# What are the properties of a Fibonacci sequence?

## 1 Introduction

One of the best architects of the 20th century modernism, Le Corbusier believed that architecture built on the basis of Fibonacci sequence would be prefect and aestheticaly pleasing, thus many of his best known projects such as Villa Savoye, Villa Garche, Chapel of Ronchamp or the headquaters of United Nations in New York include golden ratio, the ratio of the terms of Fibonacci sequence, in their proportions.[1] Additionally, it is believed that even ancient temples were built on the golden proportion. As I plan to study architecture, I believe that it is vital for me to understand better the rules used by architects over the centuries, and how did they define beauty in their projects. As golden proportion is one of the basic, simple architectural "tools", I am motivated to investigate the Fibonacci sequence, so to comprehend the origin of the ratio and other characteristics of the sequence.

Fibonacci sequence is probably one of the most known sequences. Golden ratio and golden spiral derived on its basis are commonly found in nature, art, music and architecture[2]. What is intriguing is that it is not only so common in the surronding world, but also in a variety of areas of mathematics. There is a wide range of mathematical tools and methods of proving can be used in its analysis. In my exploration

[1] Meisner, G. (2014) UN Secretatiat Building, Le Corbusier and the Golden Ratio.

[2] Lamb, R. The Golden ratio in Nature. Palmer, L. (2015) See How Artists Discover Simplicity as an Art Form in Works Which Reflect the Golden Ratio. What is interesting, number of petals of many plants, e.g. lilies and wild roses are Fibonacci numbers. Similarily number of spirals in the pinecones, a lot of vegetables, friuts and flowers is also Fibonacci. Shell of a marine mollusc Nautilus is a logarithimc spiral with growth factor equal to golden ratio. This may mean, that Fibonacci sequence and golden ratio reflect natural patterns. Moreover, researchers tend to frequently find the ratio in well-known works of art, such as "The Creation of Adam" by Michelangelo or "The Birth of Veus" by Botticelli, what sugest that the ratio may be the indicator of aesthetics.

I aim at investigating properties of the sequence, but what is most important is that I will present various methods of analysis, that include a proof by induction, proof with the use of geometry, manipulations of series, matrices, and finally linear diophantine equations. It will be preceded by basic analysis of the sequence: firstly the ratio between the terms $F_n$ and $F_{n-1}$ as $n$ goes to infinity, that inspired me to chose this topic, and then finding explicit formula of the sequence.

# 2 Analysis

## 2.1 Definitions

I will start the exploration with defining crucial terms- firstly, Fibonacci sequence itself. It is an arithmetic sequence that can be defined by the formula

$F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ where $F_0 = 0 \quad F_1 = 1$.

The first terms of the sequence are: 0, 1,1,2,5,8,13,21..

This type of formula is called recursive formula (alternatively reccurence relation). It descibes $n^{th}$ term of the sequence using the previous terms. It is not a perfect way to describe the sequence, as to find $100^{th}$ term, $99^{th}$, $98^{th}$ and possibly the preceeding ones must be known.

It can be manipulated to obtain the explicit formula for the $n^{th}$ term in the terms of $n$. Closed form equation of the Fibbonaci sequence can be found by solving second order linear homogeneous reccurence relation.

When comes to a golden ratio, it is a ratio between the consequitve Fibonacci numbers $F_n$ and $F_{n-1}$ as $n$ goes to infinity. It will be analyzed in the following section.

## 2.2 Golden ratio

Let's approximate the

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}}$$

by the substituting values of F.
$\frac{F_3}{F_2} = 2 \quad \frac{F_6}{F_5} = 1.6 \quad \frac{F_7}{F_6} = 1.625 \quad \frac{F_8}{F_7} \approx 1.615385... \quad \frac{F_9}{F_8} \approx 1.619047...$

$\frac{F_{300}}{F_{299}} \approx 1.618033...$ $\frac{F_{10000}}{F_{9999}} \approx 1.1.618033...$

The approximations of a couple of terms sugested that the value of golden ratio is somewhere between 1.61 and 1.62. Now I will find find its exact value.

