SECTION A

1. (a) the total area under the graph of the pdf is unity ***(A1)***

area $= c\int_0^1 x - x^2 \, dx$

$= c\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1$ ***A1***

$= \frac{c}{6}$

$\Rightarrow c = 6$ ***A1***

(b) $E(X) = 6\int_0^1 x^2 - x^3 \, dx$ ***(M1)***

$= 6\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2}$ ***A1***

Note: Allow an answer obtained by a symmetry argument.

[5 marks]

2. (a) attempt at completing the square ***(M1)***

$3x^2 - 6x + 5 = 3(x^2 - 2x) + 5 = 3\left[(x-1)^2 - 1\right] + 5$ ***(A1)***

$= 3(x-1)^2 + 2$ ***A1***

$(a = 3,\ b = -1,\ c = 2)$

(b) definition of suitable basic transformations:

T_1 = stretch in y direction scale factor 3 ***A1***

T_2 = translation $\begin{pmatrix}1\\0\end{pmatrix}$ ***A1***

T_3 = translation $\begin{pmatrix}0\\2\end{pmatrix}$ ***A1***

[6 marks]

3. (a) $2y+8x=4$ ***M1***

$-3x+2y=-7$ ***A1***

$2x+6-2x=6$

Note:	Award ***M1*** for attempt at components, ***A1*** for two correct equations. No penalty for not checking the third equation.

solving : $x=1,\ y=-2$ ***A1***

(b) $$\left|\boldsymbol{a}+2\boldsymbol{b}\right| = \left|\begin{pmatrix}-4\\-3\\2\end{pmatrix}+2\begin{pmatrix}4\\-2\\2\end{pmatrix}\right|$$

$$=\left|\begin{pmatrix}4\\-7\\6\end{pmatrix}\right|$$

$$\Rightarrow\left|\boldsymbol{a}+2\boldsymbol{b}\right|=\sqrt{4^2+(-7)^2+6^2}$$ ***(M1)***

$$=\sqrt{101}$$ ***A1***

[5 marks]

4. recognition of $X\sim \mathrm{B}\left(6,\frac{4}{7}\right)$ ***(M1)***

$$\mathrm{P}(X=3)=\binom{6}{3}\left(\frac{4}{7}\right)^3\left(\frac{3}{7}\right)^3\left(=20\times\frac{4^3\times3^3}{7^6}\right)$$ ***A1***

$$\mathrm{P}(X=2)=\binom{6}{2}\left(\frac{4}{7}\right)^2\left(\frac{3}{7}\right)^4\left(=15\times\frac{4^2\times3^4}{7^6}\right)$$ ***A1***

$$\frac{\mathrm{P}(X=3)}{\mathrm{P}(X=2)}=\frac{80}{45}\left(=\frac{16}{9}\right)$$ ***A1***

[4 marks]

5. (a) $\boldsymbol{BA} = \left(\begin{pmatrix} 1 & 3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}\right) = \begin{pmatrix} 18 & -14 \\ -4 & 4 \end{pmatrix}$ ***A2***

Note: Award ***A1*** for one error, ***A0*** for two or more errors.

(b) $\det(\boldsymbol{BA}) = (72 - 56) = 16$ ***(M1)A1***

(c) **EITHER**

$\boldsymbol{A}(\boldsymbol{A}^{-1}\boldsymbol{B} + 2\boldsymbol{A}^{-1})\boldsymbol{A} = \boldsymbol{BA} + 2\boldsymbol{A}$ ***(M1)(A1)***

$= \begin{pmatrix} 24 & -18 \\ 6 & -4 \end{pmatrix}$ ***A1***

OR

$\boldsymbol{A}^{-1} = -\frac{1}{2}\begin{pmatrix} -4 & 2 \\ -5 & 3 \end{pmatrix}$ ***(A1)***

an attempt to evaluate ***(M1)***

$\boldsymbol{A}^{-1}\boldsymbol{B} + 2\boldsymbol{A}^{-1} = -\frac{1}{2}\begin{pmatrix} 0 & -16 \\ 1 & -21 \end{pmatrix} - \begin{pmatrix} -4 & 2 \\ -5 & 3 \end{pmatrix}$

$= \begin{pmatrix} 4 & 6 \\ 4.5 & 7.5 \end{pmatrix}$

$\boldsymbol{A}(\boldsymbol{A}^{-1}\boldsymbol{B} + 2\boldsymbol{A}^{-1})\boldsymbol{A} = \begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}\begin{pmatrix} 4 & 6 \\ 4.5 & 7.5 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}$

