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MARKSCHEME

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MATHEMATICS

Higher Level

Paper 1

18 pages

SECTION A

1. (a) (i) $P(A \cup B) = P(A) + P(B) = 0.7$ ***A1***

(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ***(M1)***

$= P(A) + P(B) - P(A)P(B)$ ***(M1)***

$= 0.3 + 0.4 - 0.12 = 0.58$ ***A1***

(b) $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$= 0.3 + 0.4 - 0.6 = 0.1$ ***A1***

$P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ ***(M1)***

$= \dfrac{0.1}{0.4} = 0.25$ ***A1***

[7 marks]

2. **METHOD 1**

$z = (2 - \mathrm{i})(z + 2)$ ***M1***

$= 2z + 4 - \mathrm{i}z - 2\mathrm{i}$

$z(1 - \mathrm{i}) = -4 + 2\mathrm{i}$

$z = \dfrac{-4 + 2\mathrm{i}}{1 - \mathrm{i}}$ ***A1***

$z = \dfrac{-4 + 2\mathrm{i}}{1 - \mathrm{i}} \times \dfrac{1 + \mathrm{i}}{1 + \mathrm{i}}$ ***M1***

$= -3 - \mathrm{i}$ ***A1***

METHOD 2

let $z = a + \mathrm{i}b$

$\dfrac{a + \mathrm{i}b}{a + ib + 2} = 2 - \mathrm{i}$ ***M1***

$a + \mathrm{i}b = (2 - \mathrm{i})\big((a + 2) + \mathrm{i}b\big)$

$a + \mathrm{i}b = 2(a + 2) + 2b\mathrm{i} - \mathrm{i}(a + 2) + b$

$a + \mathrm{i}b = 2a + b + 4 + (2b - a - 2)\mathrm{i}$

attempt to equate real and imaginary parts ***M1***

$a = 2a + b + 4 (\Rightarrow a + b + 4 = 0)$

and $b = 2b - a - 2 (\Rightarrow -a + b - 2 = 0)$ ***A1***

Note: Award **A**1 for two correct equations.

$b = -1;\ a = -3$ ***A1***

$z = -3 - \mathrm{i}$

[4 marks]

3. (a) $u_1 = 27$

$\frac{81}{2} = \frac{27}{1-r}$ ***M1***

$r = \frac{1}{3}$ ***A1***

(b) $v_2 = 9$

$v_4 = 1$

$2d = -8 \Rightarrow d = -4$ ***(A1)***

$v_1 = 13$ ***(A1)***

$\frac{N}{2}\left(2\times 13 - 4(N-1)\right) > 0$ (accept equality) ***M1***

$\frac{N}{2}(30-4N) > 0$

$N(15-2N) > 0$

$N < 7.5$ ***(M1)***

$N = 7$ ***A1***

Note: $13+9+5+1-3-7-11 > 0 \Rightarrow N = 7$ or equivalent receives full marks.

[7 marks]

4. (a) $\vec{AB} = \boldsymbol{b} - \boldsymbol{a}$ ***A1***

$\vec{CB} = \boldsymbol{a} + \boldsymbol{b}$ ***A1***

(b) $\vec{AB}\bullet\vec{CB} = (\boldsymbol{b}-\boldsymbol{a})\bullet(\boldsymbol{b}+\boldsymbol{a})$ ***M1***

$= |\boldsymbol{b}|^2 - |\boldsymbol{a}|^2$ ***A1***

$= 0$ since $|\boldsymbol{b}| = |\boldsymbol{a}|$ ***R1***

Note: Only award the ***A1*** and ***R1*** if working indicates that they understand that they are working with vectors.

so $\vec{AB}$ is perpendicular to $\vec{CB}$ *i.e.* $A\hat{B}C$ is a right angle ***AG***

[5 marks]

5. (a) $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \dfrac{2\sin\theta\cos\theta}{1+2\cos^2\theta - 1}$ ***M1***

Note: Award ***M1*** for use of double angle formulae.

