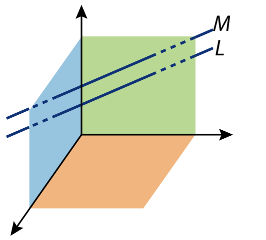


Figure 14.16b Two parallel lines.

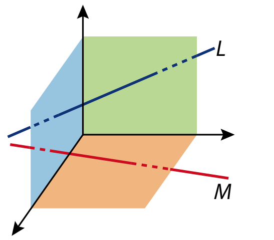


Figure 14.16c Two skew straight lines.

How do we know whether two lines are parallel?

If the 'direction' vectors are parallel, then the lines are. Check to see if one of the vectors is a scalar multiple of the other. Alternatively, you can find the angle between them, and if it is 0° or 180°, the lines are either parallel or coincident. The case for coincidence is always there, and you need to check it by examining a point on one of the lines to see whether it is also on the other line.

Example 27

Show that the following two lines are parallel.

$$L_1\!: x = 2 - 3t,\ y = t,\ z = -1 + 2t$$

$$L_2\!: x = 1 + 6s,\ y = 2 - 2s,\ z = 2 - 4s$$

Solution

Let $\mathbf{l}_1$ be the vector parallel to L_1 and $\mathbf{l}_2$ be the vector parallel to L_2.

$$\mathbf{l}_1 = -3\mathbf{i} + \mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{l}_2 = 6\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}.$$

Now you can easily see that $\mathbf{l}_2 = -2\mathbf{l}_1$, and hence the vectors are parallel.

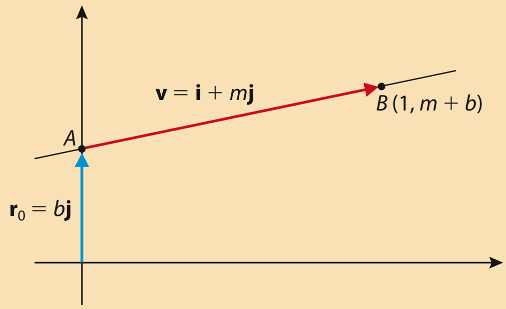
To check whether the lines coincide, we examine the point $(2, 0, -1)$, which is on the first line, and see whether it lies on the second line too.

If we choose $y = 0$, then $0 = 2 - 2s$, so $s = 1$; and when we substitute $s = 1$ into $x = 1 + 6s$ we find out that x must be 7 in order for the point $(2, 0, -1)$ to be on L_2. Therefore, the lines cannot intersect, and their 'direction' vectors are parallel, so they must be parallel.

Are the lines intersecting or skew?

If the direction vectors are not parallel, the lines either intersect or are skew. For the purposes of this course, the method starts by examining whether the lines intersect. If they do, we can find the coordinates of the point of intersection; if they do not intersect, we cannot find the coordinates of the point of intersection. Finding the coordinates of the point of intersection is a straightforward method that you already know: solving systems of equations. This can best be explained with an example.

The vector equation of a line has an interesting application in proving a result that you already know – the condition for two lines to be perpendicular.



Consider the slope-intercept form of the equation of the line is $y = mx + b$. We can think of the fixed point on the line to be its y-intercept, i.e. $\mathbf{r}_0 = b\mathbf{j}$, and another point $(1, m + b)$ on the line $\Rightarrow \mathbf{r} = \mathbf{i} + (m + b)\mathbf{j}$. So, the direction vector of the line is $\mathbf{v} = \mathbf{r} - \mathbf{r}_0 = (\mathbf{i} + (m + b)\mathbf{j}) - b\mathbf{j} = \mathbf{i} + m\mathbf{j}$.

Now, if you have two lines with slopes m_1 and m_2, their direction vectors can be written as $\mathbf{v}_1 = \mathbf{i} + m_1\mathbf{j}$ and $\mathbf{v}_2 = \mathbf{i} + m_2\mathbf{j}$. For the two lines to be perpendicular, their direction vectors will also be perpendicular, and hence

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (\mathbf{i} + m_1\mathbf{j}) \cdot (\mathbf{i} + m_2\mathbf{j}) = 1 + m_1m_2 = 0$$
$$\Rightarrow m_1m_2 = -1.$$

Example 28

The lines L_1 and L_2 have the following equations:

$$L_1 : x = 1 + 4t, y = 5 - 4t, z = -1 + 5t$$
$$L_2 : x = 2 + 8s, y = 4 - 3s, z = 5 + s$$

Show that the lines are skew.

Solution

We first examine whether the lines are parallel. Since the vector parallel to L_1 is $\mathbf{l}_1 = (4, -4, 5)$ and the vector parallel to L_2 is $\mathbf{l}_2 = (8, -3, 1)$, they are not scalar multiples of each other and the vectors and consequently the lines are not parallel.

For the lines to intersect, there should be some point $M(x_0, y_0, z_0)$ which satisfies the equations of both lines for some values of t and s. That is,

$$x_0 = 1 + 4t = 2 + 8s;\ y_0 = 5 - 4t = 4 - 3s;\ z_0 = -1 + 5t = 5 + s.$$

This leads to a set of three simultaneous equations in two unknowns: s and t.

By solving the first two equations:

$$\left.\begin{array}{l} 1 + 4t = 2 + 8s \\ 5 - 4t = 4 - 3s \end{array}\right\} \Rightarrow 6 = 6 + 5s \Rightarrow s = 0,\ t = \tfrac{1}{4}$$

For the system to be consistent, these values must satisfy the third equation, i.e. $-1 + \frac{5}{4} = 5 + 0$, which is false. Hence, the system is inconsistent and the lines are skew.

Example 29

The lines L_1 and L_2 have the following equations:

$$L_1 : x = 1 + 2t,\ y = 3 - 4t,\ z = -2 + 4t$$
$$L_2 : x = 4 + 3s,\ y = 4 + s,\ z = -4 - 2s$$

Show that the lines intersect.

Solution

We first examine whether the lines are parallel. Since the vector parallel to L_1 is $\mathbf{l}_1 = (2, -4, 4)$ and the vector parallel to L_2 is $\mathbf{l}_2 = (3, 1, -2)$, they are not scalar multiples of each other and the vectors and consequently the lines are not parallel.

For the lines to intersect, there should be some point $M(x_0, y_0, z_0)$ which satisfies the equations of both lines for some values of t and s. That is,

$$x_0 = 1 + 2t = 4 + 3s;\ y_0 = 3 - 4t = 4 + s;\ z_0 = -2 + 4t = -4 - 2s.$$

This leads to a set of three simultaneous equations in two unknowns: s and t.

By solving the first two equations:

$$\left.\begin{array}{l} 1 + 2t = 4 + 3s \\ 3 - 4t = 4 + s \end{array}\right\} \Rightarrow 5 = 12 + 7s \Rightarrow s = -1,\ t = 0$$

For the system to be consistent, these values must satisfy the third equation, i.e. $-2 + 4(0) = -4 - 2(-1) \Rightarrow -2 = -2$, which is a correct statement. Hence, the two lines intersect.

The point of intersection can be found through substitution of the value of the parameter into the corresponding line equation:

$$L_1: (1\ , 3, -2) \text{ and } L_2: (4 - 3, 4 - 1, -4 - 2(-1)) = (1, 3, -2)$$

Note: In vector form, finding the point of intersection, if it exists, follows a similar approach. For example, the vector equations of lines L_1 and L_2 are

L_1: $\mathbf{r} = (1, 3, -2) + t(2, -4, 4)$

L_2: $\mathbf{r} = (4, 4, -4) + s(3, 1, -2)$

The condition of intersection is therefore

$$(1, 3, -2) + t(2, -4, 4) = (4, 4, -4) + s(3, 1, -2),$$

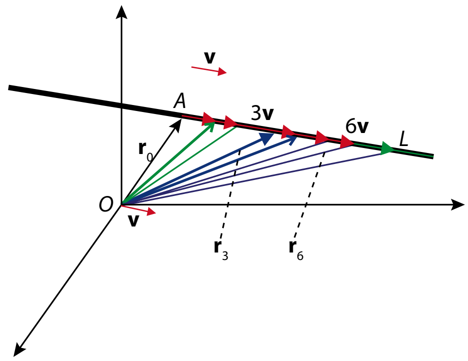
which leads to the same conclusion as in the case of parametric equations.

Application of lines to motion

The vector form of the equation of a line in space is more revealing when we think of the line as the path of an object, placed in an appropriate coordinate system and starting at position $A(x_0, y_0, z_0)$ and moving in the direction of $\mathbf{v}$.

Generally speaking, you find an object at an initial location A, represented by $\mathbf{r}_0$. The object moves on its path with a velocity vector $\mathbf{v} = (a, b, c)$. The object's position at any point in time after the start can then be described by $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$.

Figure 14.17



Assuming the unit of time is seconds, the equation tells us that for every second, the object moves a units in the x direction, b in the y direction and c in the z direction. So, for example, after 2 seconds you find the object at $\mathbf{r} = \mathbf{r}_0 + 2\mathbf{v}$.

The speed of the object is then $|\mathbf{v}|$ in the $\mathbf{v}$ direction.

In general, we can write the vector equation in a slightly modified form.

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

$$= \underbrace{\mathbf{r}_0}_{\text{initial position}} + \underbrace{t}_{\text{time}} \cdot \underbrace{|\mathbf{v}|}_{\text{speed}} \cdot \underbrace{\frac{\mathbf{v}}{|\mathbf{v}|}}_{\text{direction}}$$

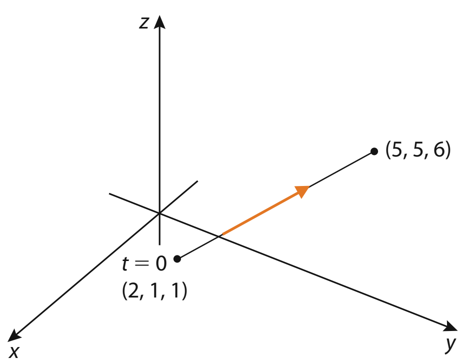
In other words, the position of an object at time t is the *initial position* plus its *rate* × *time* (distance moved) in the *direction* $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$ of its straight-line motion.

Example 30

A model plane is to fly directly from a platform at a reference point (2, 1, 1) toward a point (5, 5, 6) at a speed of 60 m/min. What is the position of the plane (to the nearest metre) after 10 minutes?

Solution

The unit vector in the direction of the flight is $\mathbf{u} = \frac{3}{5\sqrt{2}}\mathbf{i} + \frac{4}{5\sqrt{2}}\mathbf{j} + \frac{5}{5\sqrt{2}}\mathbf{k}$.



The position of the plane at any time t is

$$\mathbf{r}(t)_0 = \mathbf{r}_0 + t(\mathit{speed})(\mathbf{u})$$

$$= (2\mathbf{i} + \mathbf{j} + \mathbf{k}) + (10)(60)\left(\frac{3}{5\sqrt{2}}\mathbf{i} + \frac{4}{5\sqrt{2}}\mathbf{j} + \frac{5}{5\sqrt{2}}\mathbf{k}\right)$$

$$= (2\mathbf{i} + \mathbf{j} + \mathbf{k}) + \left(\frac{360}{\sqrt{2}}\mathbf{i} + \frac{480}{\sqrt{2}}\mathbf{j} + \frac{600}{\sqrt{2}}\mathbf{k}\right).$$

So, the plane is approximately at (257, 340, 425).

Example 31

An object is moving in the plane of an appropriately fitted coordinate system such that its position is given by

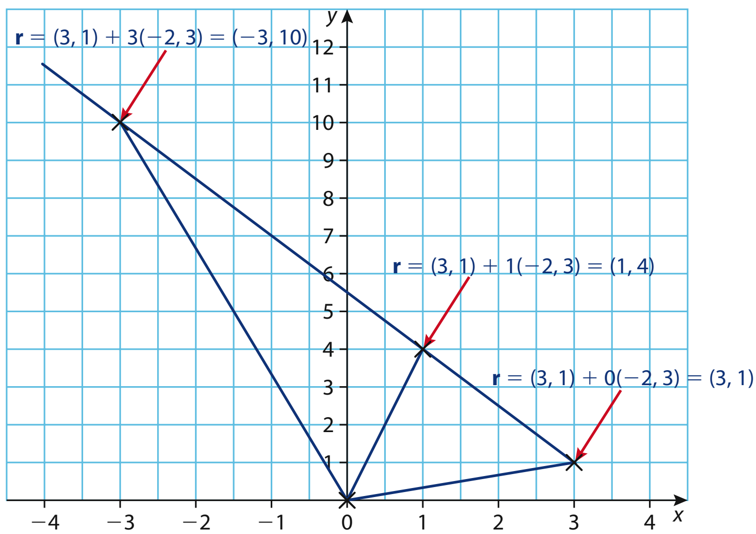
$$\mathbf{r} = (3, 1) + t(-2, 3),$$

where t stands for time in hours after start and distances are measured in km.

a) Find the initial position of the object.
b) Show the position of the object on a graph at start, 1 hour and 3 hours after start.
c) Find the velocity and speed of the object.

Solution

a) Initial position is when $t = 0$. This is the point $(3, 1)$.
b) See graph.



c) The velocity vector is $\mathbf{v} = (-2, 3)$, which means that every hour the object moves 2 units west and 3 units north.

The speed is $|\mathbf{v}| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$ km/h.

We can also express the velocity as $\sqrt{13}$ km/h in the direction of $(-2, 3)$.

Note: We can also express the direction in terms of the unit vector in the direction of $\mathbf{v}$ instead. That is, we can say that the speed is $\sqrt{13}$ km/h in the direction of $\left(\frac{-2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)$, or, equivalently, at an angle of $\cos^{-1}\left(\frac{-2}{\sqrt{13}}\right) \approx 124°$ to the positive x-direction.