$F_n = F_{n-1} + F_{n-2}(\text{ for } n \geq 2)$

$\frac{F_n}{F_{n-1}} = 1 + \frac{F_{n-2}}{F_{n-1}}$

Let $\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \Phi$. As the formula for $F_n$ is recursive, we know that $\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \lim_{n\to\infty} \frac{F_{n-1}}{F_{n-2}} = \Phi$. Hence, $\lim_{n\to\infty} \frac{F_{n-2}}{F_{n-1}} = \frac{1}{\Phi}$.

Now I will use the equation above to find $\Phi$.

We know that $\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \lim_{n\to\infty}(1 + \frac{F_{n-2}}{F_{n-1}})$

Hence, $\Phi = 1 + \frac{1}{\Phi}$.

Multiplying both sides of the equation by $\Phi$:

$\Phi^2 = \Phi + 1$

$\Phi^2 - \Phi - 1 = 0$

$\Delta = 1 - (-4) = 5$

$\Phi = \frac{1\pm\sqrt{5}}{2}$

$\phi = \frac{1+\sqrt{5}}{2}$ $\psi = \frac{1-\sqrt{5}}{2}$

We got the two solutions, of which one is negative. As Fibbonaci numbers are natural, the ratio must be also positive, while $\psi \approx -0.618$. The solution $\phi = \Phi$, and is approximately 1.618.

A characterisitc of the ratio I find beautiful is how the equation $\Phi = 1 + \frac{1}{\Phi}$ can be manipulated to create a continued fraction, having all coefficinets equal to one. In the following expansion, every time for $\Phi$ I substitute $\Phi = 1 + \frac{1}{\Phi}$.

$$\Phi = 1 + \frac{1}{\Phi} = 1 + \cfrac{1}{1 + \cfrac{1}{\Phi}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\Phi}}} =$$

$$= 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}}$$

If we stop this fraction, we get an approximation of $\Phi$- the further we go, the more accurate.

$\Phi_1 = 1 + \frac{1}{1} = \frac{2}{1}$ $\qquad \Phi_2 = 1 + \cfrac{1}{1 + \cfrac{1}{1}} = 1 + \frac{1}{2} = \frac{3}{2}$

$\Phi_3 = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}} = 1 + \cfrac{1}{1 + \cfrac{1}{2}} = 1 + \frac{2}{3} = \frac{5}{3}$

It can be seen that the following fractions are the ratios of the subsequent terms of the sequence. "Contiuning" this continued fractions means in fact adding 1 to the reciprocal of the previous fraction. Knowing that golden ratio $\Phi$ is the limit $\lim_{n\to\infty} \frac{F_n}{F_{n-1}}$, the relation mentioned above can be proved the following way:

$1 + \frac{1}{\Phi} = 1 + \frac{1}{\frac{F_n}{F_{n-1}}} = 1 + \frac{F_{n-1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = \frac{F_{n+1}}{F_n}$

This is thanks to the charcterisitc of the sequence, that the succeding terms added up form the next one. [3]

Property of the ratio that could have made it useful in architecture is the presence of golden rectangles.

[3]Knott, R. (2016) The Golden section ratio: Phi.

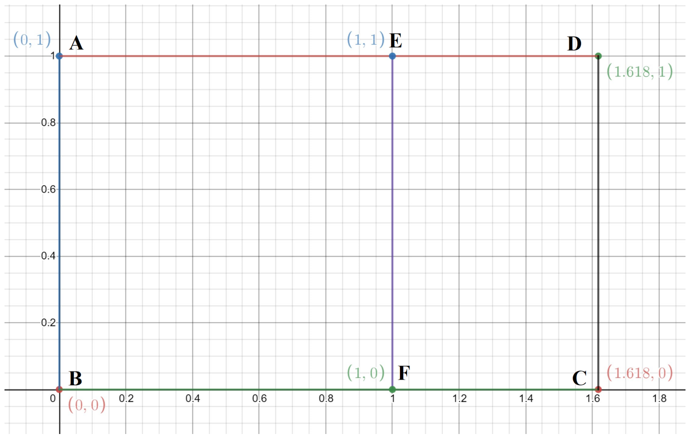


Figure 1: Golden rectangles ABCD and EFCD [4]

If golden ratio is a ratio of the sides of a rectangle, it characterized by self-similarity, what means that if I substract a square from the rectangle, what is formed is another rectangle with the proportion of side lenghts $\Phi : 1$.