$= \dfrac{2\sin\theta\cos\theta}{2\cos^2\theta}$ ***A1***

$= \dfrac{\sin\theta}{\cos\theta}$

$= \tan\theta$ ***AG***

(b) $\tan\dfrac{\pi}{8} = \dfrac{\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}$ ***(M1)***

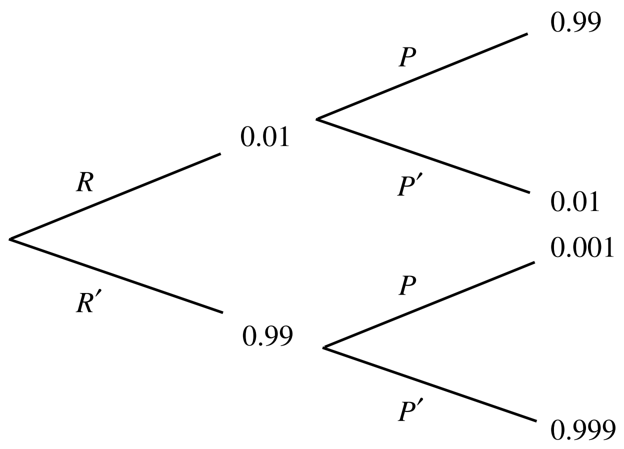
$\cot\dfrac{\pi}{8} = \dfrac{1+\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}}$ ***M1***

$= \dfrac{1+\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$

$= 1+\sqrt{2}$ ***A1***

[5 marks]

6. R is 'rabbit with the disease'
P is 'rabbit testing positive for the disease'



(a) $\mathrm{P}(P) = \mathrm{P}(R \cap P) + \mathrm{P}(R' \cap P)$

$= 0.01 \times 0.99 + 0.99 \times 0.001$ ***M1***

$= 0.01089 (= 0.0109)$ ***A1***

Note: Award ***M1*** for a correct tree diagram with correct probability values shown.

(b) $\mathrm{P}(R' \mid P) = \dfrac{0.001 \times 0.99}{0.001 \times 0.99 + 0.01 \times 0.99} \left(= \dfrac{0.00099}{0.01089} \right)$ ***M1 A1***

$\dfrac{0.00099}{0.01089} < \dfrac{0.001}{0.01} = 10\ \%$ (or other valid argument) ***R1***

[5 marks]

7. **METHOD 1**

area $= \int_0^{\sqrt{3}} \arctan x \, dx$ ***A1***

attempting to integrate by parts ***M1***

$= \left[x \arctan x\right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} x \frac{1}{1+x^2} dx$ ***A1A1***

$= \left[x \arctan x\right]_0^{\sqrt{3}} - \left[\frac{1}{2}\ln(1+x^2)\right]_0^{\sqrt{3}}$ ***A1***

Note: Award ***A1*** even if limits are absent.

$= \frac{\pi}{\sqrt{3}} - \frac{1}{2}\ln 4$ ***A1***

$\left(= \frac{\pi\sqrt{3}}{3} - \ln 2\right)$

METHOD 2

area $= \frac{\pi\sqrt{3}}{3} - \int_0^{\frac{\pi}{3}} \tan y \, dy$ ***M1A1A1***

$= \frac{\pi\sqrt{3}}{3} + [\ln|\cos y|]_0^{\frac{\pi}{3}}$ ***M1A1***

$= \frac{\pi\sqrt{3}}{3} + \ln\frac{1}{2} \quad \left(= \frac{\pi\sqrt{3}}{3} - \ln 2\right)$ ***A1***

[6 marks]

8. (a) (i) $(g \circ f)(x) = \frac{1}{2x+3},\ x \neq -\frac{3}{2}$ (or equivalent) ***A1***

(ii) $(f \circ g)(x) = \frac{2}{x} + 3,\ x \neq 0$ (or equivalent) ***A1***

(b) **EITHER**

$f(x) = (g^{-1} \circ f \circ g)(x) \Rightarrow (g \circ f)(x) = (f \circ g)(x)$ ***(M1)***