$= \begin{pmatrix} 3 & 3 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix} = \begin{pmatrix} 24 & -18 \\ 6 & -4 \end{pmatrix}$ ***A1***

[7 marks]

6. $\sin\left(x+\frac{\pi}{3}\right)=\sin x\cos\left(\frac{\pi}{3}\right)+\cos x\sin\left(\frac{\pi}{3}\right)$ ***(M1)***

$\sin x\cos\left(\frac{\pi}{3}\right)+\cos x\sin\left(\frac{\pi}{3}\right)=2\sin x\sin\left(\frac{\pi}{3}\right)$

$\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x=2\times\frac{\sqrt{3}}{2}\sin x$ ***A1***

dividing by $\cos x$ and rearranging ***M1***

$\tan x=\frac{\sqrt{3}}{2\sqrt{3}-1}$ ***A1***

rationalizing the denominator ***M1***

$11\tan x=6+\sqrt{3}$ ***A1***

[6 marks]

7. (a) $\left(u=x^2-2x-1.5;\ \frac{\mathrm{d}u}{\mathrm{d}x}=2x-2\right)$

$\frac{\mathrm{d}f}{\mathrm{d}x}=\frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}=\mathrm{e}^u(2x-2)$ ***(M1)***

$=2(x-1)\mathrm{e}^{x^2-2x-1.5}$ ***A1***

(b) $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(x-1)\times 2(x-1)\mathrm{e}^{x^2-2x-1.5}-1\times\mathrm{e}^{x^2-2x-1.5}}{(x-1)^2}$ ***M1A1***

$=\frac{2x^2-4x+1}{(x-1)^2}\mathrm{e}^{x^2-2x-1.5}$ ***(A1)***

minimum occurs when $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ ***(M1)***

$x=1\pm\sqrt{\frac{1}{2}}$ $\left(\text{accept } x=\frac{4\pm\sqrt{8}}{4}\right)$ ***A1***

$a=1+\sqrt{\frac{1}{2}}$ $\left(\text{accept } a=\frac{4+\sqrt{8}}{4}\right)$ ***R1***

[8 marks]

8. **EITHER**

differentiating implicitly:

$$1\times e^{-y} - xe^{-y}\frac{dy}{dx} + e^{y}\frac{dy}{dx} = 1$$ ***M1A1***

at the point $(c, \ln c)$

$$\frac{1}{c} - c\times\frac{1}{c}\frac{dy}{dx} + c\frac{dy}{dx} = 1$$ ***M1***

$$\frac{dy}{dx} = \frac{1}{c} \quad (c \neq 1)$$ ***(A1)***

OR

reasonable attempt to make expression explicit ***(M1)***

$xe^{-y} + e^{y} = 1 + x$

$x + e^{2y} = e^{y}(1+x)$

$e^{2y} - e^{y}(1+x) + x = 0$

$(e^{y} - 1)(e^{y} - x) = 0$ ***(A1)***

$e^{y} = 1$, $e^{y} = x$

$y = 0$, $y = \ln x$ ***A1***

Note: Do not penalize if $y = 0$ not stated.

$$\frac{dy}{dx} = \frac{1}{x}$$

gradient of tangent $= \frac{1}{c}$ ***A1***

Note: If candidate starts with $y = \ln x$ with no justification, award ***(M0)(A0)A1A1.***

THEN

the equation of the normal is

$y - \ln c = -c(x - c)$ ***M1***

$x = 0$, $y = c^2 + 1$

$c^2 + 1 - \ln c = c^2$ ***(A1)***

$\ln c = 1$

$c = e$ ***A1***

[7 marks]

9. **EITHER**

attempt at integration by substitution ***(M1)***

using $u = t + 1$, $\mathrm{d}u = \mathrm{d}t$, the integral becomes

$\int_1^2 (u-1)\ln u \,\mathrm{d}u$ ***A1***

then using integration by parts ***M1***

$\int_1^2 (u-1)\ln u \,\mathrm{d}u = \left[\left(\frac{u^2}{2} - u\right)\ln u\right]_1^2 - \int_1^2 \left(\frac{u^2}{2} - u\right) \times \frac{1}{u}\mathrm{d}u$ ***A1***