Proof:

Suppose there is a rectangle ABCD with $AB = 1$ and $AD = \Phi$. On the side AD there is marked a point E, and on BC point F, so that ABFE is a square. The rectangle is golden, if the rectangle EFCD is similar to ABCD, so the ratio of the sides ED to DC is the same as AB to AD. The lenght of the side ED is $\Phi - 1$. They would be equal if $\frac{\Phi}{1} = \frac{1}{\Phi - 1}$. Then

$1 = \Phi^2 - \Phi$
$\Phi^2 - \Phi - 1 = 0$
$\Delta = 5$
$\phi = \frac{1+\sqrt{5}}{2} \quad \psi = \frac{1-\sqrt{5}}{2}$

I reject the second solution as lenght of a side must be positive. Then LHS=RHS what proves that the rectangles are similar.

[4]Sketched in Desmos, Graphing Calculator

## 2.3 Finding explicit formula of Fibonacci sequence

In this section I am going to find the explicit formula for the $n^{th}$ term of the Fibonaci sequence by solving second order homogenous reccurence relation.

$F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$
so $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$

Suppose $c\alpha^n$ is a solution to the above reccurence relation. Then

$c\alpha^{n+2} = c\alpha^{n+1} + c\alpha^n$
$c\alpha^n(\alpha^2 - \alpha - 1) = 0$, what is a characteristic equation of this recurrence relation.
$\alpha_1 = \frac{1+\sqrt{5}}{2} \quad \alpha_2 = \frac{1-\sqrt{5}}{2} \quad \alpha_3 = 0$
As the roots $\alpha_1$, $\alpha_2$ and $\alpha_3$ are distinct, the recurrence relation has closed form solution

$F_n = A(\frac{1+\sqrt{5}}{2})^n + B(\frac{1-\sqrt{5}}{2})^n + C \times 0^n$,

but $C\alpha_3$ does not change anything, as it is equal to 0, so I will not be considering it in my calculations. To solve the recurrence, I substitute the initial conditions of this relation that are $F_0 = 0$ and $F_1 = 1$.
$0 = A + B$
$1 = A(\frac{1+\sqrt{5}}{2}) + B(\frac{1-\sqrt{5}}{2})$
$A = -B$
$1 = B(\frac{-1-\sqrt{5}}{2}) + B(\frac{1-\sqrt{5}}{2})$
$1 = B(\frac{-2\sqrt{5}}{2})$
$B = \frac{-1}{\sqrt{5}}$
$A = -B = \frac{1}{\sqrt{5}}$ so, substituting calculated values of A and B we get a formula:
$F_n = (\frac{1}{\sqrt{5}})((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n) = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$

This way I have derived the equation for the $n^{th}$ term of the Fibonacci sequence known as Binet's formula.

Binet's formula can be also derived in an alternative way with the use of a golden ratio $\phi$ and $\psi$. $\phi$ and $\psi$ are the solutions of the quadratic $x^2 - x - 1 = 0$. By rearranging it we get:

$x^2 = x + 1$ Now multiply both sides by $x$
$x^3 = x^2 + x$, and by substituting the first equation we get
$x^3 = 2x + 1$

I substitute the value $\phi$ and repeat the process up to the $10^{th}$ power, trying to observe any realtion between the following experssions.