$\frac{1}{2x+3} = \frac{2}{x} + 3$ ***A1***

OR

$(g^{-1} \circ f \circ g)(x) = \frac{1}{\frac{2}{x}+3}$ ***A1***

$2x+3 = \frac{1}{\frac{2}{x}+3}$ ***M1***

THEN

$6x^2 + 12x + 6 = 0$ (or equivalent) ***A1***

$x = -1,\ y = 1$ (coordinates are $(-1, 1)$) ***A1***

[6 marks]

9. attempt at implicit differentiation ***M1***

$e^{(x+y)}\left(1 + \frac{dy}{dx}\right) = -\sin(xy)\left(x\frac{dy}{dx} + y\right)$ ***A1A1***

let $x = 0,\ y = 0$ ***M1***

$e^0\left(1 + \frac{dy}{dx}\right) = 0$

$\frac{dy}{dx} = -1$ ***A1***

let $x = \sqrt{2\pi},\ y = -\sqrt{2\pi}$

$e^0\left(1 + \frac{dy}{dx}\right) = -\sin(-2\pi)\left(x\frac{dy}{dx} + y\right) = 0$

so $\frac{dy}{dx} = -1$ ***A1***

since both points lie on the line $y = -x$ this is a common tangent ***R1***

Note: $y = -x$ must be seen for the final ***R1***. It is not sufficient to note that the gradients are equal.

[7 marks]

10. (a) $f(x-a) \neq b$ ***(M1)***

$x \neq 0$ and $x \neq 2a$ (or equivalent) ***A1***

(b) vertical asymptotes $x=0$, $x=2a$ ***A1***

horizontal asymptote $y=0$ ***A1***

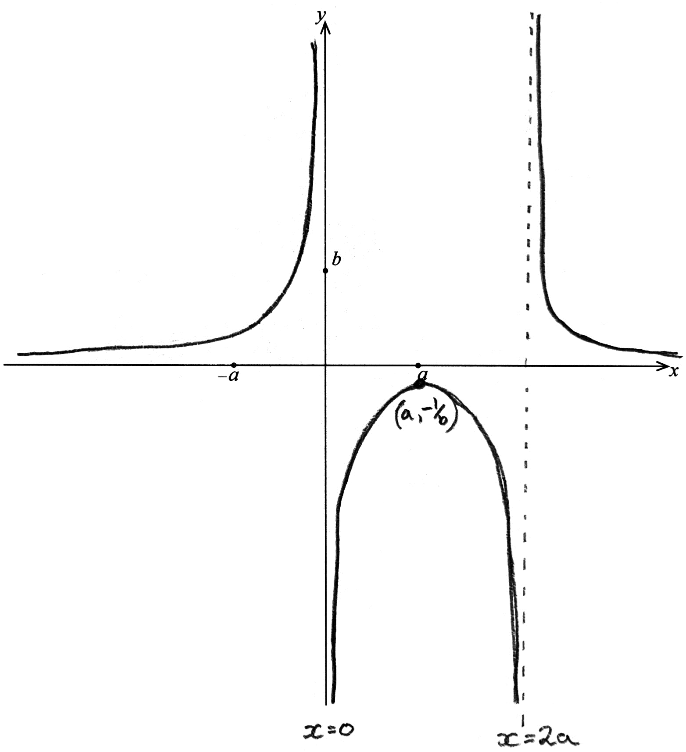
Note: Equations must be seen to award these marks.

maximum $\left(a, -\frac{1}{b}\right)$ ***A1A1***

Note: Award ***A1*** for correct x-coordinate and ***A1*** for correct y-coordinate.

one branch correct shape ***A1***

other 2 branches correct shape ***A1***



[8 marks]

SECTION B

11. (a) $\vec{AB} = \begin{pmatrix} -4 \\ -1 \\ 3 \end{pmatrix}$, $\vec{AC} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$ ***A1A1***

Note: Accept row vectors.

