Investigating the rising level of water in a bathtub

1. Introduction

I started my investigation in a slightly different way than other students did. At the beginning, I defined the main goal and later on I tried to find the topic. I knew that I want to base my research on the things I am using in my everyday life. Applying mathematics outside the classroom, so that I can perceive myself the results obtained as useful and meaningful.

The idea came unexpectedly, while preparing to the activity that millions of people are doing every day – taking a bath. Observing the rising water level and wondering how fast is it changing over time. A perfect case to apply the knowledge gained after hours spent on exploring in depth differential calculus.

To find the value I am interested it, so $\frac{dh}{dt}$, I will have to undertake several steps. Firstly, I will model my bathtub. Then I will establish an equation that will allow me to calculate the desired value. The volume of water that will be flowing out of the tap and filling the bathtub will be established experimentally. At the end, I will check how the results obtained apply to the reality. To the equation for the rising level of water per time I will substitute the dimensions of my own bathtub. The theoretical results will be compared to the ones that will be obtained through my practical measurements.

2. Model of the bathtub

Nowadays, the market offers a wide range of bathtubs: oval, rectangular, clover shaped, triangular etc. However, to make my investigation possible I had to choose such a one, which can be modelled easily. Therefore, I decided to analyze a bathtub that is a truncated pyramid. The simplification that was made applies to the bottom, which is usually slightly angled. I made it flat, so the level of water is rising evenly in the whole bathtub. To obtain this model, firstly I had to create a pyramid. It was done using GeoGebra. The pyramid in the figure below has a rectangular base with surface area S_1. It can be thought that a bathtub results after a removal of a smaller pyramid. It has a base that is parallel to the bigger one which surface area is marked as S_2.

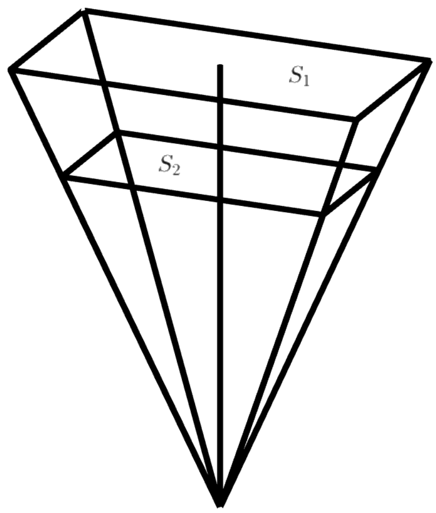

Figure 1: Pyramid used to model the bathtub.

As my aim is to find an expression for the rising level of water S_1 and S_2 will not denote the surface areas of the bathtub. S_2 will be a constant value that indicates the surface area of the lower base of water, which is equivalent to the surface area of the lower base of the bathtub, whereas S_1 will express the surface area of the upper base of water that is variable, depending on water level.

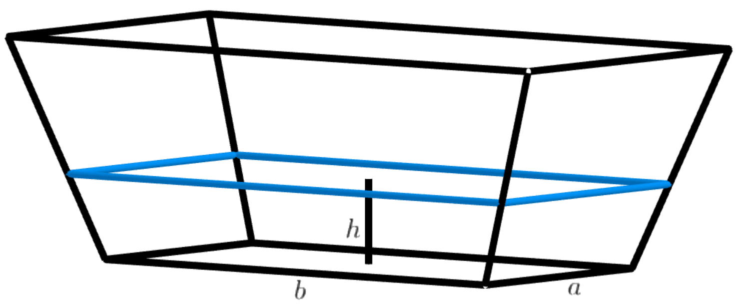

Figure 2: Bathtub with marked level of water.

Figure 2 shows the obtained bathtub. The lower base of the water has dimensions a X b and the height of water is denoted as h.