Example 32

At 12:00 midday a plane A is passing in the vicinity of an airport at a height of 12 km and a speed of 800 km/h. The direction of the plane is $(4, 3, 0)$. [Consider that $(1, 0, 0)$ is a displacement of 1 km due east, $(0, 1, 0)$ due north, and $(0, 0, 1)$ is an altitude of 1 km.]

a) Using the airport as the origin, find the position vector **r** of the plane t hours after midday.
b) Find the position of the plane 1 hour after midday.
c) Another plane B is heading towards the airport with velocity vector $(-300, -400, 0)$ from a location $(600, 480, 12)$. Is there a danger of collision?

Solution

a) The position vector at midday is $(0, 0, 12)$. The direction of the velocity vector is given by the unit vector $\frac{1}{5}(4, 3, 0)$. So, the velocity vector of this plane is $800 \cdot \frac{1}{5}(4, 3, 0) = (640, 480, 0)$.
The position vector of the plane is $\mathbf{r} = (0, 0, 12) + t(640, 480, 0)$.
b) $\mathbf{r} = (0, 0, 12) + (640, 480, 0) = (640, 480, 12)$
c) A collision can happen if the two planes pass the same point at the same time.
The position vector for the second plane is $\mathbf{r} = (600, 480, 12) + t(-300, -400, 0)$.
If the two paths intersect, they may intersect at instances corresponding to t_1 and t_2 and they should have the same position, i.e.

$$(0, 0, 12) + t_1(640, 480, 0) = (600, 480, 12) + t_2(-300, -400, 0).$$

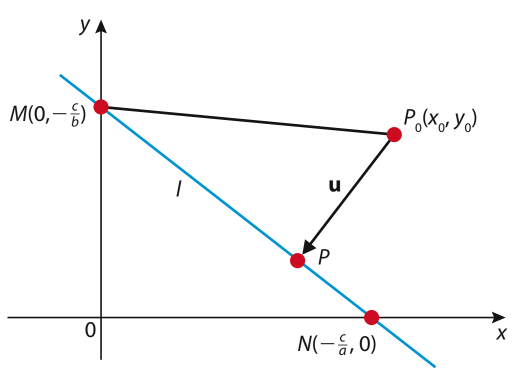
This gives rise to a set of three equations in two variables:

$$\left.\begin{aligned} 640t_1 &= 600 - 300t_2 \\ 480t_1 &= 480 - 400t_2 \\ 12 &= 12 \end{aligned}\right\}$$

Solving the system of equations simultaneously will give $t_1 = \frac{6}{7}$ and $t_2 = \frac{6}{35}$.
This means that the planes' paths will cross at $(548.57, 411.43, 12)$.
There is no collision though because plane A will pass that point at 12:51 while plane B will pass this point at 12.10!

Distance from a point to a line (optional)

2-space



Theorem: If the equation of a line l is written in the form $ax + by + c = 0$, then the distance from a point $P_0(x_0, y_0)$ to the line l is given by

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

There are several methods of proving this theorem. We will follow a vector approach, leaving some other interesting methods for the website.

The x- and y-intercepts of the line l are

$$N\left(-\frac{c}{a}, 0\right) \text{ and } M\left(0, -\frac{c}{b}\right).$$

So, a vector parallel to l can be any vector in the direction of $\overrightarrow{NM} = \left(\frac{c}{a}, -\frac{c}{b}\right)$.

For convenience we will consider the vector $\mathbf{L} = \left(\frac{1}{a}, -\frac{1}{b}\right)$.

Consider a vector in the direction of $\overrightarrow{P_0P}$ perpendicular to l. Consider vector $\mathbf{u} = (a, b)$ which is perpendicular to l because $\mathbf{d} \cdot \mathbf{L} = 0$ and hence parallel to $\overrightarrow{P_0P}$, then in triangle MPP_0, the distance $|\overrightarrow{P_0P}|$ is

$$|\overrightarrow{MP_0}|\cos(MP_0P).$$

$$\overrightarrow{MP_0} = \left(x_0, y_0 + \frac{c}{b}\right)$$

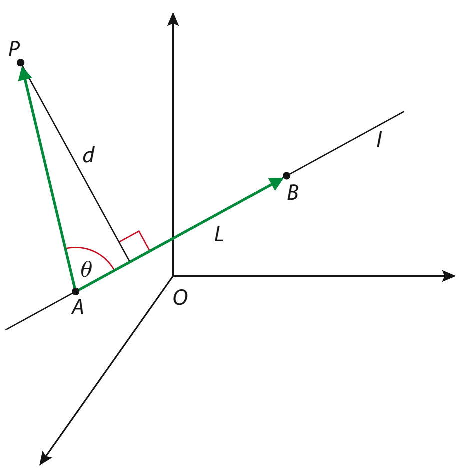
$$|\overrightarrow{P_0P}| = \left||\overrightarrow{MP_0}| \cdot \cos M\hat{P_0}P\right| = \left|\frac{|\overrightarrow{MP_0}| \cdot \overrightarrow{MP_0} \cdot \overrightarrow{P_0P}}{|\overrightarrow{MP_0}| \cdot |\overrightarrow{P_0P}|}\right| = \left|\frac{\overrightarrow{MP_0} \cdot \mathbf{u}}{|\mathbf{u}|}\right|$$

$$= \left|\frac{\left(x_0, y_0 + \frac{c}{b}\right) \cdot (a, b)}{\sqrt{a^2 + b^2}}\right| = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

3-space

$$d = |\overrightarrow{AP_0}| \cdot \sin(\theta) = |\overrightarrow{AP_0}| \cdot \frac{|\mathbf{L} \times \overrightarrow{AP_0}|}{|\mathbf{L}| \cdot |\overrightarrow{AP_0}|} = \frac{|\mathbf{L} \times \overrightarrow{AP_0}|}{|\mathbf{L}|}$$

where A is any point on line l and $\mathbf{L}$ is a vector parallel to l.



Example 33

Find the distance from the point (1, 3) to the line with equation

$$2x - y = 7.$$

Solution

The equation can be written as $2x - y - 7 = 0$ and hence the distance is

$$d = \frac{|2(1) - 3 - 7|}{\sqrt{2^2 + 1}} = \frac{8}{\sqrt{5}}.$$

Example 34

Find the distance from the point $P(8, 1, -3)$ to the line containing $M(3, 0, 6)$ and $N(5, -2, 7)$.

Solution

We have:

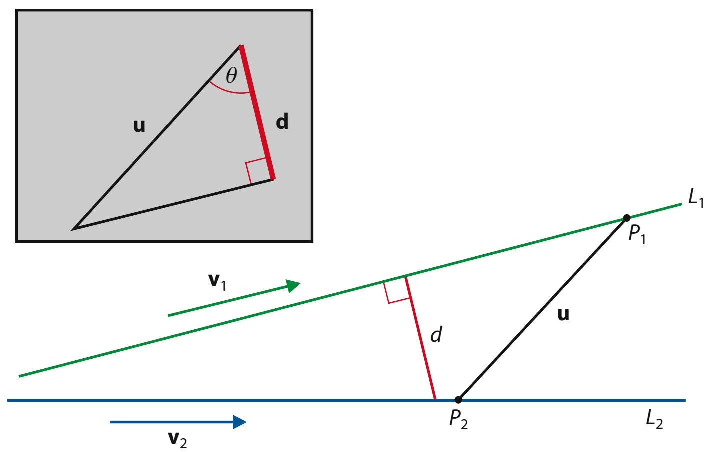
$$\overrightarrow{MN} = (2, -2, 1) \text{ and } \overrightarrow{MP} = (5, 1, -9) \Rightarrow$$

$$d = \frac{|(2, -2, 1) \times (5, 1, -9)|}{|(2, -2, 1)|} = \frac{|(17, 23, 12)|}{\sqrt{2^2 + (-2)^2 + 1}}$$

$$= \frac{\sqrt{17^2 + 23^2 + 12^2}}{3} = \frac{\sqrt{962}}{3}$$

Distance between two skew straight lines

In the diagram below, two skew straight lines L_1 and L_2 are given with direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. We need to find the distance d, defined as the length of the 'common perpendicular', between them.



Consider any two fixed points P_1 on L_1 and P_2 on L_2. The distance d is the length of the orthogonal projection of vector $\mathbf{u} = \overrightarrow{P_1P_2}$ on a direction perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$. This direction can be determined by $\mathbf{v}_1 \times \mathbf{v}_2$. So,

$$d = \left||\overrightarrow{P_1P_2}|\cos\theta\right| = \left||\overrightarrow{P_1P_2}|\frac{\overrightarrow{P_1P_2} \cdot (\mathbf{v}_1 \times \mathbf{v}_2)}{|\overrightarrow{P_1P_2}||\mathbf{v}_1 \times \mathbf{v}_2|}\right| = \left|\frac{\overrightarrow{P_1P_2} \cdot (\mathbf{v}_1 \times \mathbf{v}_2)}{|\mathbf{v}_1 \times \mathbf{v}_2|}\right|.$$

Example 35

Find the distance between the following skew lines:

$$L_1: \mathbf{r} = (2, 3, 1) + t(1, 2, -3);\ L_2: \mathbf{r} = 4\mathbf{i} + 2\mathbf{j} + s(3\mathbf{i} - \mathbf{j} + \mathbf{k})$$

Solution

The two fixed points could be taken as (2, 3, 1) and (4, 2, 0) while the vectors are $\mathbf{v}_1 = (1, 2, -3)$ and $\mathbf{v}_2 = (3, -1, 1)$.

$$d = \left|\frac{(2, -1, -1)(-1, -10, -7)}{|(-1, -10, -7)|}\right|$$

$$= \left|\frac{-2 + 10 + 7}{\sqrt{1 + 100 + 49}}\right| = \frac{15}{\sqrt{150}} = \frac{\sqrt{6}}{2}$$

Note: The minimum distance could be found using other methods too. One of them would be to consider the line going from any point on L_1 to any point on L_2. This will give a parametric equation in s and t. Then considering that this line will be perpendicular to both L_1 and L_2, i.e. $\mathbf{u} \cdot \mathbf{v}_1 = 0$, $\mathbf{u} \cdot \mathbf{v}_2 = 0$, enables us to set up a system of two equations that could be solved for s and t. Lastly, we get the distance between the points corresponding to the specific values we just established.

Exercise 14.4

1 Find a vector equation, a set of parametric equations and a set of Cartesian equations of the line containing the point A and parallel to the vector $\mathbf{u}$.

a) $A(-1, 0, 2)$, $\mathbf{u} = (1, 5, -4)$
b) $A(3, -1, 2)$, $\mathbf{u} = (2, 5, -1)$
c) $A(1, -2, 6)$, $\mathbf{u} = (3, 5, -11)$

2 Find all three forms of the equation of the line that passes through the points A and B.

a) $A(-1, 4, 2)$, $B(7, 5, 0)$
b) $A(4, 2, -3)$, $B(0, -2, 1)$
c) $A(1, 3, -3)$, $B(5, 1, 2)$

3 a) Write the equation of the line through the points $(3, -2)$ and $(5, 1)$ in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.
b) Write the equation of the line through the points $(0, -2)$ and $(5, 0)$ in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.

4 The equation of a line in 2-space is given by $\mathbf{r} = (2, 1) + t(3, -2)$. Write the equation in the form $ax + by = c$.

5 Find the equation of a line through $(2, -3)$ that is parallel to the line with equation $\mathbf{r} = 3\mathbf{i} - 7\mathbf{j} + \lambda(4\mathbf{i} - 3\mathbf{j})$.

6 Find the equation of a line through $(-2, 1, 4)$ and parallel to the vector $3\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$.

7 In each of the following, find the point of intersection of the two given lines, and if they do not intersect, explain why.

a) L_1: $\mathbf{r} = (2, 2, 3) + t(1, 3, 1)$
L_2: $\mathbf{r} = (2, 3, 4) + t(1, 4, 2)$
b) L_1: $\mathbf{r} = (-1, 3, 1) + t(4, 1, 0)$
L_2: $\mathbf{r} = (-13, 1, 2) + t(12, 6, 3)$
c) L_1: $\mathbf{r} = (1, 3, 5) + t(7, 1, -3)$
L_2: $\mathbf{r} = (4, 6, 7) + t(-1, 0, 2)$

d) $L_1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$

$L_2: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix} + s\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix}$

8 Find the vector and parametric equations of each line:

a) through the points $(2, -1)$ and $(3, 2)$

b) through the point $(2, -1)$ and parallel to the vector $\begin{pmatrix} -3 \\ 7 \end{pmatrix}$

c) through the point $(2, -1)$ and perpendicular to the vector $\begin{pmatrix} -3 \\ 7 \end{pmatrix}$

d) with y-intercept $(0, 2)$ and in the direction of $2\mathbf{i} - 4\mathbf{j}$

9 Consider the line with equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}.$$

a) For what value of t does this line pass through the point $\left(0, \frac{11}{2}, \frac{9}{2}\right)$?

b) Does the point $(-1, 4, 6)$ lie on this line?

c) For what value of m does the point $\left(\frac{1-2m}{2}, 2m, 3\right)$ lie on the given line?

10 Consider the following equations representing the paths of cars after starting time $t \geqslant 0$, where distances are measured in km and time in hours. For each car, determine

(i) starting position

(ii) the velocity vector

(iii) the speed.

a) $\mathbf{r} = (3, -4) + t\begin{pmatrix} 7 \\ 24 \end{pmatrix}$

b) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 5 \\ -12 \end{pmatrix}$

c) $(x, y) = (5, -2) + t(24, -7)$

11 Find the velocity vector of each of the following racing cars taking part in the Paris–Dakar rally:

a) direction $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ with a speed of 160 km/h

b) direction $\begin{pmatrix} 12 \\ -5 \end{pmatrix}$ with a speed of 170 km/h

12 After leaving an intersection of roads located at 3 km east and 2 km north of a city, a car is moving towards a traffic light 7 km east and 5 km north of the city at a speed of 30 km/h. (Consider the city as the origin for an appropriate coordinate system.)

a) What is the velocity vector of the car?

b) Write down the equation of the position of the car after t hours.

c) When will the car reach the traffic light?