[2 marks]

(b) $\vec{AB} \times \vec{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -4 & -1 & 3 \\ 4 & -3 & 1 \end{vmatrix} = \begin{pmatrix} 8 \\ 16 \\ 16 \end{pmatrix}$ ***M1A1***

normal $\boldsymbol{n} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ so $\boldsymbol{r} \bullet \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ ***(M1)***

$x + 2y + 2z = 7$ ***A1***

Note: If attempt to solve by a system of equations: Award ***A1*** for 3 correct equations, ***A1*** for eliminating a variable and ***A2*** for the correct answer.

[4 marks]

(c) $\boldsymbol{r} = \begin{pmatrix} 5 \\ 3 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ (or equivalent) ***A1***

$1(5 + \lambda) + 2(3 + 2\lambda) + 2(7 + 2\lambda) = 7$ ***M1***

$9\lambda = -18$

$\lambda = -2$ ***A1***

Note: $\lambda = -\frac{1}{4}$ if $\begin{pmatrix} 8 \\ 16 \\ 16 \end{pmatrix}$ is used.

distance $= 2\sqrt{1^2 + 2^2 + 2^2}$ ***(M1)***

$= 6$ ***A1***

[5 marks]

continued …

Question 11 continued

(d) (i) area $= \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right| = \frac{1}{2}\sqrt{8^2 + 16^2 + 16^2}$ ***(M1)***

$= 12$ (accept $\frac{1}{2}\sqrt{576}$) ***A1***

(ii) **EITHER**

volume $= \frac{1}{3} \times \text{area} \times \text{height}$ ***(M1)***

$= \frac{1}{3} \times 12 \times 6 = 24$ ***A1***

OR

volume $= \frac{1}{6}\left(\vec{AD} \bullet (\vec{AB} \times \vec{AC})\right)$ ***M1***

$= 24$ ***A1***

[4 marks]

(e) $\left|\vec{AB} \times \vec{AC}\right| = \sqrt{8^2 + 16^2 + 16^2}$

$\left|\vec{AC} \times \vec{AD}\right| = \left\|\begin{matrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 4 & -3 & 1 \\ 4 & 1 & 6 \end{matrix}\right\|$ ***M1***

$= \left|-19\boldsymbol{i} - 20\boldsymbol{j} + 16\boldsymbol{k}\right|$ ***A1***

EITHER

$\frac{1}{2}\sqrt{19^2 + 20^2 + 16^2} > \frac{1}{2}\sqrt{8^2 + 16^2 + 16^2}$ ***M1***

therefore since area of ACD bigger than area ABC implies that B is closer to opposite face than D ***R1***

OR

correct calculation of second distance as $\frac{144}{\sqrt{19^2 + 20^2 + 16^2}}$ ***A1***

which is smaller than 6 ***R1***

Note: Only award final ***R1*** in each case if the calculations are correct.

[4 marks]

Total [19 marks]

12. (a) (i) $f'(x)=\dfrac{x\frac{1}{x}-\ln x}{x^2}$ ***M1A1***

$=\dfrac{1-\ln x}{x^2}$

so $f'(x)=0$ when $\ln x=1$, *i.e.* $x=\mathrm{e}$ ***A1***

(ii) $f'(x)>0$ when $x<\mathrm{e}$ and $f'(x)<0$ when $x>\mathrm{e}$ ***R1***

hence local maximum ***AG***

Note: Accept argument using correct second derivative.

(iii) $y\le\dfrac{1}{\mathrm{e}}$ ***A1***

[5 marks]

(b) $f''(x)=\dfrac{x^2\frac{-1}{x}-(1-\ln x)2x}{x^4}$ ***M1***

$=\dfrac{-x-2x+2x\ln x}{x^4}$

$=\dfrac{-3+2\ln x}{x^3}$ ***A1***

Note: May be seen in part (a).