$\phi^4 = 2\phi^2 + \phi = 3\phi + 2$
$\phi^5 = 3\phi^2 + 2\phi = 5\phi + 3$
$\phi^6 = 5\phi^2 + 3\phi = 8\phi + 5$
$\phi^7 = 8\phi^2 + 5\phi = 13\phi + 8$
$\phi^8 = 13\phi^2 + 8\phi = 21\phi + 13$
$\phi^9 = 21\phi^2 + 13\phi = 34\phi + 21$
$\phi^{10} = 34\phi^2 + 21\phi = 55\phi + 34$

It can be observed that the coefficients are the following Fibbonaci numbers, so it can be supposed that

$\phi^n = F_n\phi + F_{n-1}$, and as the situation looks the analogical for $\psi$, then

$$\psi^n = F_n\psi + F_{n-1}.$$

This relation can be proved by induction.
$\phi^n = F_n\phi + F_{n-1}\ n \in \mathbb{Z}^+$

Step 1. $n = 1$
$\phi^1 = F_1\phi + F_{n-1} = \phi + 0 = \phi$

Step 2. $n = k$
Assume $\phi^k = F_k\phi + F_{k-1}$

Step 3. $n = k + 1$
Need to show $\phi^{k+1} = F_{k+1}\phi + F_k$

$LHS = \phi^k\phi = (F_k\phi + F_{k-1})\phi = F_k\phi^2 + F_{k-1}\phi = F_k(\phi + 1) + F_{k-1}\phi = F_k\phi + F_k + F_{k-1}\phi = \phi(F_k + F_{k-1}) + F_k = \phi F_{k+1} + F_k = RHS$

As $\phi^n = F_n\phi + F_{n-1}$ is true for $n = 1$, and if true for $n = k$, then true for $n = k+1$, then by the principle of mathematical induction it is true for all $n \in \mathbb{N}$. The same proof is valid for $\psi$, so there is no need to repeat it.

To derive Binet's formula I substract $\psi^n$ from $\phi^n$:

$$\phi^n - \psi^n = F_n(\phi - \psi)$$
$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi}$$

Then I substitute the values of $\phi$ and $\psi$ calculated in the previous part:

$$F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}$$
$$F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$$

In this section I have focused on the basic analysis and manipulation of the sequence. I have started with the definitions, as they are crucial for understanding of my further methods. After finding an exact value of $\Phi$, I have derived an explicit formula of the sequence through solving reccurence relation. In the next step I have derived it in another way, this time with the use of a golden ratio and the second solution a a quadratic, $\psi$, so to present a method that is beyond the curriculum. In the following section I will concentrate on proving chosen characterisitcs of the Fibonacci sequence with the use of various tools.

## 2.4 Properties of the Fibbonaci sequence

In this section I will prove some of the features of the sequence with the use of different methods.

**1. Manipulations of series**

Equation for the sum of $n$ Fibonacci numbers, can be derived after simple manipulations involving the relations between the consecutive terms of the sequence.

Theorem: $\sum_{i=1}^{n} F_n = F_{n+2} - 1$

I write the terms that I want to add as differences of the two following Fibonacci numbers, from $F_n$ to $F_1$. $F_0 = 0$, so it does not count.

$F_n = F_{n+2} - F_{n+1}$
$F_{n-1} = F_{n+1} - F_n$
$F_{n-2} = F_n - F_{n-1}$
$F_{n-3} = F_{n-1} - F_{n-2}$
$F_{n-4} = F_{n-2} - F_{n-3}$
...

$F_2 = F_4 - F_3$
$F_1 = F_3 - F_2$

And now I am summing the terms. Notice that most of the elements repeats with negative sign, thus they are cancelled. The terms that remain, as they "have nothing to be cancelled with", are $F_{n+2}$ and $F_2(=1)$. Thus

$\sum_{i=1}^{n} F_n = F_{n+2} - 1$

**2. Geometric approach**

The equation for the sum of squares of the Fibbonaci numbers is the product of two consecutive terms, and this can be showed with the geometric representation.