13 Consider the vectors $\mathbf{u} = (1, a, b)$, $\mathbf{v} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = -2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

a) Find a and b so that $\mathbf{u}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$.

b) If O is the origin, P a point whose position vector is $\mathbf{v}$ and Q is with position vector $\mathbf{w}$, find the cosine of the angle between $\mathbf{v}$ and $\mathbf{w}$.

c) Hence, find the sine of the angle and use it to find the area of the triangle OPQ.

14 The triangle ABC has vertices at the points $A(-1, 2, 3)$, $B(-1, 3, 5)$ and $C(0, -1, 1)$.

a) Find the size of the angle θ between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b) Hence, or otherwise, find the area of triangle ABC.

Let L_1 be the line parallel to $\overrightarrow{AB}$ which passes through $D(2, -1, 0)$, and L_2 be the line parallel to $\overrightarrow{AC}$ which passes through $E(-1, 1, 1)$.

c) (i) Find the equations of the lines L_1 and L_2.

(ii) Hence, show that L_1 and L_2 do not intersect.

15 Consider the points $A(1, 3, -17)$ and $B(6, -7, 8)$ which lie on the line l.

a) Find an equation of line l, giving the answer in parametric form.

b) The point P is on l such that $\overrightarrow{OP}$ is perpendicular to l. Find the coordinates of P.

16 a) Starting with the equation of a line in the form $mx + ny = p$, find a vector equation of the line.

b) (i) Starting with a vector equation of a line where $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, with $\mathbf{r}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, find an equation of the line in the form $mx + ny = p$.

(ii) What is the relationship between the components of the direction vector $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ and the slope of the line?

17 Find a parametrization for the line segment between points A and B in each of the following questions.

(i) $A(0, 0, 0)$, $B(1, 1, 3)$

(ii) $A(-1, 0, 1)$, $B(1, 1, -2)$

(iii) $A(1, 0, -1)$, $B(0, 3, 0)$

18 Find a vector equation and a set of parametric equations of the line through the point $(0, 2, 3)$ and parallel to the line $\mathbf{r} = (\mathbf{i} - 2\mathbf{j}) + 2t\mathbf{k}$.

19 Find a vector equation and a set of parametric equations of the line through the point $(1, 2, -1)$ and parallel to the line $\mathbf{r} = t(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$.

20 Find a vector equation and a set of parametric equations of the line through the origin and the point $A(x_0, y_0, z_0)$.

21 Find a vector equation and a set of parametric equations of the line through $(3, 2, -3)$ and perpendicular to

a) the xz-plane

b) the yz-plane.

22 Write a set of symmetric equations for the line through the origin and the point $A(x_0, y_0, z_0)$, $x_0, y_0, z_0 \neq 0$.

In questions 23–29, determine whether the lines l_1 and l_2 are parallel, skew or intersecting. If they intersect, find the coordinates of the point of intersection.

23 l_1: $x - 3 = 1 - y = \dfrac{z - 5}{2}$, l_2: $\mathbf{r} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{j} + \mathbf{k})$

24 l_1: $\begin{cases} x = -1 + s \\ y = 2 - 3s \\ z = 1 + 2s \end{cases}$ $\quad l_2$: $\begin{cases} x = 2 - 2m \\ y = -1 + 6m \\ z = -4m \end{cases}$

25 l_1: $\dfrac{x - 3}{2} = \dfrac{1 + y}{4} = 2 - z$, l_2: $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

26 l_1: $x - 1 = \dfrac{y - 1}{3} = \dfrac{z + 4}{2}$, l_2: $1 - x = -1 - y = \dfrac{z}{2}$

27 l_1: $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(-6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k})$, l_2: $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + m(2\mathbf{i} - 3\mathbf{j} + \mathbf{k})$

28 $\frac{x-2}{5} = y - 1 = \frac{z-2}{3}$ and $\frac{x+4}{3} = \frac{7-y}{3} = \frac{10-z}{4}$

29 $x = 1 + t, y = 2 - 2t, z = t + 5$ and $x = 2 + 2t, y = 5 - 9t, z = 2 + 6t$

30 Find the point on the line

$$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} + t(-3\mathbf{i} + \mathbf{j} + \mathbf{k})$$

that is closest to the origin. (Hint: use the parametric form and the distance formula and minimize the distance using derivatives!)

31 Find the point on the line

$$\mathbf{r} = 4\mathbf{j} + 5\mathbf{k} + t(\mathbf{i} - 3\mathbf{j} + \mathbf{k})$$

that is closest to the origin.

32 Find the point on the line

$$\mathbf{r} = 5\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(\mathbf{i} - 3\mathbf{j} + \mathbf{k})$$

that is closest to the point $(-1, 4, 1)$.

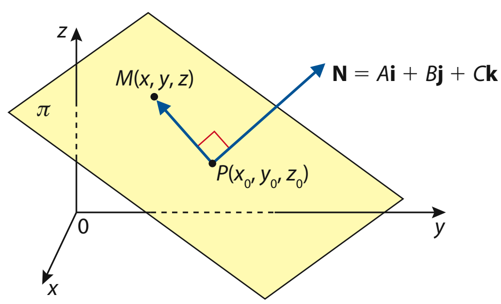
14.5 Planes

To define/specify a plane is to identify it in a way that makes it unique. One way is to set up an equation in a frame that will identify every point that belongs to the plane. There are several ways of specifying a plane but we will only mention four of them here. The rest will be cases that we address in some problems later. For more helpful geometric concepts please refer to the book's website.

A plane can be defined

- by three non-collinear points
- by two intersecting straight lines
- to be perpendicular to a certain direction and at a specific distance from the origin (for example)
- by being drawn through a given point and perpendicular to a given direction.

A direction, for our purposes, can be defined by a vector. In the case of a plane, the vector determining the direction is perpendicular to the plane and is said to be **normal to the plane**.



Equations of a plane

From the many ways of defining a plane above, the last two are mostly appropriate for deriving equations of a plane.

Cartesian (scalar) equation of a plane

Consider a plane π and a fixed point $P(x_0, y_0, z_0)$ on that plane. A vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, called the normal vector to the plane, is a vector perpendicular to the plane.

To find an equation for the plane, consider an arbitrary point $M(x, y, z)$ in space. Recalling that a line perpendicular to the plane is perpendicular to every line in the plane, we can conclude that for the point M to be on the plane, the vector $\mathbf{N}$ must be perpendicular to $\overrightarrow{PM}$.

Hence,

$$\mathbf{N} \cdot \overrightarrow{PM} = 0, \text{ but } \overrightarrow{PM} = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}, \text{ and}$$

$$\mathbf{N} \cdot \overrightarrow{PM} = 0$$

$$\Leftrightarrow (A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot ((x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}) = 0$$

Using the scalar product definition this can be simplified to

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

This is a Cartesian equation of a plane that passes through a point $P(x_0, y_0, z_0)$ and has a normal vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$.

Note: If $\mathbf{N}$ is normal to a given plane, then any vector parallel to $\mathbf{N}$ will be normal to the plane. Suppose we have chosen $3\mathbf{N}$ as our normal, then

$$3A(x - x_0) + 3B(y - y_0) + 3C(z - z_0) = 0$$

$$\Leftrightarrow A(x - x_0) + B(y - y_0) + C(z - z_0)$$

Specifically, the unit vector $\mathbf{n}$ in the same direction as $\mathbf{N}$ is of particular importance, as we will see soon.

Note: The above equation can be simplified further.
$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$
$\Leftrightarrow Ax + By + Cz = Ax_0 + By_0 + Cz_0$, and setting $Ax_0 + By_0 + Cz_0 = D$ will give us a more concise form of the equation,

$$Ax + By + Cz = D$$

which is similar to the equation of a line in the plane, i.e. $Ax + By = C$.

Note: In many sources, the equation of the plane is given in the form

$$Ax + By + Cz + D = 0.$$

This is the case when we set the quantity $Ax_0 + By_0 + Cz_0 = -D$. Each form has some advantage in using it. We will adhere to the previous form for reasons that will be clear in the following discussion.

Example 36

Write an equation for the plane that contains $(2, -3, 5)$ and has normal $\mathbf{N} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.

Solution

A Cartesian equation for the plane is of the form:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$\Rightarrow 2(x - 2) + (y + 3) - 3(z - 5) = 0$$

$$\Rightarrow 2x + y - 3z = -14$$

Alternatively, since $\mathbf{N} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ is the normal to the plane, then

$2x + y - 3z = D$, but the line contains the point $(2, -3, 5)$, and thus

$$2(2) - 3 - 3(5) = D \Rightarrow -14 = D.$$

And therefore

$$2x + y - 3z = -14 \text{ as before.}$$

Example 37

Show that every equation of the form $Ax + By + Cz = D$ with $A^2 + B^2 + C^2 \neq 0$ represents a plane in space.

Solution

The equation $Ax + By + Cz = D$ is a linear equation in 3 variables, x, y and z. This means that it has an infinite number of solutions, and hence we can be confident that there exist numbers x_0, y_0, z_0 such that $Ax_0 + By_0 + Cz_0 = D$.

Since the equation $Ax + By + Cz = D$ is also true, then

$$Ax + By + Cz = Ax_0 + By_0 + Cz_0 = D$$
$$\Leftrightarrow Ax + By + Cz - (Ax_0 + By_0 + Cz_0) = 0$$
$$\Leftrightarrow A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

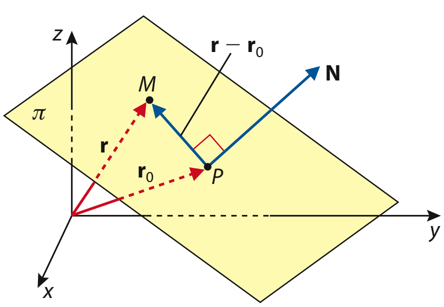
The last equation represents the equation of a plane through a fixed point $P(x_0, y_0, z_0)$ with a normal vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. The condition $A^2 + B^2 + C^2 \neq 0$ guarantees that $\mathbf{N} \neq \mathbf{0}$.

Vector equation of a plane

We can write the equation of the plane in vector notation. Using the same set up as before: the normal vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, a fixed point $P(x_0, y_0, z_0)$ with a position vector $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$, and an arbitrary point $M(x, y, z)$ with a position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{i} + z\mathbf{k}$.

The equation $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ can be interpreted as the scalar product $\mathbf{N} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$ which you can also see in the diagram. The normal $\mathbf{N}$ is perpendicular to $PM = \mathbf{r} - \mathbf{r}_0$ and hence their dot product must be zero. Using the distributive property of the scalar product, we have

$$\mathbf{N} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \Leftrightarrow \mathbf{N} \cdot \mathbf{r} - \mathbf{N} \cdot r_0 = 0$$
$$\Leftrightarrow \mathbf{N} \cdot \mathbf{r} = \mathbf{N} \cdot \mathbf{r}_0$$

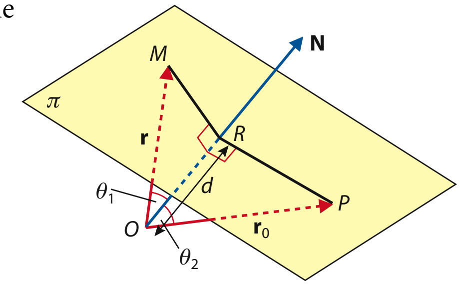


This is one form of the vector equation of a plane that passes through a point with position vector $\mathbf{r}_0$ and has a normal $\mathbf{N}$.

Note: Notice here that $\mathbf{N} \cdot \mathbf{r} = Ax + By + Cz$ and $\mathbf{N} \cdot \mathbf{r}_0 = Ax_0 + By_0 + Cz_0 = D$, which shows that $\mathbf{N} \cdot \mathbf{r} = \mathbf{N} \cdot \mathbf{r}_0$ is another way of stating $Ax + By + Cz = D$.

Unit vector equation of a plane

The diagram shows vector **N** as drawn from the origin O, along with the position vectors **r** and $\mathbf{r}_0$.



The vector equation $\mathbf{N} \cdot \mathbf{r} = \mathbf{N} \cdot \mathbf{r}_0$ can be investigated further.

$$\mathbf{N} \cdot \mathbf{r} = |\mathbf{N}||\mathbf{r}|\cos\theta_1 = |\mathbf{N}|OR = |\mathbf{N}|d$$

where OR $= d$ is the distance from the origin to the plane.

Also

$$\mathbf{N} \cdot \mathbf{r}_0 = |\mathbf{N}||\mathbf{r}_0|\cos\theta_2 = |\mathbf{N}|d.$$

In both cases, the result is of course the same. Both sides of the equation $\mathbf{N} \cdot \mathbf{r} = \mathbf{N} \cdot \mathbf{r}_0$ are equal to the same value: the magnitude of the normal multiplied by the distance from the origin.

Furthermore, if we divide each side by $|\mathbf{N}|$, we get the distance from the origin to the plane.

That is,

$$\mathbf{N} \cdot \mathbf{r} = |\mathbf{N}|d \Rightarrow d = \frac{\mathbf{N} \cdot \mathbf{r}}{|\mathbf{N}|}$$

as well as

$$\mathbf{N} \cdot \mathbf{r}_0 = |\mathbf{N}|d \Rightarrow d = \frac{\mathbf{N} \cdot \mathbf{r}_0}{|\mathbf{N}|}.$$

This last result gives us the basis for forming a new vector equation of the plane in terms of a unit vector perpendicular to it.