$f''(x)=0$ ***(M1)***

$-3+2\ln x=0$

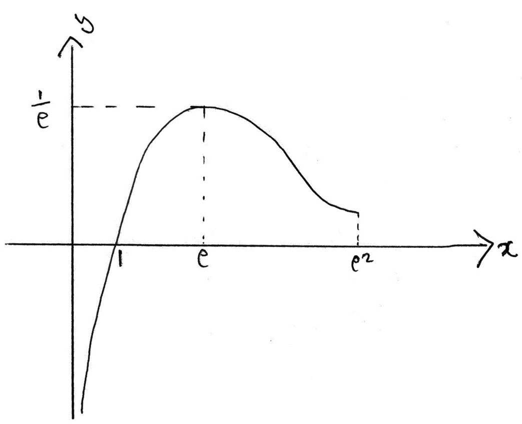
$x=\mathrm{e}^{\frac{3}{2}}$

since $f''(x)<0$ when $x<\mathrm{e}^{\frac{3}{2}}$ and $f''(x)>0$ when $x>\mathrm{e}^{\frac{3}{2}}$ ***R1***

then point of inflexion $\left(\mathrm{e}^{\frac{3}{2}},\dfrac{3}{2\mathrm{e}^{\frac{3}{2}}}\right)$ ***A1***

[5 marks]

(c)



A1A1A1

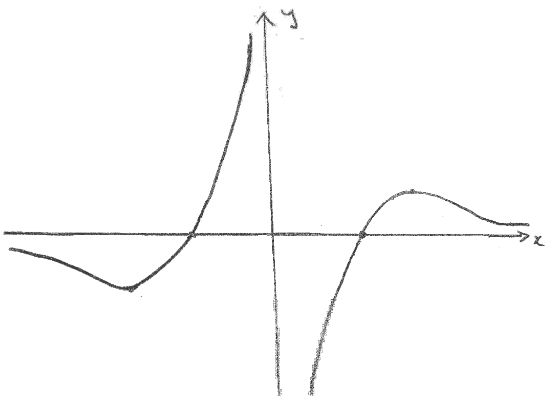
Note: Award ***A1*** for the maximum and intercept, ***A1*** for a vertical asymptote and ***A1*** for shape (including turning concave up).

[3 marks]

continued …

Question 12 continued

(d) (i)

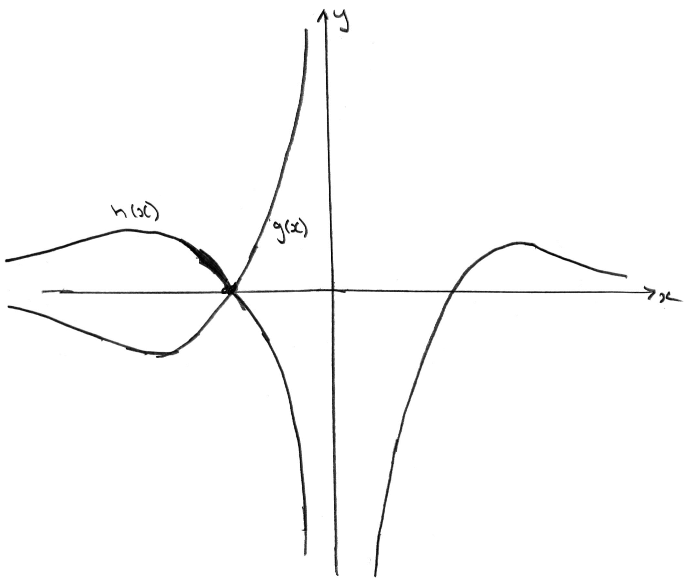


A1A1

Note: Award ***A1*** for each correct branch.

(ii) all real values ***A1***

(iii)



(M1)(A1)

Note: Award ***(M1)(A1)*** for sketching the graph of h, ignoring any graph of g.