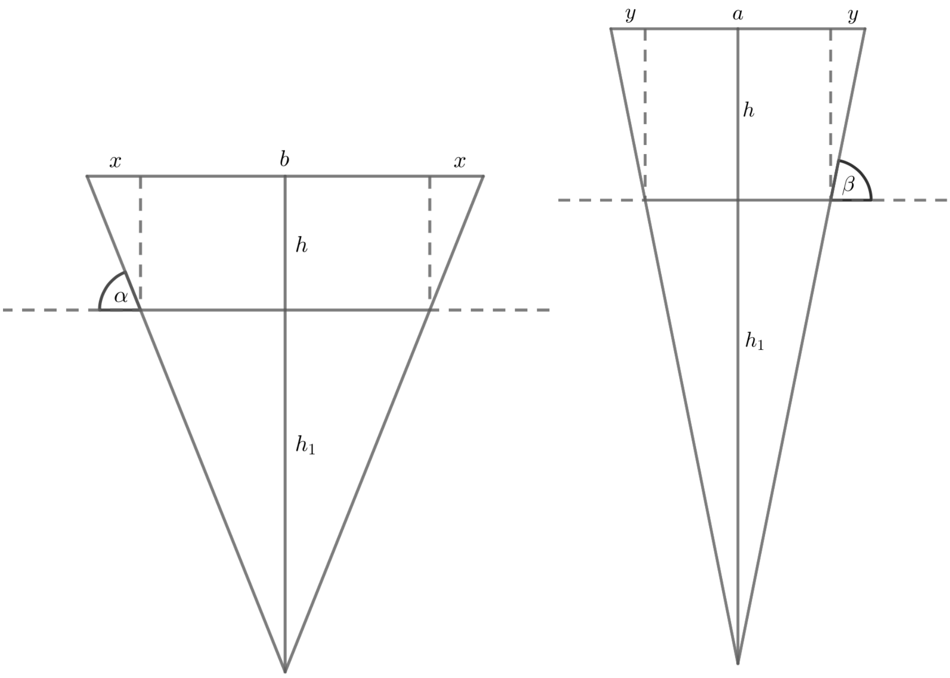

Figure 3: Cross sections of the pyramid with marked angles of inclination (diagrams not to scale).

The sketch on the left shows the cross section when looking on the pyramid from the front side. Its height is expressed as $h_1 + h$. The angle at which the bathtub inclines to the left and to the right is marked as α. x represents half of the difference between the length b and the length of the upper base. Similarly, on the sketch on the right, y expresses half of the difference between the width of the upper and the lower base of the bathtub. β is the angle at which the bathtub widens to the front and backward.

Now, having the models and variables established it is necessary to create an equation that will relate all of these. This will be done in the section below.

3. Equation

The increasing level of water per unit of time can be calculated using the following equation.

$$\frac{dV}{dt} = \frac{dh}{dt} \times \frac{dV}{dh} \qquad (1)$$

$\frac{dV}{dt}$ is a constant that will be established experimentally using the materials and procedure listed below.

Materials:

- stopwatch [±0.01 s]
- measuring bowl [±50 ml]

Procedure:

1. Prepare the measuring bowl [±50 ml] and the stopwatch [±0.01 s].
2. Turn the tap on to the maximum.
3. At the same time place the measuring bowl [±50 ml] under the tap and turn on the stopwatch [±0.01 s].
4. Turn it off when the water level in the bowl will reach 1l.
5. Note down the time into a table.
6. Repeat the experiment five times.

The table below shows the values of the measurements taken.

Table 1: The times in which the measuring bowl was filled out with 1l of water.

trial	time[s][±0.01 s]
1	4.02
2	4.26
3	3.74
4	4.20
5	4.25

The mean from these values equals $1l/4.09s$, which gives the value of $\frac{dV}{dt} = 244.5cm^3/s$.

One of the variables needed to find an expression for $\frac{dh}{dt}$ has been already established. In the next section I will be deriving the formula, which later on will enable me to obtain $\frac{dV}{dh}$.

4. Volume of the water

To find $\frac{dV}{dh}$ the formula for the volume of the water is needed. This corresponds to the volume of the bathtub when it is completely filled out with water. My bathtub is a truncated pyramid. Its volume can be calculated by subtracting the volume of the pyramid cut out from the volume of the whole pyramid. This can be written as below.

$$\begin{aligned} V &= \frac{1}{3}S_1(h + h_1) - \frac{1}{3}S_2 h_1 \\ &= \frac{1}{3}S_1 h + \frac{1}{3}S_1 h_1 - \frac{1}{3}S_2 h_1 \\ &= \frac{1}{3}S_1 h + \frac{1}{3}h_1(S_1 - S_2) \qquad (2) \end{aligned}$$