Let us call the unit vector normal to the plane **n**. So, using the results just established, we can write

$d = \dfrac{\mathbf{N} \cdot \mathbf{r}}{|\mathbf{N}|} = \dfrac{\mathbf{N} \cdot \mathbf{r}_0}{|\mathbf{N}|}$, which in turn can be simplified to

$\dfrac{\mathbf{N}}{|\mathbf{N}|} \cdot \mathbf{r} = \dfrac{\mathbf{N}}{|\mathbf{N}|} \cdot \mathbf{r}_0$, and since $\dfrac{\mathbf{N}}{|\mathbf{N}|} = \mathbf{n}$, then obviously

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0 = d.$$

This equation is very practical when we need to find the distance from the origin to the plane. The distance from the origin to a plane is the scalar product between the unit normal and the position vector of any point on the plane.

This will be shown in the examples below.

Example 38

Write a vector equation for the plane that contains $(2, -3, 5)$ and has normal $\mathbf{N} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.

Solution

We apply the results of the previous discussion:

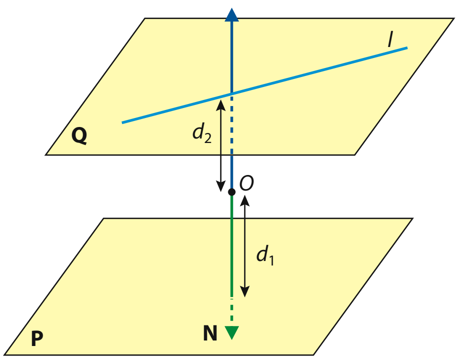
$$\mathbf{N} \cdot \mathbf{r} = \mathbf{N} \cdot \mathbf{r}_0$$
$$\Rightarrow (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \cdot \mathbf{r} = (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k})$$
$$\Rightarrow (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \cdot \mathbf{r} = -14$$

Notice that this result can easily transfer into Cartesian form by expanding the scalar product on the left.

$$(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \cdot \mathbf{r} = -14$$
$$\Rightarrow (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = -14$$
$$\Rightarrow 2x + y - 3z = -14$$

Example 39

Show that the line l with equation $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k} + k(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ is parallel to the plane P whose equation is $\mathbf{r} \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = -3$ and find the distance between them.



Solution

Since $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ is the direction vector of the line l, and since this vector is perpendicular to $\mathbf{N} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, the normal to P, as $(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 6 - 4 - 2 = 0$, then the line l must be parallel to plane P.

To find the distance between the line and the plane P, we may find the distance between a point on l, $(2, -1, 5)$ for example, and plane P. One way would be to consider a plane Q containing the given point and parallel to P. The equation of plane Q can be found using the last result:

$$\mathbf{N} \cdot \mathbf{r} = \mathbf{N} \cdot \mathbf{r}_0$$
$$\Rightarrow (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot \mathbf{r} = (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$$
$$\Rightarrow (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot \mathbf{r} = 11$$

The distance from the origin to P is given by

$$d = \left|\frac{\mathbf{N} \cdot \mathbf{r_0}}{|\mathbf{N}|}\right| = \frac{3}{\sqrt{4 + 4 + 1}} = 1$$

while the distance from the origin to Q is

$$d = \left|\frac{\mathbf{N} \cdot \mathbf{r_0}}{|\mathbf{N}|}\right| = \frac{11}{\sqrt{4 + 4 + 1}} = \frac{11}{3}.$$

The distance between the two planes is the sum of these two distances since the planes are on opposite sides of the origin (see note), and hence the required distance is $\frac{14}{3}$.

Note: The vector equation for P is $\mathbf{r} \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = -3$, or $\mathbf{r} \cdot (-2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 3$ and the equation for Q is $(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \cdot \mathbf{r} = 11$. The normals to the two planes are opposite, and hence they are on opposite sides of the origin. If the two normals are in the same direction, then the distance between them will be the difference of the two distances from the origin.

Example 40

Show that the plane with vector equation $\mathbf{r} \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = -3$ contains the line with equation $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + \mathbf{k} + k(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$.

Solution

We have several methods available to us at this stage. One method is to check whether two points are common to the line and the plane. One point on the line is $(1, 3, 1)$. Since $(1, 3, 1) \cdot (2, -2, 1) = 2 - 6 + 1 = -3$, then the point is on the plane. Another point on the line can be found by choosing any value for k, say $k = 1$. Thus, another point has the position vector

$$\mathbf{r} = \mathbf{i} + 3\mathbf{j} + \mathbf{k} + 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} = 4\mathbf{i} + 5\mathbf{j} - \mathbf{k}.$$

Since $(4\mathbf{i} + 5\mathbf{j} - \mathbf{k}) \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 8 - 10 - 1 = -3$, this point will also lie on the plane and therefore the plane will contain the whole line.

Another method would be to check only one point and prove that the line is parallel to the plane as in Example 39 above.

Example 41

Find the vector equation of the line through $(1, 2, 3)$ that is perpendicular to the plane with vector equation $\mathbf{r} \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = -3$ and find their point of intersection.

Solution

A vector parallel to the required line must be parallel to the normal vector to the plane. Hence, a vector equation of the line is
$\mathbf{r} = (1, 2, 3) + k(2, -2, 1)$.

To find the point of intersection, we consider any point on the line. Such a point would have the position vector $(1 + 2k, 2 - 2k, 3 + k)$. For this point to be on the plane, the following equation must be true:

$$(1 + 2k, 2 - 2k, 3 + k) \cdot (2, -2, 1) = -3;$$

i.e. $2 + 4k - 4 + 4k + 3 + k = -3$, so

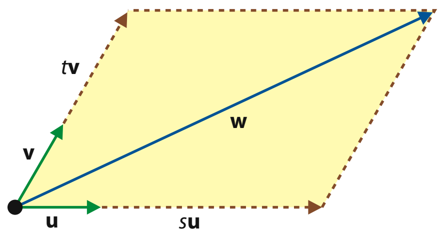
$9k = -4$, and $k = -\frac{4}{9}$ giving the point of intersection of the line and the plane as

$$\left(1 + 2 \times -\tfrac{4}{9}, 2 - 2 \times -\tfrac{4}{9}, 3 - \tfrac{4}{9}\right) = \left(\tfrac{1}{9}, \tfrac{26}{9}, \tfrac{23}{9}\right).$$

Parametric form for the equation of a plane

We start this section with an example that demonstrates the following theorem:

Three coplanar vectors **u**, **v** *and* **w** *are given. If* **u** *and* **v** *are not parallel, then* **w** *can always be expressed as a linear combination of* **u** *and* **v**, *i.e. it is always possible to find two scalars s and t such that* $\mathbf{w} = s\mathbf{u} + t\mathbf{v}$. (The proof of the theorem is not included in this book.)



Using the diagram on the left, this means that it is always possible to construct a parallelogram whose diagonal is **w** and whose sides are the non-parallel vectors **u** and **v** or their multiples.

For example, given the two non-parallel vectors $\mathbf{u} = (1, 0, -2)$ and $\mathbf{v} = (3, 1, -9)$, then we can always find the scalars s and t so that vector $\mathbf{w} = (2, 1, -7)$ can be expressed as a linear combination of **u** and **v**. Thus,

$$(2, 1, -7) = s(1, 0, -2) + t(3, 1, -9).$$

To find s and t we solve the system of equations:

$$\begin{cases} s + 3t = 2 \\ t = 1 \\ -2s - 9t = -7 \end{cases}$$

This system is consistent and yields the solution $s = -1$ and $t = 1$. Thus,

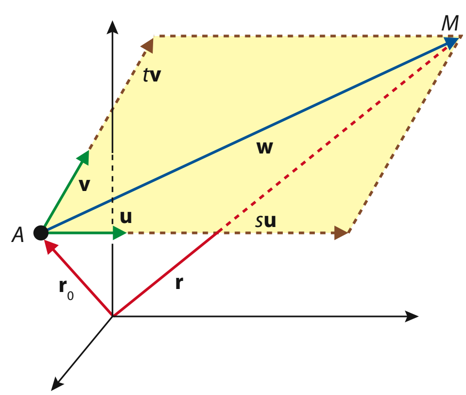
$$\mathbf{w} = -\mathbf{u} + \mathbf{v}.$$

Now, consider the plane which is parallel to vectors **u** and **v** and which contains the point A whose position vector is $\mathbf{r}_0$. As the figure below shows, the two vectors determine the direction of the plane and A 'fixes' it in space. So, if M is any point in this plane then, according to the previous theorem, $\overrightarrow{AM} = s\mathbf{u} + t\mathbf{v}$ where s and t are two scalars.

If **r** is the position vector of M, then

$$\mathbf{r} = \mathbf{r}_0 + \overrightarrow{AM} = \mathbf{r}_0 + s\mathbf{u} + t\mathbf{v}.$$

Thus, any equation of the form $\mathbf{r} = \mathbf{r}_0 + s\mathbf{u} + t\mathbf{v}$, where s and t are independent scalars, represents the equation of a plane parallel to **u** and **v** and contains the point with position vector $\mathbf{r}_0$.



Note that this equation is not unique. This is because one can start at any other fixed point on the plane other than A and may choose any number of intersecting vectors in the plane other than **u** and **v**. The parametric form of the equation of the plane is seldom needed or used.

Example 42

Find an equation of a plane with normal $\mathbf{q} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and that contains the point $A(2, 1, 1)$. Use all forms you learned.

Solution

The Cartesian equation

Consider the equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$\Rightarrow 2(x - 2) - 3(y - 1) + (z - 1) = 0$$

The equation would be: $2x - 3y + z - 2 = 0$.

Or, start the equation:

$$Ax + By + Cz = D$$

$$\Rightarrow 2x - 3y + z = D$$

Since the plane contains the point $(2, 1, 1)$, then

$2(2) - 3(1) + 1 = D$, and thus $D = 2$.

Vector equations

Finding the vector equation can also be achieved by applying:

$$\mathbf{N} \cdot \mathbf{r} = \mathbf{N} \cdot \mathbf{r}_0 \Rightarrow (2, -3, 1) \cdot (x, y, z) = (2, -3, 1) \cdot (2, 1, 1)$$

$$\Rightarrow (2, -3, 1) \cdot (x, y, z) = 2$$

Note: It is easy to transform the vector equation into Cartesian form by simply performing the dot product. The opposite is also true.

Parametric equation

A parametric equation of this plane is not as straightforward and may not be the most efficient way of doing this problem. However, for the sake of giving an example we present a way of doing it.

The parametric form requires that we have two vectors parallel to the plane. We may find the two vectors by considering that they have to be perpendicular to $(2, -3, 1)$. So, take a vector $(1, 1, z)$ and find z so that this vector is perpendicular to $(2, -3, 1)$:

$$\Rightarrow 2 - 3 + z = 0 \text{ and } z = 1$$

Do the same with $(1, 0, z)$, i.e. $2 + 0 + z = 0$ and $z = -2$. Therefore, two vectors that are perpendicular to $(2, -3, 1)$ are $(1, 1, 1)$ and $(1, 0, -2)$, and a parametric equation of the plane is

$$\mathbf{r} = \mathbf{r}_0 + s\mathbf{u} + t\mathbf{v} \Rightarrow \mathbf{r} = (2, 1, 1) + s(1, 1, 1) + t(1, 0, -2).$$

Observe that the choice of the vectors is arbitrary and hence the parametric form is not unique.

Example 43

Find the equation of the plane that contains the following three points:

$A(1, 3, 0)$, $B(-2, 1, 2)$ and $C(1, -2, -1)$.

Solution

Consider any point $M(x, y, z)$ on this plane. For this point to belong to the plane, the following vectors must be coplanar: $\overrightarrow{AM}$, $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

This means that the parallelepiped with these vectors as edges is flat, i.e. with volume zero. Since we know that the volume of the parallelepiped is the absolute value of the scalar triple product, we equate that value to zero and get the equation. Here are the details:

$$\overrightarrow{AM} = (x-1, y-3, z), \overrightarrow{AB} = (-3, -2, 2), \overrightarrow{AC} = (0, -5, -1)$$

$$\Rightarrow \overrightarrow{AM} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = \begin{vmatrix} x-1 & y-3 & z \\ -3 & -2 & 2 \\ 0 & -5 & -1 \end{vmatrix} = 0$$

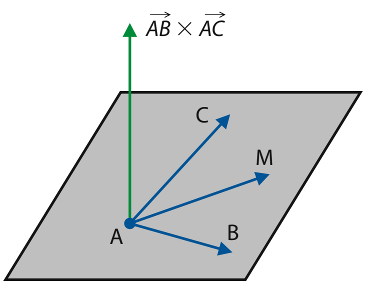
$$\Rightarrow 3(4x - y + 5z - 1) = 0$$

So, the Cartesian equation of the plane is $4x - y + 5z - 1 = 0$.

We can also deduce the vector equation for this plane by writing it in scalar product form:

$$(4\mathbf{i} - \mathbf{j} + 5\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 1.$$

The vector form could also be achieved if we think of the problem as a plane containing a fixed point and normal to a given vector.



The normal can be found by computing the cross product of two vectors in the plane; in this case, we can take $\overrightarrow{AB}$ and $\overrightarrow{AC}$. So,

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -2 & 2 \\ 0 & -5 & -1 \end{vmatrix} = 3(4\mathbf{i} - \mathbf{j} + 5\mathbf{k}).$$

The equation of the plane is then

$$(4\mathbf{i} - \mathbf{j} + 5\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = (4\mathbf{i} - \mathbf{j} + 5\mathbf{k}) \cdot (\mathbf{i} + 3\mathbf{j})$$

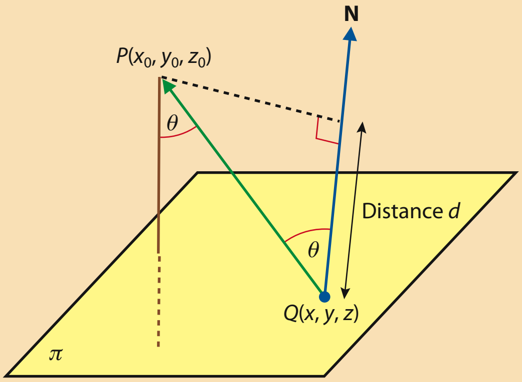
$$(4\mathbf{i} - \mathbf{j} + 5\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 1.$$

This is the same as above.