$-e^2 < x < -1$ (accept $x < -1$) ***A1***

[6 marks]

Total [19 marks]

13. (a) $(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta(\mathrm{i}\sin\theta) + 3\cos\theta(\mathrm{i}\sin\theta)^2 + (\mathrm{i}\sin\theta)^3$ ***(M1)***

$= \cos^3\theta - 3\cos\theta\sin^2\theta + \mathrm{i}(3\cos^2\theta\sin\theta - \sin^3\theta)$ ***A1***

[2 marks]

(b) from De Moivre's theorem

$(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos 3\theta + \mathrm{i}\sin 3\theta$ ***(M1)***

$\cos 3\theta + \mathrm{i}\sin 3\theta = (\cos^3\theta - 3\cos\theta\sin^2\theta) + \mathrm{i}(3\cos^2\theta\sin\theta - \sin^3\theta)$

equating real parts ***M1***

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$ ***A1***

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$ ***AG***

Note: Do not award marks if part (a) is not used.

[3 marks]

(c) $(\cos\theta + \mathrm{i}\sin\theta)^5 =$

$\cos^5\theta + 5\cos^4\theta(\mathrm{i}\sin\theta) + 10\cos^3\theta(\mathrm{i}\sin\theta)^2 + 10\cos^2\theta(\mathrm{i}\sin\theta)^3 + 5\cos\theta(\mathrm{i}\sin\theta)^4 + (\mathrm{i}\sin\theta)^5$

(A1)

from De Moivre's theorem

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ ***M1***

$= \cos^5\theta - 10\cos^3\theta(1 - \cos^2\theta) + 5\cos\theta(1 - \cos^2\theta)^2$ ***A1***

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$\therefore \cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ ***AG***

Note: If compound angles used in (b) and (c), then marks can be allocated in (c) only.

[3 marks]

(d) $\cos 5\theta + \cos 3\theta + \cos\theta$

$= (16\cos^5\theta - 20\cos^3\theta + 5\cos\theta) + (4\cos^3\theta - 3\cos\theta) + \cos\theta = 0$ ***M1***

$16\cos^5\theta - 16\cos^3\theta + 3\cos\theta = 0$ ***A1***

$\cos\theta(16\cos^4\theta - 16\cos^2\theta + 3) = 0$

$\cos\theta(4\cos^2\theta - 3)(4\cos^2\theta - 1) = 0$ ***A1***

$\therefore \cos\theta = 0; \pm\frac{\sqrt{3}}{2}; \pm\frac{1}{2}$ ***A1***

$\therefore \theta = \pm\frac{\pi}{6}; \pm\frac{\pi}{3}; \pm\frac{\pi}{2}$ ***A2***

[6 marks]

continued...

Question 13 continued

(e) $\cos 5\theta = 0$

$5\theta = \ldots\frac{\pi}{2};\left(\frac{3\pi}{2};\frac{5\pi}{2}\right);\frac{7\pi}{2};\ldots$ ***(M1)***

$\theta = \ldots\frac{\pi}{10};\left(\frac{3\pi}{10};\frac{5\pi}{10}\right);\frac{7\pi}{10};\ldots$ ***(M1)***

Note: These marks can be awarded for verifications later in the question.

now consider $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$ ***M1***

$\cos\theta(16\cos^4\theta - 20\cos^2\theta + 5) = 0$

$\cos^2\theta = \frac{20 \pm \sqrt{400 - 4(16)(5)}}{32}$; $\cos\theta = 0$ ***A1***

$\cos\theta = \pm\sqrt{\frac{20 \pm \sqrt{400 - 4(16)(5)}}{32}}$

$\cos\frac{\pi}{10} = \sqrt{\frac{20 + \sqrt{400 - 4(16)(5)}}{32}}$ since max value of cosine $\Rightarrow$ angle closest to zero

R1

$\cos\frac{\pi}{10} = \sqrt{\frac{4.5 + 4\sqrt{25 - 4(5)}}{4.8}} = \sqrt{\frac{5 + \sqrt{5}}{8}}$ ***A1***

$\cos\frac{7\pi}{10} = -\sqrt{\frac{5 - \sqrt{5}}{8}}$ ***A1A1***

[8 marks]

Total [22 marks]