Now it is necessary to find an expression for $h_1$in terms of h, $S_1$and S_2. As the figure that was cut out to obtain the bathtub is similar to the whole pyramid, using the ratios of heights and areas of the base an equation can be set up. Taking into consideration that the ratio of the areas of the base gives the scale squared the following results:

$$\left(\frac{h_1}{h + h_1}\right)^2 = \frac{S_2}{S_1}$$

$$\frac{h_1}{h + h_1} = \frac{\sqrt{S_2}}{\sqrt{S_1}}$$

$$h_1\sqrt{S_1} = h_1\sqrt{S_2} + h\sqrt{S_2}$$

Subtracting $h_1\sqrt{S_2}$ from both sides and factoring h_1 out

$$h_1\left(\sqrt{S_1} - \sqrt{S_2}\right) = h\sqrt{S_2}$$

This gives the following expression for h_1

$$h_1 = h \times \frac{\sqrt{S_2}}{\sqrt{S_1} - \sqrt{S_2}} \qquad (3)$$

Substituting expression for h_1 (equation 3) into equation 2

$$V = \frac{1}{3}S_1 h + \frac{1}{3}h\frac{\sqrt{S_2}}{\sqrt{S_1} - \sqrt{S_2}} \times (S_1 - S_2)$$

$$= \frac{1}{3}h(S_1 + \frac{\sqrt{S_2}\,S_1}{\sqrt{S_1} - \sqrt{S_2}} - \frac{\sqrt{S_2}\,S_2}{\sqrt{S_1} - \sqrt{S_2}})$$

After bringing the fractions to the common denominator

$$V = \frac{h}{3}\left(\frac{S_1\sqrt{S_1} - S_1\sqrt{S_2} + S_1\sqrt{S_2} - S_2\sqrt{S_2}}{\sqrt{S_1} - \sqrt{S_2}}\right)$$

$$= \frac{h}{3}\left(\frac{S_1\sqrt{S_1} - S_2\sqrt{S_2}}{\sqrt{S_1} - \sqrt{S_2}}\right)$$

After rationalizing the denominator

$$V = \frac{h}{3}\left(\frac{(S_1\sqrt{S_1} - S_2\sqrt{S_2})(\sqrt{S_1} + \sqrt{S_2})}{\sqrt{S_1} - \sqrt{S_2}}\right)$$

$$= \frac{h}{3}\left(\frac{{S_1}^2 + S_1\sqrt{S_1 S_2} - S_2\sqrt{S_1 S_2} - {S_2}^2}{S_1 - S_2}\right)$$

Using the differences of squares the following equation can be written as

$$V = \frac{h}{3}\left(\frac{(S_1 - S_2)(S_1 + S_2) + \sqrt{S_1 S_2}(S_1 - S_2)}{S_1 - S_2}\right)$$

Therefore, the volume of the water is given by the formula

$$V = \frac{1}{3}h\left(S_1 + S_2 + \sqrt{S_1 S_2}\right) \qquad (4)$$

In the next section I will differentiate equation 4, so then I will have all values needed to express $\frac{dh}{dt}$.

5. Differentiating the formula for the volume of water

The formula for the volume cannot be differentiated straightforward. S_2 which is the surface area of the lower base of water is constant. However, as mentioned before the surface area of the upper base of water depends on h. Therefore, firstly it must be expressed in terms of h, S_2 and the angles α and β. The figure below shows the plan view of the bathtub which is a truncated pyramid.