Theorem: $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$

Proof:
I take $n$ squares of the lenghts of sides equal to the consecutive Fibbonaci numbers. Those would be $1 \times 1$, $1 \times 1$, $2 \times 2$, $3 \times 3$, $5 \times 5$, ... $F_n \times F_n$. Then I place them one next to another creating rectangles. On each stage of addition, the dimensions of the rectangles created are the lenght of the newly added square $\times$ (the previous square + newly added square), what means literally $F_n \times (F_n + F_{n-1})$. Since $F_n + F_{n-1} = F_{n+1}$, so the sum of squares of $n$ Fibbonaci numbers is equal to $F_n F_{n+1}$.[5]

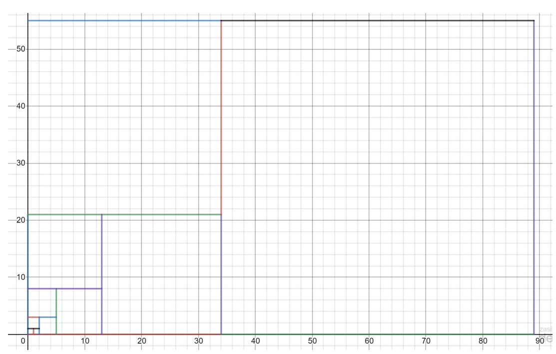


Figure 2: Rectangle of the side lenghts $F_{10} \times F_{11}$ [6]

In this case there are 10 sqares, and the dimensions of the rectangle are $55 \times 89$, so in other words $F_{10} \times F_{11}$. The area of this rectangle, is the same as $\sum_{i=1}^{10} F_i^2 = F_{10} F_{11}$,

[5]Honsberger, R.(1985) Mathematical Gems III.
[6]Sketched in Desmos, Graphing Calculator

and equal to 4895.

### 3. Proof with the use of matrix algebra

Cassini's identity is a property of the sequence that can be easily proved by induction with the use of matrices. It states that

$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ for $n \geq 1$

Matrix is a tool used to solve linear equations, and is represented in a form similar to a table in brackets, where its elements are arranged in rows and columns. For the Fibonacci sequence, the matrix $Q$ is of the size $2 \times 2$, and is defined by

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix}$$ [7]

Determinant of a matrix is the value that can be derived from the elements of a matrix, and is denoted by $\det Q$. For small matrices of the size $2 \times 2$, its value is calculated the following way:

$$\det \begin{pmatrix} x & y \\ z & v \end{pmatrix} = xv - yz$$ [8]

In my case, $\det(Q) = 0 - 1 = -1$ The use of matrices makes some things easier to be observed, than with the use of another methods, and that is why I am going to use it to prove Cassini's identity with its use.

I will try to prove by induction that for $n \geq 1$, $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$

Step 1.

$$Q^1 = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

For $n = 1$ I did not have to do any additional calulations. As I want to present how matrices $2 \times 2$ are multiplied, I will find the matrix $Q^2$ as well.

$$Q^2 = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} \times \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Multiplication of matrices looks similar to a dot product I know form vectors. Firstyl

[7] Wolfram Math World. Fibonacci Q-Matrix

[8] Wikipedia, Free Encyclopedia. Determinant.

I mulitply the first number of the first row by the first of the first column of the second matrix, then the second of the first row by the second of the first column, etc. In this case it looks the following:

$$\begin{pmatrix} (1\times1+1\times1) & (0\times1+1\times0) \\ (1\times1+0\times1) & (1\times1+0\times0) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \checkmark$$

Step 2. Assume true for $n = k$.

$$Q^k = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}$$

Step 3. Show it it true for $n = k + 1$

$$Q^{k+1} = Q^k \times Q = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} (F_{k+1}\times1+\mathrm{F}_k\times1) & (F_{k-1}\times1+\mathrm{F}_k\times0) \\ (F_k\times1+\mathrm{F}_{k-1}\times1) & (F_k\times1+\mathrm{F}_{k+1}\times0) \end{pmatrix} = \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix}$$

As $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ is true for $n = 1$, and assuming it is true for $n = k$, then true for $n = k + 1$, therefore by the principle of mathematical induction, $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$ is true for all $n \geq 1$