$= -\left[\frac{u^2}{4} - u\right]_1^2$ ***(A1)***

$= \frac{1}{4}$ (accept 0.25) ***A1***

OR

attempt to integrate by parts ***(M1)***
correct choice of variables to integrate and differentiate ***M1***

$\int_0^1 t\ln(t+1)\,\mathrm{d}t = \left[\frac{t^2}{2}\ln(t+1)\right]_0^1 - \int_0^1 \frac{t^2}{2} \times \frac{1}{t+1}\mathrm{d}t$ ***A1***

$= \left[\frac{t^2}{2}\ln(t+1)\right]_0^1 - \frac{1}{2}\int_0^1 t - 1 + \frac{1}{t+1}\mathrm{d}t$ ***A1***

$= \left[\frac{t^2}{2}\ln(t+1)\right]_0^1 - \frac{1}{2}\left[\frac{t^2}{2} - t + \ln(t+1)\right]_0^1$ ***(A1)***

$= \frac{1}{4}$ (accept 0.25) ***A1***

[6 marks]

10. (a) **Note:** Interchange of variables may take place at any stage.

for the inverse, solve for *x* in

$y = \frac{2x-3}{x-1}$

$y(x-1) = 2x-3$ ***M1***

$yx - 2x = y - 3$

$x(y-2) = y-3$ ***(A1)***

$x = \frac{y-3}{y-2}$

$\Rightarrow f^{-1}(x) = \frac{x-3}{x-2} \quad (x \neq 2)$ ***A1***

Note: Do not award final ***A1*** unless written in the form $f^{-1}(x) = \ldots$

(b) $\pm f^{-1}(x) = 1 + f^{-1}(x)$ leads to

$2\frac{x-3}{x-2} = -1$ ***(M1)A1***

$x = \frac{8}{3}$ ***A1***

[6 marks]

SECTION B

11. (a) (i) $1\times 2+2\times 3+...+n(n+1)=\frac{1}{3}n(n+1)(n+2)$ ***R1***

(ii) $\text{LHS}=40;\ \text{RHS}=40$ ***A1***

[2 marks]

(b) the sequence of values are:
5, 7, 11, 19, 35 … or an example ***A1***
35 is not prime, so Bill's conjecture is false ***R1AG***

[2 marks]

(c) $\text{P}(n): 5\times 7^n+1$ is divisible by 6
$\text{P}(1):$ 36 is divisible by 6 $\Rightarrow \text{P}(1)$ true ***A1***
assume $\text{P}(k)$ is true $(5\times 7^k+1=6r)$ ***M1***

Note:	Do **not** award ***M1*** for statement starting 'let $n=k$'. Subsequent marks are independent of this ***M1***.

consider $5\times 7^{k+1}+1$ ***M1***
$=7(6r-1)+1$ ***(A1)***
$=6(7r-1)\Rightarrow \text{P}(k+1)$ is true ***A1***
$\text{P}(1)$ true and $\text{P}(k)$ true $\Rightarrow \text{P}(k+1)$ true, so by MI $\text{P}(n)$ is true for all $n\in\mathbb{Z}^+$ ***R1***

Note:	Only award ***R1*** if there is consideration of $\text{P}(1)$, $\text{P}(k)$ and $\text{P}(k+1)$ in the final statement. Only award ***R1*** if at least one of the two preceding ***A*** marks has been awarded.

[6 marks]

Total [10 marks]

12. (a) (i) use of $\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ *(M1)*

$\boldsymbol{a} \bullet \boldsymbol{b} = -1$ *(A1)*

$|\boldsymbol{a}| = 7, |\boldsymbol{b}| = 5$ *(A1)*

$\cos\theta = -\frac{1}{35}$ *A1*

(ii) the required cross product is

$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 6 & 3 & 2 \\ 0 & -3 & 4 \end{vmatrix} = 18\boldsymbol{i} - 24\boldsymbol{j} - 18\boldsymbol{k}$ *M1A1*

(iii) using $\boldsymbol{r} \bullet \boldsymbol{n} = \boldsymbol{p} \bullet \boldsymbol{n}$ the equation of the plane is *(M1)*

$18x - 24y - 18z = 12 \quad (3x - 4y - 3z = 2)$ *A1*

(iv) recognizing that $z = 0$ *(M1)*

x-intercept $= \frac{2}{3}$, y-intercept $= -\frac{1}{2}$ *(A1)*

area $= \left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{6}$ *A1*

[11 marks]