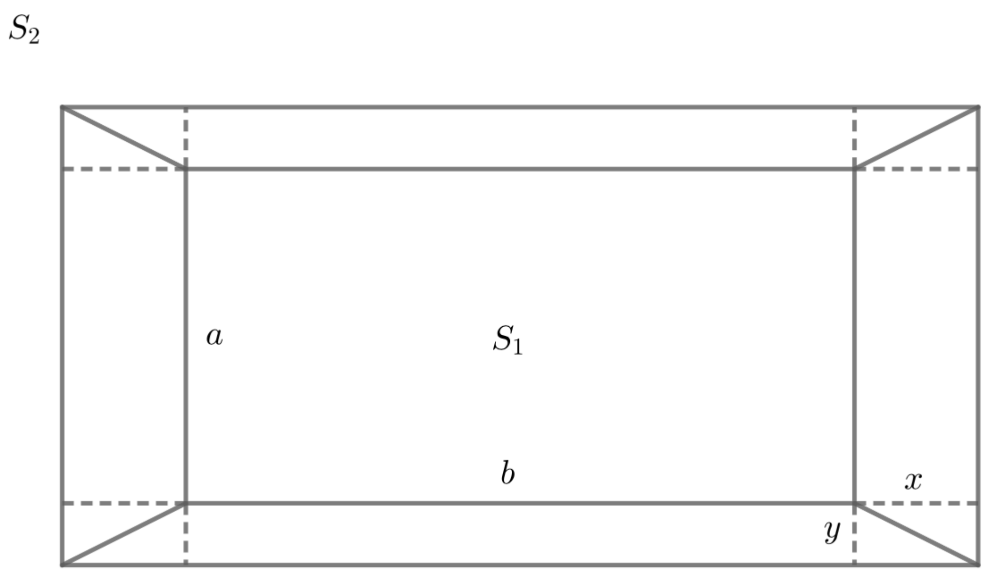

Figure 4: Bathtub - plan view.

The area S_2 is the sum of the area S_1, the base area of four corners, area of two rectangles with one of the sides length a and area of two rectangles with one of the sides length b.

The figures that have a rectangular face with one of the side length a looks like on the sketch below. The sketches below show two figures. One that has rectangular face with one side length a and second that has rectangular face with one side being b.

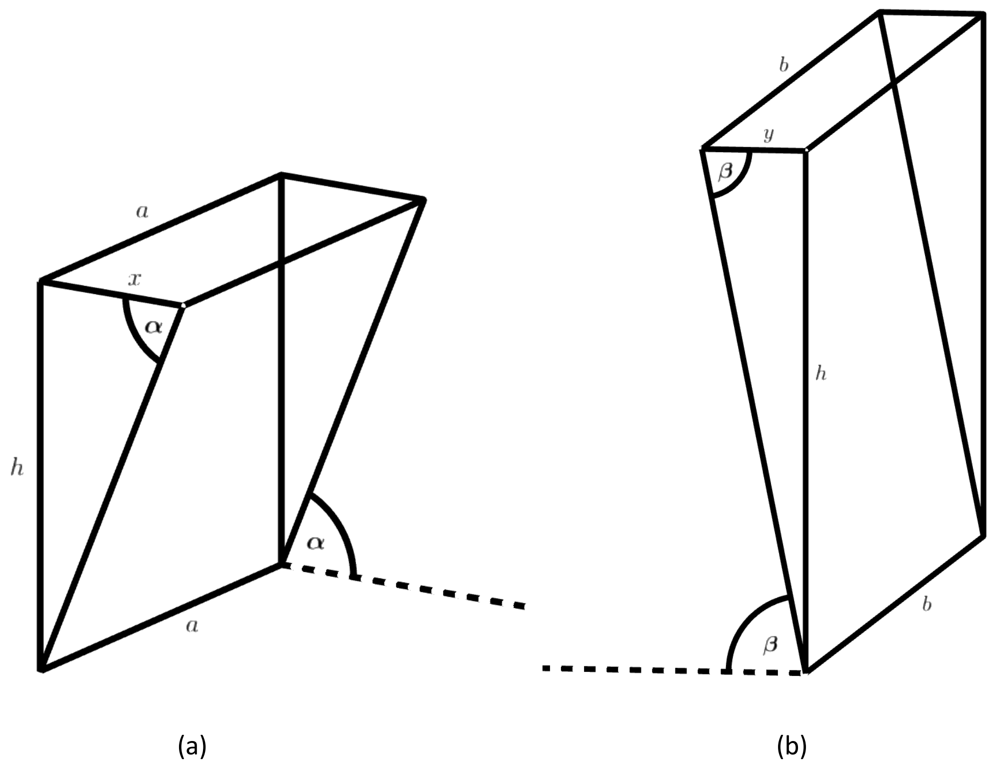

Figure 5: (a) 3D view of the triangular prism with a rectangular face with one of the sides being a.

(b) 3D view of the triangular prism with a rectangular face with one of the sides being b.

(diagrams not to scale)

Now, it is possible to express x in terms of the angle α and h. Tangent is the trigonometric function that relates these two sides. So, the following equation can be set up.

$\tan \alpha = \frac{h}{x}$, therefore x can be written as $x = \frac{h}{\tan \alpha}$

Applying the formula for the area of the rectangle the following results $Area = \frac{ah}{\tan \alpha}$.