Distance between a point and a plane

The distance between a point $P(x_0, y_0, z_0)$ and a plane with equation $Ax + By + Cz = D$ is given by

$$d = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$



In the diagram above, let $Q(x, y, z)$ be any point on the plane and $\mathbf{N}(A, B, C)$ be a normal to the plane. The distance we are looking for is d. Then,

$$d = \left| |\overrightarrow{QP}| \cos\theta \right| = \left| |\overrightarrow{QP}| \frac{\overrightarrow{QP} \cdot \mathbf{N}}{|\overrightarrow{QP}||\mathbf{N}|} \right|$$

$$= \left| \frac{\overrightarrow{QP} \cdot \mathbf{N}}{|\mathbf{N}|} \right| = \left| \frac{(A, B, C) \cdot (x_0 - x, y_0 - y, z_0 - z)}{\sqrt{A^2 + B^2 + C^2}} \right|$$

$$= \frac{|A(x_0 - x) + B(y_0 - y) + C(z_0 - z)|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|Ax_0 + By_0 + Cz_0 - (Ax + Bx + Cz)|}{\sqrt{A^2 + B^2 + C^2}}$$

Since $Q(x, y, z)$ is on the plane, then $Ax + By + Cz = D$, so replacing this expression in the result above will yield

$$d = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

This formula is similar to the distance between a point and a line in 2-space.

Example 44

Show that the line l with equation $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k} + k(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ is parallel to the plane P whose equation is $\mathbf{r} \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = -3$ and find the distance between them.

Solution

In a previous example we showed that the line is parallel to the plane because it is perpendicular to the normal of the plane. To find the distance, we used a relatively complex approach. At this moment we can utilize the distance formula just established to find the required distance.

A point on the line is $(2, -1, 5)$ and the Cartesian equation of the plane is simply

$$2x - 2y + z = -3.$$

Hence, the distance is

$$d = \frac{|2(2) - 2(-1) + 1(5) - (-3)|}{\sqrt{4 + 4 + 1}} = \frac{14}{3}.$$

Example 45

Find the distance between the two parallel planes: $x + 2y - 2z = 3$ and $2x + 4y - 4z = 7$.

Solution

It is enough to find the distance from one point on one of the planes to the other plane since all points are equidistant.

Take the point $(1, 1, z)$ on the first plane:

$$1 + 2 - 2z = 3, \text{ so } z = 0.$$

Thus, the point is $(1, 1, 0)$ and the distance between the planes is

$$d = \frac{|2(1) + 4(1) - 4(0) - 7|}{\sqrt{4 + 16 + 16}} = \frac{1}{6}.$$

Example 46

Find the distance between the two skew straight lines

$$L_1: \mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -4 \\ 5 \end{pmatrix} \text{ and } L_2: \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 8 \\ -3 \\ 1 \end{pmatrix}.$$

Solution

We can reduce this problem to the type in the previous example by creating two planes that contain the given lines and are parallel to each other.

For the two planes to be parallel, they must be perpendicular to the same vector. Hence, by finding the cross product of the direction vectors of the lines we would have found a vector perpendicular to both.

$$l_1 \times l_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -4 & 5 \\ 8 & -3 & 1 \end{vmatrix} = 11\mathbf{i} + 36\mathbf{j} + 20\mathbf{k}$$

Considering the point $(2, 4, 5)$ on L_2, the plane containing this line will be

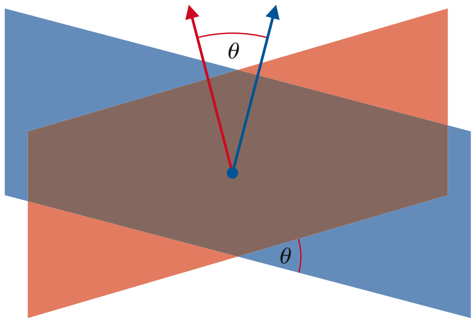
$$\begin{pmatrix} 11 \\ 36 \\ 20 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ 36 \\ 20 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$

or $11x + 36y + 20z = 266$ and the distance between $(1, 5, -1)$ on L_1 to this plane will be

$$d = \frac{|11(1) + 36(5) + 20(-1) - 266|}{\sqrt{11^2 + 36^2 + 20^2}} = \frac{95}{\sqrt{1817}}.$$

The angle between two planes

The angle between two planes is defined to be the *acute* angle between them as you see in the figure below.



Consider two planes **P** and **Q** with unit normals $\mathbf{n}_1$ and $\mathbf{n}_2$ respectively. Their vector equations are of the form

$$\mathbf{r} \cdot \mathbf{n}_1 = d_1 \text{ and } \mathbf{r} \cdot \mathbf{n}_2 = d_2.$$

The angle between the planes is equal to the angle between the normal units $\mathbf{n}_1$ and $\mathbf{n}_2$ or the normals $\mathbf{N}_1$ and $\mathbf{N}_2$, and hence

$$\cos\theta = \mathbf{n}_1 \cdot \mathbf{n}_2 = \frac{\mathbf{N}_1 \cdot \mathbf{N}_2}{|\mathbf{N}_1|\,|\mathbf{N}_2|}.$$

For example, the angle between the planes with vector equations

$$\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 3 \text{ and } \mathbf{r} \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 2$$

is given by

$$\cos\theta = \frac{(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \cdot (2\mathbf{i} - 2\mathbf{j} + \mathbf{k})}{\sqrt{1+1+4} \cdot \sqrt{4+4+1}} = -\frac{2}{3\sqrt{6}}.$$

This is the cosine of the obtuse angle between the two planes. The acute angle between them is $\cos^{-1}\left(\frac{2}{3\sqrt{6}}\right)$.

Example 47

Find the angle between the planes with equations

$$2x - 3y = 0 \text{ and } 3x + y - z = 4.$$

Solution

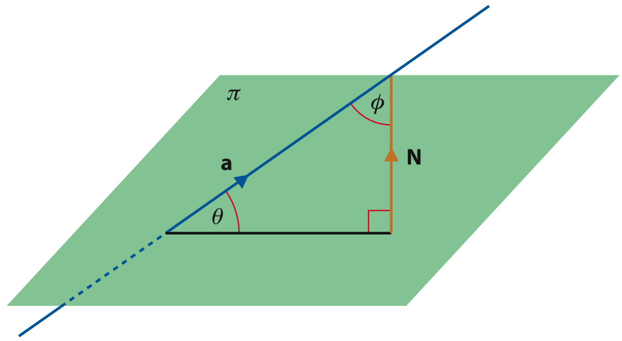
The two normals are $2\mathbf{i} - 3\mathbf{j}$ and $3\mathbf{i} + \mathbf{j} - \mathbf{k}$, and therefore the angle is given by

$$\cos\theta = \frac{(2\mathbf{i} - 3\mathbf{j}) \cdot (3\mathbf{i} + \mathbf{j} - \mathbf{k})}{\sqrt{13}\sqrt{11}} = \frac{3}{\sqrt{143}}.$$

So, the angle between the planes is $\cos^{-1}\left(\frac{3}{\sqrt{143}}\right)$.

The angle between a line and a plane

The angle between a line and a plane can be defined as the angle θ formed by the line and its projection on the plane, as shown in the figure below.



Consider the line l with equation $\mathbf{r} = \mathbf{r}_0 + \lambda\mathbf{a}$ and the plane with equation $\mathbf{r} \cdot \mathbf{N} = D$.

The acute angle ϕ between $\mathbf{N}$, the normal to the plane, and the line l can be found by using the law of cosines:

$$\cos\phi = \left|\frac{\mathbf{a} \cdot \mathbf{N}}{|\mathbf{a}|\,|\mathbf{N}|}\right|$$

If θ is the acute angle between the line and the plane then

$$\theta = \frac{\pi}{2} - \phi.$$

Therefore, to find the angle between the line and the plane, we

- either find the angle ϕ first and then find its complement, or
- since ϕ and θ are complements, then $\sin\theta = \cos\phi = \left|\frac{\mathbf{a} \cdot \mathbf{N}}{|\mathbf{a}|\,|\mathbf{N}|}\right|$.

For example: to find the angle between the line with equation

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

and the plane with equation

$$\mathbf{r} \cdot (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 3$$

we find that

$$\sin\theta = \frac{(\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} - \mathbf{j} + \mathbf{k})}{\sqrt{3} \cdot \sqrt{4}} = \frac{4}{3\sqrt{2}}$$

$$\Rightarrow \theta = \sin^{-1}\frac{4}{3\sqrt{2}}.$$

Example 48

Find the angle between the line with equations

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{2}, \text{ and the plane with equation}$$

$$2x - y - z = 7.$$

Solution

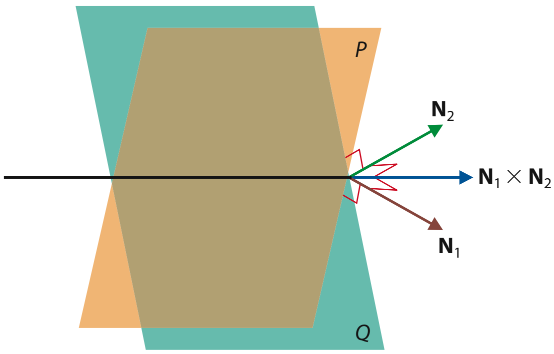
The direction of the line is given by $\mathbf{a} = (2, 3, 2)$ and the normal to the plane by

$\mathbf{N} = (2, -1, -1)$, and the angle is given by

$$\sin\theta = \left|\frac{(2, 3, 2)\cdot(2, -1, -1)}{\sqrt{17}\cdot\sqrt{6}}\right| = \frac{1}{\sqrt{102}}$$

$$\Rightarrow \theta = \sin^{-1}\frac{1}{\sqrt{102}}.$$

Line of intersection of two planes



Unless two planes are parallel they will intersect along a straight line. Consider two planes P and Q that have l as their line of intersection. Also let the planes have the following vector equations:

$$\mathbf{r}\cdot\mathbf{N}_1 = D_1 \text{ and } \mathbf{r}\cdot\mathbf{N}_2 = D_2.$$

Since the line l lies in plane P then $\mathbf{N}_1$, the normal to this plane, must be perpendicular to it. This is also true for $\mathbf{N}_2$. Therefore, the direction of line l is perpendicular to both $\mathbf{N}_1$ and $\mathbf{N}_2$.

To find the line of intersection, we will demonstrate two methods:

1 Use the cross product of $\mathbf{N}_1$ and $\mathbf{N}_2$ as the direction of l and a specific point on the line.

2 Use the fact that all points on l must satisfy the equations of both planes; i.e. we solve a system of equations.

These methods are best demonstrated when we apply them to a particular situation.

Let the planes P and Q have the equations:

$$\mathbf{r}\cdot(\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 6 \text{ and } \mathbf{r}\cdot(2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 4.$$

1 To find a vector equation of the line of intersection, we need first to find the cross product of the two normals and then find a point on the line l.

$$(\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \times (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 2\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}$$

To find a point on the line we use the fact that the points on that line must satisfy both equations. So, consider the points on both planes that have the x-coordinate zero; i.e.

$$(0, y, z) \cdot (\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 6 \text{ and } (0, y, z) \cdot (2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 4$$

$$\Rightarrow \begin{cases} y - 3z = 6 \\ -y + z = 4 \end{cases} \Rightarrow z = -5 \text{ and } y = -9$$

So, the vector equation of the line is: $\mathbf{r} = (0, -9, -5) + t(2, 7, 3)$.

2 The second method uses a system of equations to find the equation. The equations of the planes in Cartesian form are:

$$x + y - 3z = 6 \text{ and } 2x - y + z = 4.$$

Since this system has to be solved simultaneously, and since there are two equations in three variables, we should consider one of these variables as a parameter and solve for the rest. So,

$$\begin{cases} x + y - 3z = 6 \\ 2x - y + z = 4 \end{cases} \Rightarrow 3x - 2z = 10 \Rightarrow z = -5 + \tfrac{3}{2}x;\ y = -9 + \tfrac{7}{2}x.$$

Therefore, we either consider x to be the parameter or, for convenience purposes, we replace it by another parameter such as the following:

$$x = 2\lambda,\ y = 7\lambda - 9,\ z = 3\lambda - 5$$

This equation is equivalent to the one found in part (1).

3 If the equations of the planes are in parametric form it may not be necessary to convert them into Cartesian form. However, from the example below, you may notice that it may be more straightforward to follow the Cartesian method.