(b) (i) $\boldsymbol{p} \bullet \boldsymbol{p} = |\boldsymbol{p}||\boldsymbol{p}|\cos 0$ *M1A1*

$= |\boldsymbol{p}|^2$ *AG*

(ii) consider the LHS, and use of result from part (i)

$|\boldsymbol{p} + \boldsymbol{q}|^2 = (\boldsymbol{p} + \boldsymbol{q}) \bullet (\boldsymbol{p} + \boldsymbol{q})$ *M1*

$= \boldsymbol{p} \bullet \boldsymbol{p} + \boldsymbol{p} \bullet \boldsymbol{q} + \boldsymbol{q} \bullet \boldsymbol{p} + \boldsymbol{q} \bullet \boldsymbol{q}$ *(A1)*

$= \boldsymbol{p} \bullet \boldsymbol{p} + 2\boldsymbol{p} \bullet \boldsymbol{q} + \boldsymbol{q} \bullet \boldsymbol{q}$ *A1*

$= |\boldsymbol{p}|^2 + 2\boldsymbol{p} \bullet \boldsymbol{q} + |\boldsymbol{q}|^2$ *AG*

continued …

Question 12 continued

(iii) **EITHER**

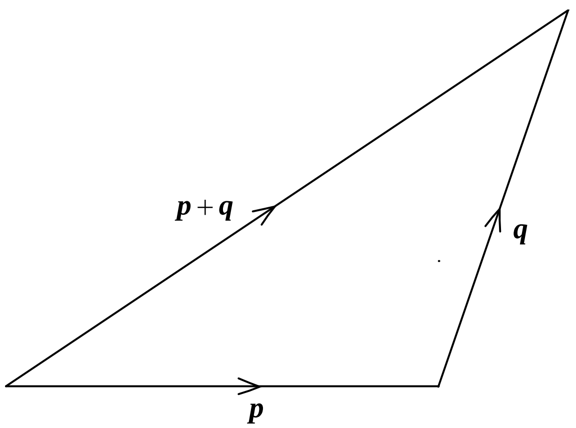
use of $\boldsymbol{p}\bullet\boldsymbol{q} \le |\boldsymbol{p}||\boldsymbol{q}|$ ***M1***

so $0 \le |\boldsymbol{p}+\boldsymbol{q}|^2 = |\boldsymbol{p}|^2 + 2\boldsymbol{p}\bullet\boldsymbol{q} + |\boldsymbol{q}|^2 \le |\boldsymbol{p}|^2 + 2|\boldsymbol{p}||\boldsymbol{q}| + |\boldsymbol{q}|^2$ ***A1***

take square root (of these positive quantities) to establish ***A1***

$|\boldsymbol{p}+\boldsymbol{q}| \le |\boldsymbol{p}| + |\boldsymbol{q}|$ ***AG***

OR



M1M1

Note:	Award ***M1*** for correct diagram and ***M1*** for correct labelling of vectors including arrows.

since the sum of any two sides of a triangle is greater than the third side,

$|\boldsymbol{p}| + |\boldsymbol{q}| > |\boldsymbol{p}+\boldsymbol{q}|$ ***A1***

when $\boldsymbol{p}$ and $\boldsymbol{q}$ are collinear $|\boldsymbol{p}| + |\boldsymbol{q}| = |\boldsymbol{p}+\boldsymbol{q}|$

$\Rightarrow |\boldsymbol{p}+\boldsymbol{q}| \le |\boldsymbol{p}| + |\boldsymbol{q}|$ ***AG***

[8 marks]

Total [19 marks]

13. (a) (i) $\omega^3 = \left(\cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right)^3$

$= \cos\left(3\times\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(3\times\frac{2\pi}{3}\right)$ ***(M1)***

$= \cos 2\pi + \mathrm{i}\sin 2\pi$ ***A1***

$= 1$ ***AG***

(ii) $1 + \omega + \omega^2 = 1 + \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right) + \cos\left(\frac{4\pi}{3}\right) + \mathrm{i}\sin\left(\frac{4\pi}{3}\right)$ ***M1A1***

$= 1 + -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2} - \frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}$ ***A1***

$= 0$ ***AG***

[5 marks]

(b) (i) $\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\left(\theta+\frac{2\pi}{3}\right)} + \mathrm{e}^{\mathrm{i}\left(\theta+\frac{4\pi}{3}\right)}$