Now let's calculate the determinant of $Q$.
$\det(Q^n) = F_{n-1}F_{n+1} - F_nF_n = F_{n-1}F_{n+1} - F_n^2$, what is a left hand side of the Cassini's identity. Now let's calculate the right hand side. To do it, it is important to mention one more property of determinant, that is $\det(AB) = \det(A)\det(B)$. With this knowledge,

$\det(A^n) = \det(A \times A^{n-1}) = \det(A) \times \det(A^{n-1}) = \det(A)^n$
so in my case $\det(Q^n) = \det(Q)^n$, and knowing that $\det(Q) = -1$,
$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$

Cassini's identiy is useful in proving the next property of Fibonacci numbers, that is, the consecutive pairs of terms are relatively prime.

## 4. Linear diophantine equations

All consecutive Fibonacci numbers are coprime, what means that their greatest common divisor is equal to 1. According to the number theory, that I have studied in Discrete Mathematics option, if $d = GCD(x, y)$, then there exists a pair of integers

$a, b$ such that $d = ax+by$ [9]. In the case of Fibonacci numbers, if $d = GCD(F_{n+1}, F_n)$, then $d = aF_{n+1} + bF_n$. Here is where I will use Cassini's identity.

$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$. Let's distinguish the two cases: if $n$ is even and if $n$ is odd.

a) $n$ is even. Then $F_{n-1}F_{n+1} - F_nF_n = 1$
b) $n$ is odd. Then $-F_{n-1}F_{n+1} + F_nF_n = 1$

If $F_{n+1}$ is my $x$, and $F_n$ is my $y$, then $\pm F_{n-1}$ becomes $a$ and $\pm F_n$ becomes $b$ (Fibonacci numbers are integers, so it is true), and the conclusion is that $d$ is found to be equal to 1.

# 3 Conclusion

In my investigation I aimed at presenting to a reader properties of a Fibonacci sequence that make it unique and interesting. I have started with a golden ratio, $\Phi$, being commonly found in the surronding world, e.g. in flowers or shells, but also architecture, as it was the main factor that grabbed my attention and enccouraged to explore this sequence.

Then I have focused on the sequence itself. Firstly I have derived its formula in the two ways, one by solving reccurence relation and second alternative, that I consider to be more difficult, because it required observing the relation between the terms, but also more attractive and less popular. Solving reccurence relation is rather a basic method, included in the curriculum of option Discrete Mathematics, that I have studied at school, so that is why I have decided to include also something more.

The next part was focused on the features of the sequence that I found worth mentioning. The chosen topic is very broad and reaches many areas of mathematics, what allowed me to choose properties that I could prove using varied methods, such as by combinatoric manipulations, geometric approach, proof with the use of matrices and linear diophantine equations, but additionally to present that one property can be used to prove another, as I did with Cassini's identity and proving that GCD of pairs of terms is 1. I have tried not to repeat similar methods of proving, but I am aware that most of the propeties could have been proved by mathematical induction. Nonetheless, I decided to use the ones that I did not study at school, so

[9]Quinn, C., et al..(2014) Mathematics for the Intenational Student. Mathematics HL (Option): Discrete Mathematics

to broaden my knowledge and interest the reader.

My interest with architecture was the main factor that made me choose this topic, as I wanted to explore the sequence adored by my favourite architect, Le Corbusier, who has even created a new metric system, The Modulor, based on Fibonacci[10]. I hoped I will be able to answer the question what is so unique about the sequence and why did he defined golden ratio as the most beautiful proportion, but now I see that without investigating other sequences and comparing their properties, but focusing only on the one I will not be able to do it, thus my methods should be changed. What was very beneficial for me is that I became familiar with new mathematical tools, like matrices and continued fractions, but also deepened my understaning of basic methods, such as solving recurrences and proving by induction, but also I have learnt how to sketch in a graphing calculator, Desmos. As the properties I have described are only a small percentage of all hidden in the sequence, I think that the topic remains open and worth of further research.

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