By analogy, the area of the rectangular face of the *Figure 5(b)* can be expressed as

$Area = \frac{bh}{\tan \beta}$.

The four corners look like on the figure below and are pyramids with rectangular bases with dimensions x X y.

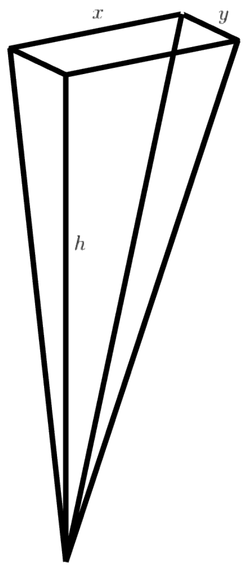

Figure 6: Pyramid illustrating the corners of the bathtub.

Using the expressions for x and y the area of the base can be written as $Area = \frac{h^2}{\tan\alpha \times \tan\beta}$.

S_2 can be now expressed in terms of h, S_1 and angles α and β by adding the calculated areas.

$$S_2 = ab + \frac{2ah}{\tan\alpha} + \frac{2bh}{\tan\beta} + \frac{4h^2}{\tan\alpha \times \tan\beta}$$

Substituting this equation in the equation (4) for the volume of water the following results.

$$V = \frac{1}{3}h\left(ab + ab + \frac{2ah}{\tan\alpha} + \frac{2bh}{\tan\beta} + \frac{4h^2}{\tan\alpha \times \tan\beta} + \sqrt{ab\left(ab + \frac{2ah}{\tan\alpha} + \frac{2bh}{\tan\beta} + \frac{4h^2}{\tan\alpha \times \tan\beta}\right)}\right)$$

$$= \frac{2abh}{3} + \frac{2ah^2}{3\tan\alpha} + \frac{2bh^2}{3\tan\beta} + \frac{4h^3}{3\tan\alpha \times \tan\beta} + \frac{1}{3}h\sqrt{a^2b^2 + \frac{2a^2bh}{\tan\alpha} + \frac{2ab^2h}{\tan\beta} + \frac{4abh^2}{\tan\alpha \times \tan\beta}}$$

$$= \frac{2abh}{3} + \frac{2ah^2}{3\tan\alpha} + \frac{2bh^2}{3\tan\beta} + \frac{4h^3}{3\tan\alpha \times \tan\beta} + \sqrt{\frac{a^2b^2h^2}{9} + \frac{2a^2bh^3}{9\tan\alpha} + \frac{2ab^2h^3}{9\tan\beta} + \frac{4abh^4}{9\tan\alpha \times \tan\beta}}$$

Now it is possible to differentiate the expression for the volume of water.

$$\frac{dV}{dh} = \frac{2ab}{3} + \frac{4ah}{3\tan\alpha} + \frac{4bh}{3\tan\beta} + \frac{12h^2}{3\tan\alpha \times \tan\beta} + \left(\sqrt{\frac{a^2b^2h^2}{9} + \frac{2a^2bh^3}{9\tan\alpha} + \frac{2ab^2h^3}{9\tan\beta} + \frac{4abh^4}{9\tan\alpha \times \tan\beta}}\right)' \quad (5)$$

The equation under the square root will be differentiated using chain rule.

$$\left(\sqrt{\frac{a^2b^2h^2}{9}+\frac{2a^2bh^3}{9\tan\alpha}+\frac{2ab^2h^3}{9\tan\beta}+\frac{4abh^4}{9\tan\alpha\times\tan\beta}}\right)'=\frac{\frac{2a^2b^2h}{9}+\frac{6a^2bh^2}{9\tan\alpha}+\frac{6ab^2h^2}{9\tan\beta}+\frac{16abh^3}{9\tan\alpha\times\tan\beta}}{2\sqrt{\frac{a^2b^2h^2}{9}+\frac{2a^2bh^3}{9\tan\alpha}+\frac{2ab^2h^3}{9\tan\beta}+\frac{4abh^4}{9\tan\alpha\times\tan\beta}}}$$