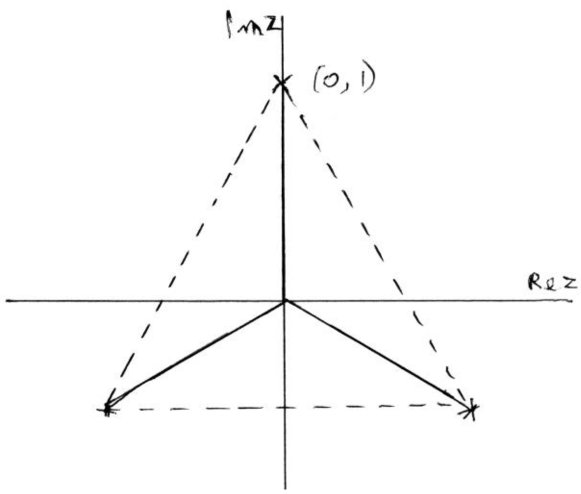
$= \mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)} + \mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{3}\right)}$ ***(M1)***

$= \left(\mathrm{e}^{\mathrm{i}\theta}\left(1 + \mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)} + \mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{3}\right)}\right)\right)$

$= \mathrm{e}^{\mathrm{i}\theta}(1 + \omega + \omega^2)$ ***A1***

$= 0$ ***AG***

(ii)



A1A1

Note:	Award ***A1*** for one point on the imaginary axis and another point marked with approximately correct modulus and argument. Award ***A1*** for third point marked to form an equilateral triangle centred on the origin.

[4 marks]

continued …

Question 13 continued

(c) (i) attempt at the expansion of at least two linear factors ***(M1)***

$(z-1)\ \ z^2 - z(\omega+\omega^2)+\omega^3$ or equivalent ***(A1)***

use of earlier result ***(M1)***

$F(z)=(z-1)(z^2+z+1)=z^3-1$ ***A1***

(ii) equation to solve is $z^3=8$ ***(M1)***

$z=2,\ 2\omega,\ 2\omega^2$ ***A2***

Note: Award ***A1*** for 2 correct solutions.

[7 marks]

Total [16 marks]

14. (a) the differential equation is separable and can be written as ***(M1)***

$\int -y^{-2}\,\mathrm{d}y=\int \cos^2 x\,\mathrm{d}x$ (or equivalent) ***A1***

$=\int \frac{1+\cos 2x}{2}\mathrm{d}x$ ***A1***

$\frac{1}{y}=\frac{1}{2}x+\frac{1}{4}\sin 2x(+C)$ ***A1A1***

when $x=0,\ y=1$ ***M1***

$C=1$

$y=\frac{1}{\frac{1}{2}x+\frac{1}{4}\sin 2x+1}$ ***A1***

[7 marks]

(b) (i) recognizing use of $(1+\tan x)^2$ ***(M1)***

$(1+\tan x)^2=1+2\tan x+\tan^2 x\ge 1+\tan^2 x=\sec^2 x$ ***A1***

(since all terms are positive)

$(1+\tan x)^2\ge \sec^2 x$

$\sec^2 x=1+\tan^2 x\ge 1$ ***A1***

$\Rightarrow (1+\tan x)^2\ge \sec^2 x\ge 1$

since all terms are positive, taking square root gives ***R1***

$1\le \sec x\le 1+\tan x$ ***AG***

(ii) $\int_0^{\frac{\pi}{4}}\mathrm{d}x\le \int_0^{\frac{\pi}{4}}\sec x\,\mathrm{d}x\le \int_0^{\frac{\pi}{4}}1+\tan x\,\mathrm{d}x$ ***M1***

$x\ {}_0^{\frac{\pi}{4}}\le \int_0^{\frac{\pi}{4}}\sec x\,\mathrm{d}x\le\ x-\ln\cos x\ {}_0^{\frac{\pi}{4}}$ ***M1A1***

$\frac{\pi}{4}\le \int_0^{\frac{\pi}{4}}\sec x\,\mathrm{d}x\le \frac{\pi}{4}-\ln\frac{1}{\sqrt{2}}$ ***A1***

$\frac{\pi}{4}\le \int_0^{\frac{\pi}{4}}\sec x\,\mathrm{d}x\le \frac{\pi}{4}+\frac{1}{2}\ln 2$ ***AG***

[8 marks]

Total [15 marks]