Find the intersection between the two planes

$$P\colon \mathbf{r} = \mathrm{i} + \mathrm{j} + \lambda(2\mathbf{i} - \mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} + \mathbf{k}), \text{ and}$$

$$Q\colon \mathbf{r} = 3\mathbf{i} - \mathbf{k} + s(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + t(\mathbf{i} + 2\mathbf{j} - \mathbf{k}).$$

A point on P will have the following coordinates: $(1 + 2\lambda + \mu, 1 - \mu, -\lambda + \mu)$, while a point on Q will have the coordinates $(3 + s + t, -s + 2t, -1 + 2s - t)$. For the collection of points on the intersection, we must have the coordinates satisfy both equations, and hence

$$1 + 2\lambda + \mu = 3 + s + t$$

$$1 - \mu = -s + 2t$$

$$-\lambda + \mu = -1 + 2s - t$$

This system of three equations in four unknowns must have an infinite number of solutions if the planes intersect – the set of points that belong to the line of intersection. We can solve this system best, after re-arranging terms, by Gaussian elimination:

$$\begin{pmatrix} 2 & 1 & -1 & -1 & 2 \\ 0 & -1 & 1 & -2 & -1 \\ -1 & 1 & -2 & 1 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{9}{2} & \frac{5}{2} \\ 0 & 0 & 1 & \frac{5}{2} & \frac{3}{2} \end{pmatrix}$$

The last result expresses λ, μ and s in terms of t. To find the equation of the line we have to substitute these values in either of the two equations above. For example, it is easier to substitute for s in terms of t in the equation for Q. So, from the result above,

$$s = \tfrac{3}{2} - \tfrac{5}{2}t$$

and the line will have the vector equation

$$\mathbf{r} = 3\mathbf{i} - \mathbf{k} + \left(\tfrac{3}{2} - \tfrac{5}{2}t\right)(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + t(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$
$$= \left(\tfrac{9}{2}\mathbf{i} - \tfrac{3}{2}\mathbf{j} + 2\mathbf{k}\right) + t\left(-\tfrac{3}{2}\mathbf{i} + \tfrac{9}{2}\mathbf{j} - 6\mathbf{k}\right).$$

If $\lambda = \frac{1}{2} + \frac{3}{2}t$ and $\mu = \frac{5}{2} - \frac{9}{2}t$ are substituted into the equation for P we will get the same result. (Try it!)

As we notice from the above discussion, the process is long and complex, even with many steps that are 'hidden' to save space. Alternatively, the Cartesian solution may be more efficient.

P: point $(1, 1, 0)$ is on the plane, and $(2, 0, -1) \times (1, -1, 1) = (-1, -3, -2)$ is perpendicular to the plane, so the Cartesian equation is

$$(x - 1) + 3(y - 1) + 2z = 0, \text{ or } x + 3y + 2z = 4.$$

Q: point $(3, 0, -1)$ is on the plane, and $(1, -1, 2) \times (1, 2, -1) = (-3, 3, 3)$ is perpendicular to the plane, so the Cartesian equation is

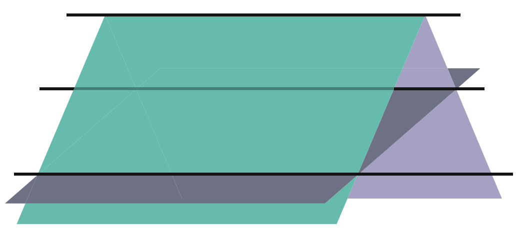
$$-3(x - 3) + 3y + 3(z + 1) = 0, \text{ or } x - y - z = 4.$$

The intersection between the planes is the result of solving the following system:

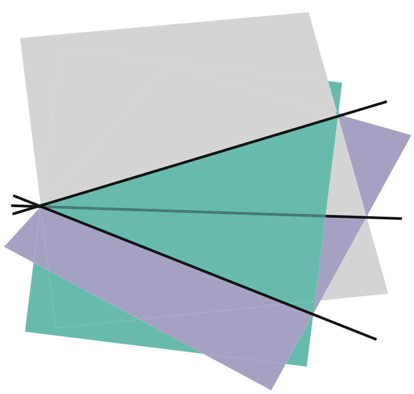
$$\begin{cases} x + 3y + 2z = 4 \\ x - y - z = 4 \end{cases} \Rightarrow x = 4 + m, y = -3m, z = 4m$$

This result compares to the previous one and appears to be more elegant!

Note:
Three planes can intersect in three lines as shown here.



The three lines of intersection are parallel. Hence, the system of equations they represent is inconsistent.



If the lines of intersection are not parallel, then the three planes meet at one point as shown. This system is consistent with a unique solution. The three planes can also all pass through one straight line. In that case, the system is consistent with an infinite number of solutions.

Exercise 14.5

1 Which of the points $A(3, -2, -1)$, $B(2, 1, -1)$, $C(1, 4, 0)$ lie in the plane

$$3x + 2y - 3z = 11?$$

2 Which of the points $A(3, 2, -3)$, $B(2, 1, -2)$, $C(1, 4, 0)$ lie in the plane

$$(\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = -6?$$

In questions 3–16, find an equation for the plane satisfying the given conditions. Give two forms for each equation out of the three forms: Cartesian, vector or parametric.

3 Contains the point $(3, -2, 4)$ and perpendicular to $2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$

4 Contains the point $(-3, 2, 1)$ and perpendicular to $2\mathbf{i} + 3\mathbf{k}$

5 Contains the point $(0, 3, 1)$ and perpendicular to $3\mathbf{k}$

6 Contains the point $(3, -2, 4)$ and parallel to the plane $5x + y - 2z = 7$

7 Contains the point $(3, 0, 1)$ and parallel to the plane $y - 2z = 11$

8 Contains the point $(3, -2, 4)$ and the line $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

9 Contains the lines $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$

10 Contains the point $(1, -3, 2)$ and the line $x = 2t, y = 2 + t, z = -1 + 3t$

11 Contains the point $M(p, q, r)$ and perpendicular to the vector $\overrightarrow{OM}$

12 Contains the three points $(1, 2, 2)$, $(3, -1, 0)$ and $(7, 0, -2)$

13 Contains the three points $(2, -2, -2)$, $(3, -1, 3)$ and $(0, 1, 5)$

14 Contains the point $(1, -2, 3)$ and the line $x - 2 = y + 1 = \frac{z - 5}{3}$

15 Contains the two parallel lines

$$\mathbf{r} = (1, -1, 5) + t(3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) \text{ and } \mathbf{r} = (-3, 4, 0) + t(3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$$

16 Contains the point $(1, 1, 0)$ and parallel to the two lines

$$x = 2 + t, y = -1, z = t \text{ and } x = s, y = 2 - s, z = -1 + s$$

In questions 17–22, find the acute angle between the given lines or planes.

17 $3x + 4y - z = 1$ and $x - 2y = 3$

18 $4x - 7y + z = 3$ and $3x + 2y + 2z = 17$

19 $x = 4$ and $x + z = 4$

20 $x - 2y + 2z = 3$ and $x = 2 - 6t, y = 4 + 3t, z = 1 - 2t$

21 $(3\mathbf{i} - \mathbf{k}) \cdot (x, y, z) = 4$ and $\mathbf{r} = (2\mathbf{j} + 3\mathbf{k}) + \lambda(-\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

22 $x + y + z = 7$ and $z = 0$

In questions 23–26 find the points of intersection of the given line and plane.

23 $\mathbf{r} = 5\mathbf{i} - 2\mathbf{k} + \lambda(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$ and $(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = -35$

24 $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix}$ and $4x - 2y + 3z - 30 = 0$

25 $x - 3 = \dfrac{y - 4}{5} = \dfrac{z - 6}{3}$ and $(2\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 5$

26 $x = t, y = 4 - \frac{1}{3}t, z = 5 - \frac{5}{3}t$ and $3x - y + 2z = 6$

In questions 27–30, find the line of intersection between the given planes.

27 $x = 10$ and $x + y + z = 3$

28 $2x - y + z = 5$ and $x + y - z = 4$

29 $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1$ and $x - y - 2z = 5$

30 $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ -8 \end{pmatrix}$ and $3x - y - z = 3$

31 Find a plane through $A(2, 1, -1)$ and perpendicular to the line of intersection of the planes $2x + y - z = 3$ and $x + 2y + z = 2$.

32 Find a plane through the points $A(1, 2, 3)$ and $B(3, 2, 1)$ and perpendicular to the plane $(4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 7$.

33 What point on the line through (1, 2, 5) and (3, 1, 1) is closest to the point (2, −1, 5)?

34 Find an equation of the plane that contains the line

$$\mathbf{r} = (-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

and is parallel to $\dfrac{x - 1}{3} = \dfrac{y + 2}{2} = \dfrac{z - 4}{4}$.

35 Find an equation of the plane that contains the line

$$x = 1 + 2t, y = 1 + 2t, z = 2 - t$$

and is parallel to $x - 1 = \dfrac{y - 2}{2} = z - 7$.

36 Show that the equation

$$\frac{x}{A} + \frac{y}{B} + \frac{z}{C} = 1$$

is the equation of a plane.

37 Find the equation of a plane that contains the point (4, −3, −1) and is perpendicular to the planes $2x - 3y + 4z = 5$ and $4x - 3z = 5$.

38 Find the equation of a plane that contains the point (2, 3, 0) and is perpendicular to the plane $2x - 3y + 4z = 5$ and parallel to the line $\mathbf{r}(t) = (t - 3)\mathbf{i} + (4 - 2t)\mathbf{j} + (1 + t)\mathbf{k}$.

Review exercise

1 Briefly discuss how you test if two vectors are parallel or perpendicular. Use more than one approach.

2 Briefly discuss how you test if three vectors are coplanar.

3 Briefly discuss how you find the angle between two vectors.

4 Briefly discuss how you find the equation of a line.

5 Briefly discuss how you find the equation of a plane.

6 Briefly discuss how you find the angle between two planes.

7 Briefly discuss how you find the angle between a line and a plane.

Find vector, parametric and Cartesian equations for the lines in questions 8–15.

8 The line through the point $(4, -3, 0)$ parallel to the vector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

9 The line through $A(-1, 1, 4)$ and $B(4, 6, -1)$.

10 The line through $A(2, 3, 0)$ and $B(0, 1, 2)$.

11 The line through the origin parallel to the vector $\mathbf{j} + 2\mathbf{k}$.

12 The line through the point $(4, -1, 2)$ parallel to the line $x = 2 + 3t$, $y = 3 - t$, $z = 4t$.

13 The line through $(1, 2, 2)$ parallel to the y-axis.

14 The line through $(3, 5, 6)$ perpendicular to the plane $4x - 8y + 7z = 23$.

15 The line through $(3, 5, 6)$ perpendicular to the vectors $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$.

Find equations for the planes in questions 16–20.

16 The plane through $A(1, 3, 0)$ normal to the vector $\mathbf{n} = 4\mathbf{i} - \mathbf{j} + \mathbf{k}$.

17 The plane through $P(2, 0, 5)$ parallel to the plane $2x + 3y - z = 11$.

18 The plane through $(2, 1, -1)$, $(3, -1, 0)$ and $(1, -1, 2)$.

19 The plane through $B(2, 5, 4)$ perpendicular to the line $x = 2 + 3t$, $y = 3 - t$, $z = 4t$.

20 The plane through $P(2, -1, 2)$ perpendicular to the vector from the origin to P.

21 Find the point of intersection of the lines $\mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}2\\3\\4\end{pmatrix}$ and $x = 2 + s$, $y = 4 + 2s$, $z = -4s - 1$.

22 Find the equation of the plane determined by the straight lines in the previous question.

23 Find the point of intersection of the lines $x = 2 - y = z - 1$ and $\frac{x-2}{2} = y - 3 = \frac{z-6}{5}$.

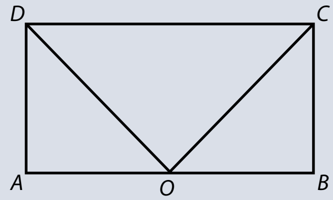
24 Find the equation of the plane determined by the straight lines in the previous question.

25 Find the equation of the plane through $M(1, -2, 1)$ and perpendicular to the vector from the origin to M.

Practice questions

1 $ABCD$ is a rectangle and O is the midpoint of $[AB]$. Express each of the following vectors in terms of $\overrightarrow{OC}$ and $\overrightarrow{OD}$

a) $\overrightarrow{CD}$ **b)** $\overrightarrow{OA}$ **c)** $\overrightarrow{AD}$



2 The vectors $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the x-axis and y-axis respectively. The vectors $\mathbf{u} = -\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = 3\mathbf{i} + 5\mathbf{j}$ are given.

a) Find $\mathbf{u} + 2\mathbf{v}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

A vector $\mathbf{w}$ has the same direction as $\mathbf{u} + 2\mathbf{v}$, and has a magnitude of 26.

b) Find $\mathbf{w}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

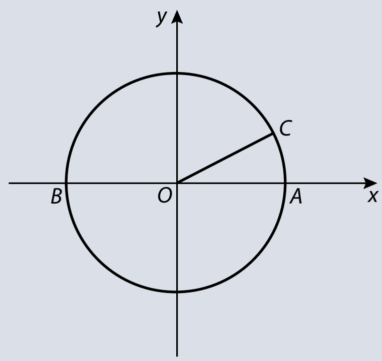
3 The circle shown has centre O and radius 6. $\overrightarrow{OA}$ is the vector $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$, $\overrightarrow{OB}$ is the vector $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$ and $\overrightarrow{OC}$ is the vector $\begin{pmatrix} 5 \\ \sqrt{11} \end{pmatrix}$.

a) Verify that A, B and C lie on the circle.

b) Find the vector $\overrightarrow{AC}$.

c) Using an appropriate scalar product, or otherwise, find the cosine of angle OAC.

d) Find the area of triangle ABC, giving your answer in the form $a\sqrt{11}$, where $a \in \mathbb{N}$.



4 The quadrilateral $OABC$ has vertices with coordinates $O(0, 0)$, $A(5, 1)$, $B(10, 5)$ and $C(2, 7)$.

a) Find the vectors $\overrightarrow{OB}$ and $\overrightarrow{AC}$.

b) Find the angle between the diagonals of the quadrilateral $OABC$.

5 The vectors $\mathbf{u}$ and $\mathbf{v}$ are given by $\mathbf{u} = 3\mathbf{i} + 5\mathbf{j}$ and $\mathbf{v} = \mathbf{i} - 2\mathbf{j}$. Find scalars a and b such that $a(\mathbf{u} + \mathbf{v}) = 8\mathbf{i} + (b - 2)\mathbf{j}$.

6 Find a vector equation of the line passing through $(-1, 4)$ and $(3, -1)$. Give your answer in the form $\mathbf{r} = \mathbf{p} + t\mathbf{d}$, where $t \in \mathbb{R}$.

7 In this question, the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents a displacement due east and the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ a displacement due north. Distances are in kilometres and time in hours.