Substituting this into equation 5

$$\frac{dV}{dh}=\frac{2ab}{3}+\frac{4ah}{3\tan\alpha}+\frac{4bh}{3\tan\beta}+\frac{4h^2}{\tan\alpha\times\tan\beta}+\frac{\frac{2a^2b^2h}{9}+\frac{6a^2bh^2}{9\tan\alpha}+\frac{6ab^2h^2}{9\tan\beta}+\frac{16abh^3}{9\tan\alpha\times\tan\beta}}{2\sqrt{\frac{a^2b^2h^2}{9}+\frac{2a^2bh^3}{9\tan\alpha}+\frac{2ab^2h^3}{9\tan\beta}+\frac{4abh^4}{9\tan\alpha\times\tan\beta}}}$$

According to equation 1

$$\frac{dh}{dt}=\frac{244.5}{\frac{2ab}{3}+\frac{4ah}{3\tan\alpha}+\frac{4bh}{3\tan\beta}+\frac{4h^2}{\tan\alpha\times\tan\beta}+\frac{\frac{2a^2b^2h}{9}+\frac{6a^2bh^2}{9\tan\alpha}+\frac{6ab^2h^2}{9\tan\beta}+\frac{16abh^3}{9\tan\alpha\times\tan\beta}}{2\sqrt{\frac{a^2b^2h^2}{9}+\frac{2a^2bh^3}{9\tan\alpha}+\frac{2ab^2h^3}{9\tan\beta}+\frac{4abh^4}{9\tan\alpha\times\tan\beta}}}}$$

6. Example

Having the equation for $\frac{dh}{dt}$ I will substitute the values that correspond to these of my own bathtub. The side a equals $45cm$, side b is $130cm$ long, $\alpha = 75°$ and $\beta = 71°$.[1] As an example I will consider the case that assumes $h = 5cm$.

$$\frac{dh}{dt}=\frac{244.5}{\frac{2\times45\times130}{3}+\frac{4\times45\times5}{3\tan 75°}+\frac{4\times130\times5}{3\tan 71°}+\frac{4\times5^2}{\tan 75°\times\tan 71°}+\frac{\frac{2\times45^2\times130^2\times5}{9}+\frac{6\times45^2\times130\times5^2}{9\tan 75°}+\frac{6\times45\times130^2\times5^2}{9\tan 71°}+\frac{16\times45\times130\times5^3}{9\tan 75°\times\tan 71°}}{2\sqrt{\frac{45^2\times130^2\times5^2}{9}+\frac{2\times45^2\times130\times5^3}{9\tan 75°}+\frac{2\times45\times130^2\times5^3}{9\tan 71°}+\frac{4\times45\times130\times5^4}{9\tan 75°\times\tan 71°}}}}=$$

$0.0381cm/s$

This theoretical value will be now compared to the one that was measured practically. To obtain it I placed a tape measure [±1mm] in the bathtub and fill it with water, so that its level equals $5cm$. Then, I turned on the tap in the same way that I was doing it while establishing

[1] The lengths of side a and b were measured with a tape measure [±1mm]. The find the values of α and β I used a special protractor that is especially popular on the construction sites, as it enables to measure the angles of inclination.

$\frac{dV}{dt}$. At the same time I turned on the stopwatch [±0.01s]. The value I measured was how long it will take the level of water to increase 1cm. It equals $24.3s$, so it can be assumed that $\frac{dh}{dt}$, when $h = 5cm$ equals $0.0412cm/s$.

To illustrate the difference between these two values I calculated the percentage difference. It shows that the results are not very divergent, as the measured value is only 8.14% greater from the one obtained from the calculations. The reason for this difference will be explained in the next section.

7. Conclusions

The value that results from my own measurements is slightly different than this calculated one. There are several ways to explain it. Firstly, the model that I created does not resemble my bathtub perfectly. As mentioned at the beginning the bottom was not angled. What is more, the corners in the real bathtub are minimized. Therefore, the actual volume of the bathtub is smaller than it results from the calculations, which allows the level of water to rise faster. Moreover, it must be remembered that the volume of water that flows out of the tap was established experimentally. The fact that follows is, that it differs slightly in each trial. However, the most important aspect is connected with the measurements that I took. It is crucial to notice that it is very difficult to measure $\frac{dh}{dt}$, so the instantaneous rate of change when h equals exactly $5cm$. The thing that was calculated is the average rate of change of the water level, $\frac{\Delta h}{\Delta t}\Big|_{\substack{h=5cm \\ \Delta h=1cm}}$. However, this value differs from $\frac{dh}{dt}$, $as\ it$ assumes that the height of water level is increasing evenly.