Exercise 15C

1 If $z = \cos\theta + \mathrm{i}\sin\theta$, find the values of each of the following.

a) $z^2 - \frac{1}{z^2}$ **b)** $z^4 + \frac{1}{z^4}$ **c)** $z^5 + \frac{1}{z^5}$ **d)** $z^2 - \frac{2}{z} + \frac{2}{z} - \frac{1}{z^2}$

2 Express each of the following in terms of z, where $z = \cos\theta + \mathrm{i}\sin\theta$.

a) $\cos 6\theta$ **b)** $\sin 5\theta$ **c)** $\cos^4\theta$ **d)** $\sin^3\theta$
e) $\sin^2 5\theta$ **f)** $\cos^4 3\theta$

3 Express each of the following in terms of $\cos\theta$.

a) $\cos 6\theta$ **b)** $\cos 4\theta$ **c)** $\frac{\sin 4\theta}{\sin\theta}$ **d)** $\frac{\sin 6\theta}{\sin\theta}$

4 Express each of the following in terms of $\sin\theta$.

a) $\sin 3\theta$ **b)** $\sin 5\theta$ **c)** $\frac{\cos 7\theta}{\cos\theta}$ **d)** $\frac{\cos 5\theta}{\cos\theta}$

5 Express each of the following in terms of sines or cosines of multiple angles.

a) $\sin^3\theta$ **b)** $\cos^3\theta$ **c)** $\cos^5\theta$ **d)** $\sin^5\theta$
e) $\cos^6\theta$

6 Prove that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$.

7 Prove that $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$. Hence solve $t^3 - 3t^2 - 3t + 1 = 0$.

8 By considering $(\cos\theta + \mathrm{i}\sin\theta)^3$, use de Moivre's theorem to establish the identity

$$\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$$

Write down the coefficient of θ^4 in the series expansion of $\cos 3\theta$.

Hence, using the identity above, obtain the coefficient of θ^4 in the series expansion of $\cos^3\theta$.

(AEB 96)

9 i) Show that $(2 + \mathrm{i})^4 = -7 + 24\mathrm{i}$.

ii) Use de Moivre's theorem to show that

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

and $$\sin 4\theta = 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$$

iii) If $t = \tan\theta$, show that

$$\tan 4\theta = \frac{4t - 4t^3}{1 - 6t^2 + t^4}$$

iv) By considering the argument of $(2 + \mathrm{i})$, explain why $t = \frac{1}{2}$ is a root of the following equation

$$\frac{4t - 4t^3}{1 - 6t^2 + t^4} = -\frac{24}{7}$$

v) Using the symmetry properties of the four roots of $z^4 = a^4$, draw an Argand diagram showing the four roots of $z^4 = -7 + 24\text{i}$.

vi) Find one other root of the equation in part **iv**. (NICCEA)

10 Use de Moivre's theorem to prove that

$$\sin 6\theta = 6\cos^5\theta \sin\theta - 20\cos^3\theta \sin^3\theta + 6\cos\theta \sin^5\theta$$

By putting $x = \sin\theta$, deduce that, for $|x| \leqslant 1$,

$$-\tfrac{1}{2} \leqslant x(16x^4 - 16x^2 + 3)\sqrt{(1 - x^2)} \leqslant \tfrac{1}{2} \qquad \text{(OCR)}$$

11 Use de Moivre's theorem to prove that

$$\cos 5\theta = \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5)$$

By considering the equation $\cos 5\theta = 0$, show that the exact value of $\cos^2(\frac{1}{10}\pi)$ is $\dfrac{5 + \sqrt{5}}{8}$.

(OCR)

12 Use de Moivre's theorem to show that

$$\tan 5\theta = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$$

where $t = \tan\theta$. (OCR)

13 Let $z = \cos\theta + \text{i}\sin\theta$.

a) Use the binomial theorem to show that the real part of z^4 is

$$\cos^4\theta - 6\cos^2\theta \sin^2\theta + \sin^4\theta$$

Obtain a similar expression for the imaginary part of z^4 in terms of θ.

b) Use de Moivre's theorem to write down an expression for z^4 in terms of 4θ.

c) Use your answers to parts **a** and **b** to express $\cos 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

d) Hence show that $\cos 4\theta$ can be written in the form $k(\cos^m\theta - \cos^n\theta) + p$, where k, m, n, p are integers. State the values of k, m, n, p. (SQA/CSYS)

14 Use de Moivre's theorem to show that

$$\sin 5\theta = a\cos^4\theta \sin\theta + b\cos^2\theta \sin^3\theta + c\sin^5\theta$$

where a, b and c are integers to be determined.

Hence show that

$$\frac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1 \quad (\theta \neq k\pi, \text{ where } k \in \mathbb{Z})$$

By means of the substitution $x = 2\cos\theta$, find, in trigonometric form, the roots of the equation

$$x^4 - 3x^2 + 1 = 0$$

Hence, or otherwise, show that

$$\cos^2(\tfrac{1}{5}\pi) + \cos^2(\tfrac{2}{5}\pi) = \tfrac{3}{4} \qquad \text{(NEAB)}$$