Two crews of workers are laying an underground cable in a north-south direction across a desert. At 06:00 each crew sets out from their base camp, which is situated at the origin (0, 0). One crew is in a Toyundai vehicle and the other in a Chryssault vehicle.

The Toyundai has velocity vector $\begin{pmatrix} 18 \\ 24 \end{pmatrix}$ and the Chryssault has velocity vector $\begin{pmatrix} 36 \\ -16 \end{pmatrix}$.

a) Find the speed of each vehicle.

b) **(i)** Find the position vectors of each vehicle at 06:30.

(ii) Hence, or otherwise, find the distance between the vehicles at 06:30.

c) At this time (06:30) the Chryssault stops and its crew begin their day's work, laying cable in a northerly direction. The Toyundai continues travelling in the same direction, at the same speed, until it is exactly north of the Chryssault. The Toyundai crew then begin their day's work, laying cable in a southerly direction. At what time does the Toyundai crew begin laying cable?

d) Each crew lays an average of 800 m of cable in an hour. If they work non-stop until their lunch break at 11:30, what is the distance between them at this time?

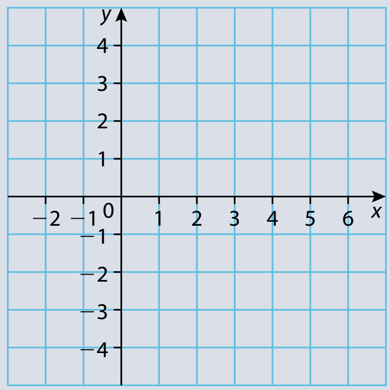
e) How long would the Toyundai take to return to base camp from its lunchtime position, assuming it travelled in a straight line and with the same average speed as on the morning journey? (Give your answer to the nearest minute.)

8 The line L passes through the origin and is parallel to the vector $2\mathbf{i} + 3\mathbf{j}$. Write down a vector equation for L.

9 The triangle ABC is defined by the following information:

$\overrightarrow{OA} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$, $\overrightarrow{AC}$ is parallel to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

a) On the grid below, draw an accurate diagram of triangle ABC.



b) Write down the vector $\overrightarrow{OC}$.

10 In this question, the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents a displacement due east and the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represents a displacement due north.

The point (0, 0) is the position of Shipple Airport. The position vector $\mathbf{r}_1$ of an aircraft, *Air One*, is given by

$$\mathbf{r}_1 = \begin{pmatrix} 16 \\ 12 \end{pmatrix} + t\begin{pmatrix} 12 \\ -5 \end{pmatrix},$$

where t is the time in minutes since 12:00.

a) Show that *Air One*

(i) is 20 km from Shipple Airport at 12:00

(ii) has a speed of 13 km/min.

b) Show that a Cartesian equation of the path of *Air One* is:

$$5x + 12y = 224.$$

The position vector $\mathbf{r}_2$ of an aircraft, *Air Two*, is given by

$$\mathbf{r}_2 = \begin{pmatrix} 23 \\ -5 \end{pmatrix} + t\begin{pmatrix} 2.5 \\ 6 \end{pmatrix},$$

where t is the time in minutes since 12:00.

c) Find the angle between the paths of the two aircraft.

d) **(i)** Find a Cartesian equation for the path of *Air Two*.

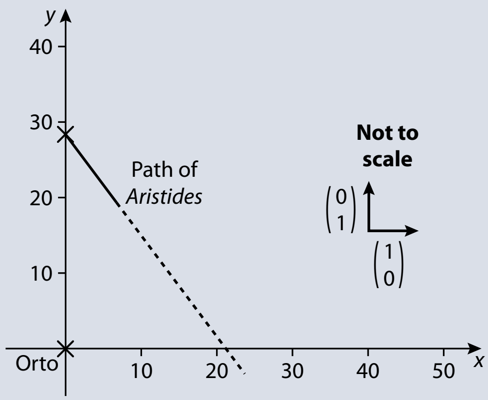
(ii) Hence, find the coordinates of the point where the two paths cross.

e) Given that the two aircraft are flying at the same height, show that they do not collide.

11 Find the size of the angle between the two vectors $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}6\\-8\end{pmatrix}$. Give your answer to the nearest degree.

12 A line passes through the point $(4, -1)$ and its direction is perpendicular to the vector $\begin{pmatrix}2\\3\end{pmatrix}$. Find the equation of the line in the form $ax + by = p$, where a, b and p are integers to be determined.

13 In this question, the vector $\begin{pmatrix}1\\0\end{pmatrix}$ represents a displacement due east and the vector $\begin{pmatrix}0\\1\end{pmatrix}$ represents a displacement due north. Distances are in kilometres.
The diagram shows the path of the oil tanker *Aristides* relative to the port of Orto, which is situated at the point (0, 0).



The position of the *Aristides* is given by the vector equation

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\28\end{pmatrix} + t\begin{pmatrix}6\\-8\end{pmatrix}$$

at a time t hours after 12:00.

a) Find the position of the *Aristides* at 13:00.

b) Find

(i) the velocity vector

(ii) the speed of the *Aristides*.

c) Find a Cartesian equation for the path of the *Aristides* in the form $ax + by = g$.

Another ship, the cargo vessel *Boadicea*, is stationary, with position vector $\begin{pmatrix}18\\4\end{pmatrix}$.

d) Show that the two ships will collide, and find the time of collision.

To avoid collision, the *Boadicea* starts to move at 13:00 with velocity vector $\begin{pmatrix}5\\12\end{pmatrix}$.

e) Show that the position of the *Boadicea* for $t \geqslant 1$ is given by

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}13\\-8\end{pmatrix} + t\begin{pmatrix}5\\12\end{pmatrix}.$$

f) Find how far apart the two ships are at 15:00.

14 Find the angle between the following vectors **a** and **b**, giving your answer to the nearest degree.

$\mathbf{a} = -4\mathbf{i} - 2\mathbf{j}$
$\mathbf{b} = \mathbf{i} - 7\mathbf{j}$

15 In this question, a unit vector represents a displacement of 1 metre.

A miniature car moves in a straight line, starting at the point (2, 0). After t seconds, its position, (x , y), is given by the vector equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0.7 \\ 1 \end{pmatrix}.$$

a) How far from the point (0, 0) is the car after 2 seconds?

b) Find the speed of the car.

c) Obtain the equation of the car's path in the form $ax + by = c$.

Another miniature vehicle, a motorcycle, starts at the point (0, 2) and travels in a straight line with constant speed. The equation of its path is

$$y = 0.6x + 2, \quad x \geqslant 0.$$

Eventually, the two miniature vehicles collide.

d) Find the coordinates of the collision point.

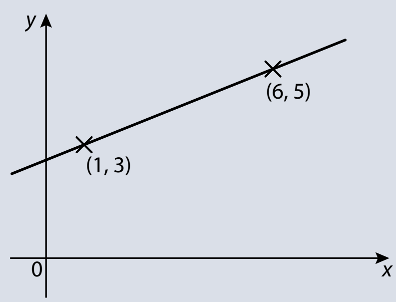
e) If the motorcycle left point (0, 2) at the same moment the car left point (2 , 0), find the speed of the motorcycle.

16 The diagram right shows a line passing through the points (1, 3) and (6, 5).

Find a vector equation for the line, giving your answer in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + t\begin{pmatrix} c \\ d \end{pmatrix},$$

where t is any real number.

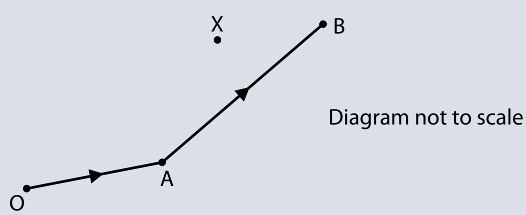


17 The vectors $\begin{pmatrix} 2x \\ x - 5 \end{pmatrix}$ and $\begin{pmatrix} x + 1 \\ 5 \end{pmatrix}$ are perpendicular for two values of x.

a) Write down the quadratic equation which the two values of x must satisfy.

b) Find the two values of x.

18 The diagram below shows the positions of towns O, A, B and X.



Town A is 240 km east and 70 km north of O.
Town B is 480 km east and 250 km north of O.
Town X is 339 km east and 238 km north of O.

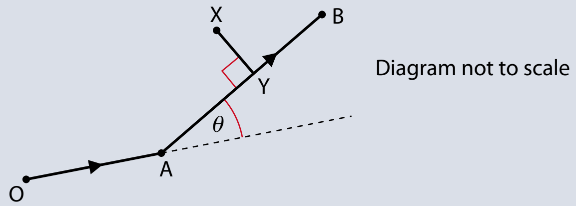
A plane flies at a constant speed of 300 km h^{-1} from O towards A.

a) **(i)** Show that a unit vector in the direction of $\overrightarrow{OA}$ is $\begin{pmatrix} 0.96 \\ 0.28 \end{pmatrix}$.

(ii) Write down the velocity vector for the plane in the form $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

(iii) How long does it take for the plane to reach A?

At A the plane changes direction so it now flies towards B. The angle between the original direction and the new direction is θ, as shown in the following diagram. This diagram also shows the point Y, between A and B, where the plane comes closest to X.



b) Use the scalar product of two vectors to find the value of θ in degrees.

c) **(i)** Write down the vector $\overrightarrow{AX}$.

(ii) Show that the vector $\mathbf{n} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ is perpendicular to $\overrightarrow{AB}$.

(iii) By finding the projection of $\overrightarrow{AX}$ in the direction of $\mathbf{n}$, calculate the distance XY.

d) How far is the plane from A when it reaches Y?

19 A vector equation of a line is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $t \in \mathbb{R}$.
Find the equation of this line in the form $ax + by = c$, where a, b and $c \in \mathbb{Z}$.

20 Three of the coordinates of the parallelogram $STUV$ are $S(-2, -2)$, $T(7, 7)$ and $U(5, 15)$.

a) Find the vector $\overrightarrow{ST}$ and hence the coordinates of V.

b) Find a vector equation of the line (UV) in the form $\mathbf{r} = \mathbf{p} + \lambda\mathbf{d}$, where $\lambda \in \mathbb{R}$.

c) Show that the point E with position vector $\begin{pmatrix} 1 \\ 11 \end{pmatrix}$ is on the line (UV), and find the value of λ for this point.

The point W has position vector $\begin{pmatrix} a \\ 17 \end{pmatrix}$, $a \in \mathbb{R}$.

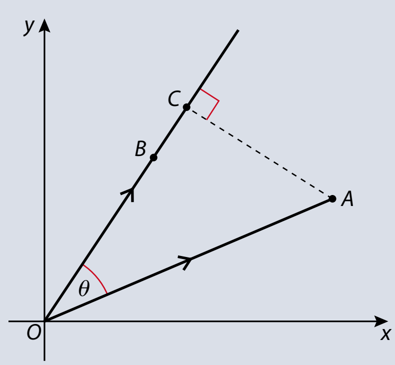
d) **(i)** If $\overrightarrow{EW} = 2\sqrt{13}$, show that one value of a is -3 and find the other possible value of a.

(ii) For $a = -3$, calculate the angle between $\overrightarrow{EW}$ and $\overrightarrow{ET}$.

21 Calculate the acute angle between the lines with equations

$\mathbf{r} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + s\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

22 The diagram on the right shows the point O with coordinates $(0, 0)$, the point A with position vector $\mathbf{a} = 12\mathbf{i} + 5\mathbf{j}$, and the point B with position vector $\mathbf{b} = 6\mathbf{i} + 8\mathbf{j}$. The angle between (OA) and (OB) is θ.



Find

a) $|\mathbf{a}|$

b) a unit vector in the direction of $\mathbf{b}$

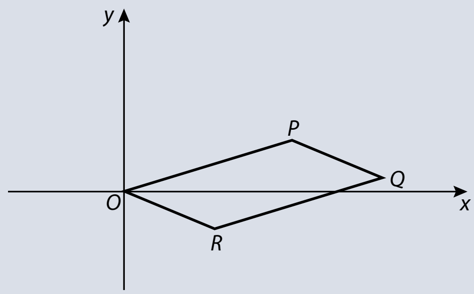
c) the exact value of $\cos\theta$ in the form $\frac{p}{q}$, where $p, q \in \mathbb{Z}$.

23 The vector equations of two lines are given below.

$$\mathbf{r}_1 = \begin{pmatrix}5\\1\end{pmatrix} + \lambda\begin{pmatrix}3\\-2\end{pmatrix},\ \mathbf{r}_2 = \begin{pmatrix}-2\\2\end{pmatrix} + t\begin{pmatrix}4\\1\end{pmatrix}$$

The lines intersect at the point P. Find the position vector of P.

24 The diagram shows a parallelogram $OPQR$ in which $\overrightarrow{OP} = \begin{pmatrix}7\\3\end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix}10\\1\end{pmatrix}$.



a) Find the vector $\overrightarrow{OR}$.

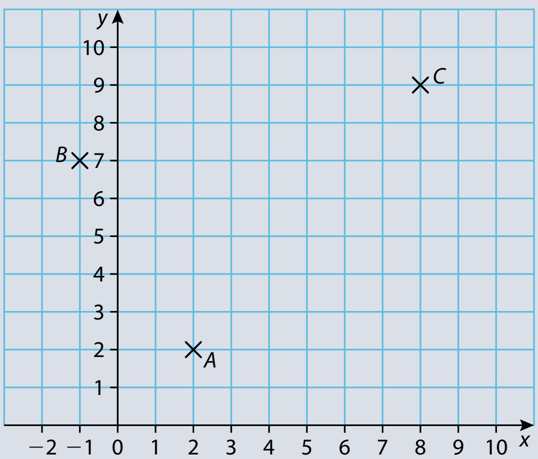
b) Use the scalar product of two vectors to show that $\cos O\hat{P}Q = -\frac{15}{\sqrt{754}}$.

c) **(i)** Explain why $\cos P\hat{Q}R = -\cos O\hat{P}Q$.

(ii) Hence, show that $\sin P\hat{Q}R = \frac{23}{\sqrt{754}}$.

(iii) Calculate the area of the parallelogram $OPQR$, giving your answer as an integer.

25 The diagram shows points A, B and C, which are three vertices of a parallelogram $ABCD$. The point A has position vector $\begin{pmatrix}2\\2\end{pmatrix}$.



a) Write down the position vector of B and C.

b) The position vector of point D is $\begin{pmatrix}d\\4\end{pmatrix}$. Find d.

c) Find $\overrightarrow{BD}$.

The line L passes through B and D.

d) **(i)** Write down a vector equation of L in the form $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-1\\7\end{pmatrix} + t\begin{pmatrix}m\\n\end{pmatrix}$.

(ii) Find the value of t at point B.

e) Let P be the point (7, 5). By finding the value of t at P, show that P lies on the line L.

f) Show that $\overrightarrow{CP}$ is perpendicular to $\overrightarrow{BD}$.

26 The points A and B have the position vectors $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$ respectively.

a) (i) Find the vector $\overrightarrow{AB}$.

(ii) Find $|\overrightarrow{AB}|$.

The point D has position vector $\begin{pmatrix} d \\ 23 \end{pmatrix}$.

b) Find the vector $\overrightarrow{AD}$ in terms of d.

The angle $B\hat{A}D$ is 90°.

c) (i) Show that $d = 7$.

(ii) Write down the position vector of the point D.

The quadrilateral $ABCD$ is a rectangle.

d) Find the position vector of the point C.

e) Find the area of the rectangle $ABCD$.

27 Points A, B and C have position vectors $4\mathbf{i} + 2\mathbf{j}$, $\mathbf{i} - 3\mathbf{j}$ and $-5\mathbf{i} - 5\mathbf{j}$, respectively. Let D be a point on the x-axis such that $ABCD$ forms a parallelogram.

a) (i) Find $\overrightarrow{BC}$.

(ii) Find the position vector of D.

b) Find the angle between $\overrightarrow{BD}$ and $\overrightarrow{AC}$.

The line L_1 passes through A and is parallel to $\mathbf{i} + 4\mathbf{j}$. The line L_2 passes through B and is parallel to $2\mathbf{i} + 7\mathbf{j}$. A vector equation of L_1 is $\mathbf{r} = (4\mathbf{i} + 2\mathbf{j}) + s(\mathbf{i} + 4\mathbf{j})$.

c) Write down a vector equation of L_2 in the form $\mathbf{r} = \mathbf{b} + t\mathbf{q}$.

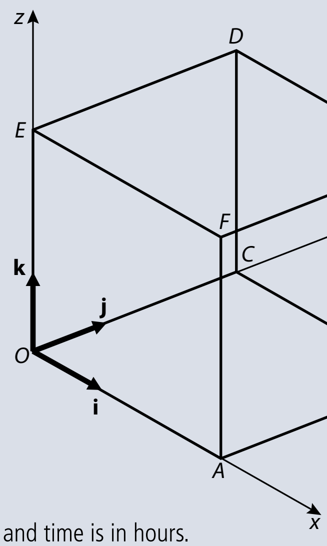
d) The lines L_1 and L_2 intersect at the point P. Find the position vector of P.

28 The diagram shows a cube, $OABCDEFG$, where the length of each edge is 5 cm. Express the following vectors in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

a) $\overrightarrow{OG}$

b) $\overrightarrow{BD}$

c) $\overrightarrow{EB}$



29 In this question, distance is in kilometres and time is in hours.

A balloon is moving at a constant height with a speed of 18 km h^{-1}, in the direction of the vector $\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$.

At time $t = 0$, the balloon is at point B with coordinates (0, 0, 5).

a) Show that the position vector $\mathbf{b}$ of the balloon at time t is given by

$$\mathbf{b} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} + \frac{18t}{5}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}.$$

At time $t = 0$, a helicopter goes to deliver a message to the balloon. The position vector **h** of the helicopter at time t is given by

$$\mathbf{h} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 49 \\ 32 \\ 0 \end{pmatrix} + t\begin{pmatrix} -48 \\ -24 \\ 6 \end{pmatrix}.$$

b) **(i)** Write down the coordinates of the starting position of the helicopter.

(ii) Find the speed of the helicopter.

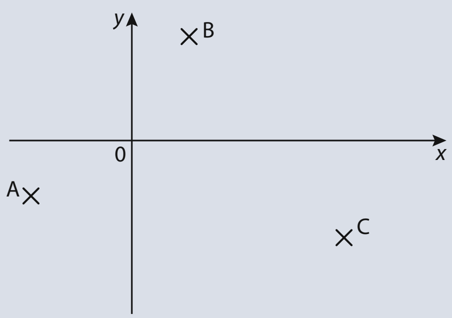
c) The helicopter reaches the balloon at point R.

(i) Find the time the helicopter takes to reach the balloon.

(ii) Find the coordinates of R.

30 In this question, the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents a displacement due east and the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represents a displacement of 1 km north.

The diagram below shows the positions of towns A, B and C in relation to an airport O, which is at the point (0, 0). An aircraft flies over the three towns at a constant speed of 250 km h^{-1}.



Town A is 600 km west and 200 km south of the airport.
Town B is 200 km east and 400 km north of the airport.
Town C is 1200 km east and 350 km south of the airport.

a) **(i)** Find $\overrightarrow{AB}$.

(ii) Show that the vector of length one unit in the direction of $\overrightarrow{AB}$ is $\begin{pmatrix} 0.8 \\ 0.6 \end{pmatrix}$.

An aircraft flies over town A at 12:00, heading towards town B at 250 km h^{-1}.

Let $\begin{pmatrix} p \\ q \end{pmatrix}$ be the velocity vector of the aircraft. Let t be the number of hours in flight after 12:00.

The position of the aircraft can be given by the vector equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -600 \\ -200 \end{pmatrix} + t\begin{pmatrix} p \\ q \end{pmatrix}.$$

b) **(i)** Show that the velocity vector is $\begin{pmatrix} 200 \\ 150 \end{pmatrix}$.

(ii) Find the position of the aircraft at 13:00.

(iii) At what time is the aircraft flying over town B?

Over town B the aircraft changes direction so it now flies towards town C. It takes five hours to travel the 1250 km between B and C. Over town A the pilot noted that she had 17 000 litres of fuel left. The aircraft uses 1800 litres of fuel per hour when travelling at 250 km h^{-1}. When the fuel gets below 1000 litres a warning light comes on.

c) How far from town C will the aircraft be when the warning light comes on?

31 The coordinates of the points P, Q, R and S are $(4, 1, -1)$, $(3, 3, 5)$, $(1, 0, 2c)$ and $(1, 1, 2)$, respectively.

a) Find the value of c so that the vectors $\overrightarrow{OR}$ and $\overrightarrow{PR}$ are orthogonal.
For the remainder of the question, use the value of c found in part **a)** for the coordinate of the point R.

b) Evaluate $\overrightarrow{PS} \times \overrightarrow{PR}$.

c) Find an equation of the line l which passes through the point Q and is parallel to the vector PR.

d) Find an equation of the plane π which contains the line l and passes through the point S.

e) Find the shortest distance between the point P and the plane π.

32 Consider the points $A(1, 2, 1)$, $B(0, -1, 2)$, $C(1, 0, 2)$ and $D(2, -1, -6)$.

a) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

b) Calculate $\overrightarrow{AB} \times \overrightarrow{BC}$.

c) Hence, or otherwise, find the area of triangle ABC.

d) Find the equation of the plane P containing the points A, B and C.

e) Find a set of parametric equations for the line through the point D and perpendicular to the plane P.

f) Find the distance from the point D to the plane P.

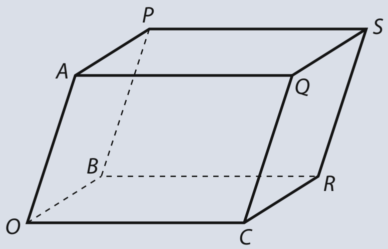
g) Find a unit vector which is perpendicular to the plane P.

h) The point E is a reflection of D in the plane P. Find the coordinates of E.

33 **a)** If $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, show that $\mathbf{u} \times \mathbf{v} = 7\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$.

b) Let $\mathbf{w} = \lambda\mathbf{u} + \mu\mathbf{v}$ where λ and μ are scalars. Show that $\mathbf{w}$ is perpendicular to the line of intersection of the planes $x + 2y + 3z = 5$ and $2x - y + 2z = 7$ for all values of λ and μ.

34 Three points A, B and C have coordinates $(2, 1, -2)$, $(2, -1, -1)$ and $(1, 2, 2)$ respectively. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, where O is the origin, form three concurrent edges of a parallelepiped $OAPBCQSR$ as shown in the following diagram.



a) Find the coordinates of P, Q, R and S.

b) Find an equation for the plane $OAPB$.

c) Calculate the volume, V, of the parallelepiped given that $V = \overrightarrow{OA} \times \overrightarrow{OB} \cdot \overrightarrow{OC}$.

35 The triangle ABC has vertices at the points $A(-l, 2, 3)$, $B(-l, 3, 5)$ and $C(0, -1, 1)$.

a) Find the size of the angle θ between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b) Hence, or otherwise, find the area of triangle ABC.

Let l_1 be the line parallel to $\overrightarrow{AB}$ which passes through $D(2, -1, 0)$ and l_2 be the line parallel to $\overrightarrow{AC}$ which passes through $E(-l, 1, 1)$.

c) **(i)** Find the equations of the lines l_1 and l_2.

(ii) Hence, show that l_1 and l_2 do not intersect.

d) Find the shortest distance between l_1 and l_2.

36 **a)** Solve the following system of linear equations:

$$x + 3y - 2z = -6$$
$$2x + y + 3z = 7$$
$$3x - y + z = 6$$

b) Find the vector $\mathbf{v} = (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) \times (2\mathbf{i} + \mathbf{j} + 3\mathbf{k})$.

c) If $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{u} = m\mathbf{a} + n\mathbf{b}$ where m, n are scalars, and $\mathbf{u} \neq 0$, show that $\mathbf{v}$ is perpendicular to $\mathbf{u}$ for all m and n.

d) The line l lies in the plane $3x - y + z = 6$, passes through the point $(1, -1, 2)$ and is perpendicular to $\mathbf{v}$. Find the equation of l.

37 The points A, B, C, D have the following coordinates: $A(1, 3, 1)$, $B(1, 2, 4)$, $C(2, 3, 6)$, $D(5, -2, 1)$.

a) **(i)** Evaluate the vector product $\overrightarrow{AB} \times \overrightarrow{AC}$, giving your answer in terms of the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

(ii) Find the area of the triangle ABC.

The plane containing the points A, B, C is denoted by Π and the line passing through D perpendicular to Π is denoted by L. The point of intersection of L and Π is denoted by P.

b) **(i)** Find the Cartesian equation of Π.

(ii) Find the Cartesian equation of L.

c) Determine the coordinates of P.

d) Find the perpendicular distance of D from Π.

38 The point $A(2, 5, -1)$ is on the line L, which is perpendicular to the plane with equation $x + y + z - 1 = 0$.

a) Find the Cartesian equation of the line L.

b) Find the point of intersection of the line L and the plane.

c) The point A is reflected in the plane. Find the coordinates of the image of A.

d) Calculate the distance from the point $B(2, 0, 6)$ to the line L.

39 **a)** The point $P(1, 2, 11)$ lies in the plane π_1. The vector $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ is perpendicular to π_1. Find the Cartesian equation of π_1.

b) The plane π_2 has equation $x + 3y - z = -4$.

(i) Show that the point P also lies in the plane π_2.

(ii) Find a vector equation of the line of intersection of π_1 and π_2.

c) Find the acute angle between π_1 and π_2.

40 A line l_1 has equation $\frac{x+2}{3} = \frac{y}{1} = \frac{z-9}{-2}$.

a) Let M be a point on l_1 with parameter μ. Express the coordinates of M in terms of μ.

b) The line l_2 is parallel to l_1 and passes through $P(4, 0, -3)$.

(i) Write down an equation for l_2.

(ii) Express $\overrightarrow{PM}$ in terms of μ.

c) The vector $\overrightarrow{PM}$ is perpendicular to l_1.

(i) Find the value of μ.

(ii) Find the distance between l_1 and l_2.

d) The plane π_1 contains l_1 and l_2. Find an equation for π_1, giving your answer in the form $Ax + By + Cz = D$.

e) The plane π_2 has equation $x - 5y - z = -11$. Verify that l_1 is the line of intersection of the planes π_1 and π_2.

41 a) Show that the lines $\frac{x-2}{1}=\frac{y-2}{3}=\frac{z-3}{1}$ and $\frac{x-2}{1}=\frac{y-3}{4}=\frac{z-4}{2}$ intersect and find the coordinates of P, the point of intersection.

b) Find the Cartesian equation of the plane π that contains the two lines.

c) The point $Q(3, 4, 3)$ lies on π. The line L passes through the midpoint of $[PQ]$. Point S is on L such that $|\overrightarrow{PS}| = |\overrightarrow{QS}| = 3$, and the triangle PQS is normal to the plane π. Given that there are two possible positions for S, find their coordinates.

42 a) The plane π_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ -9 \end{pmatrix}$.

The plane π_2 has the equation $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(i) For points which lie in π_1 and π_2, show that $\lambda = \mu$.

(ii) Hence, or otherwise, find a vector equation of the line of intersection of π_1 and π_2.

b) The plane π_3 contains the line $\frac{2-x}{3}=\frac{y}{-4}=z+1$ and is perpendicular to $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

Find the Cartesian equation of π_3.

c) Find the intersection of π_1, π_2 and π_3.