M11/5/MATHL/HP1/ENG/TZ2/XX/M

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MARKSCHEME

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MATHEMATICS

Higher Level

Paper 1

20 pages

SECTION A

1. (a) **METHOD 1**

$f'(x) = q - 2x = 0$ *M1*

$f'(3) = q - 6 = 0$

$q = 6$ *A1*

$f(3) = p + 18 - 9 = 5$ *M1*

$p = -4$ *A1*

METHOD 2

$f(x) = -(x-3)^2 + 5$ *M1A1*

$= -x^2 + 6x - 4$

$q = 6,\ p = -4$ *A1A1*

(b) $g(x) = -4 + 6(x-3) - (x-3)^2\ (= -31 + 12x - x^2)$ *M1A1*

Note: Accept any alternative form which is correct. Award ***M1A0*** for a substitution of $(x+3)$.

[6 marks]

2. (a) $A^2 = \begin{pmatrix} 2a & -2 \\ -a & 2a+1 \end{pmatrix}$ *(M1)A1*

(b) **METHOD 1**

$\det A^2 = 4a^2 + 2a - 2a = 4a^2$ *M1*

$a = \pm 2$ *A1A1* *N2*

METHOD 2

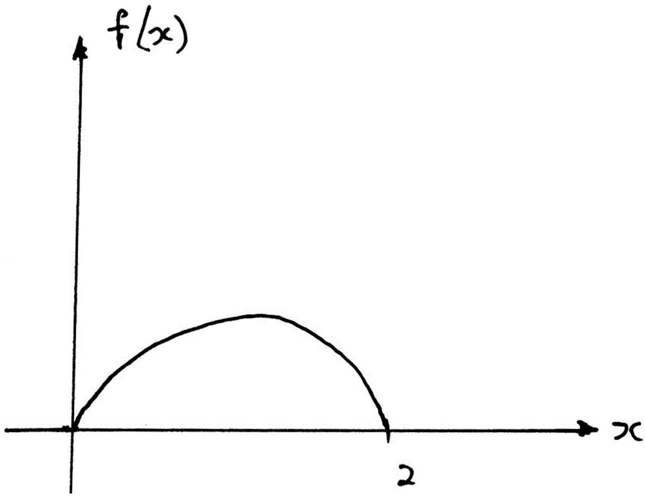
$\det A = -2a$ *M1*

$\det A = \pm 4$

$a = \pm 2$ *A1A1* *N2*

[5 marks]

3. (a)



A1

Note: Award ***A1*** for intercepts of 0 and 2 and a concave down curve in the given domain .

Note: Award ***A0*** if the cubic graph is extended outside the domain $[0, 2]$.

(b) $\int_0^2 kx(x+1)(2-x)\,\mathrm{d}x = 1$ ***(M1)***

Note: The correct limits and =1 must be seen but may be seen later.

$k\int_0^2 (-x^3 + x^2 + 2x)\,\mathrm{d}x = 1$ ***A1***

$k\left[-\frac{1}{4}x^4 + \frac{1}{3}x^3 + x^2\right]_0^2 = 1$ ***M1***

$k\left(-4 + \frac{8}{3} + 4\right) = 1$ ***(A1)***

$k = \frac{3}{8}$ ***A1***

[6 marks]

4. (a) $AB = \sqrt{1^2 + (2-\sqrt{3})^2}$ ***M1***

$= \sqrt{8 - 4\sqrt{3}}$ ***A1***

$= 2\sqrt{2-\sqrt{3}}$ ***A1***

(b) **METHOD 1**

$\arg z_1 = -\frac{\pi}{4}$ $\arg z_2 = -\frac{\pi}{3}$ ***A1A1***

Note: Allow $\frac{\pi}{4}$ and $\frac{\pi}{3}$.

Note: Allow degrees at this stage.

$A\hat{O}B = \frac{\pi}{3} - \frac{\pi}{4}$

$= \frac{\pi}{12}$ (accept $-\frac{\pi}{12}$) ***A1***

Note: Allow ***FT*** for final ***A1***.

METHOD 2

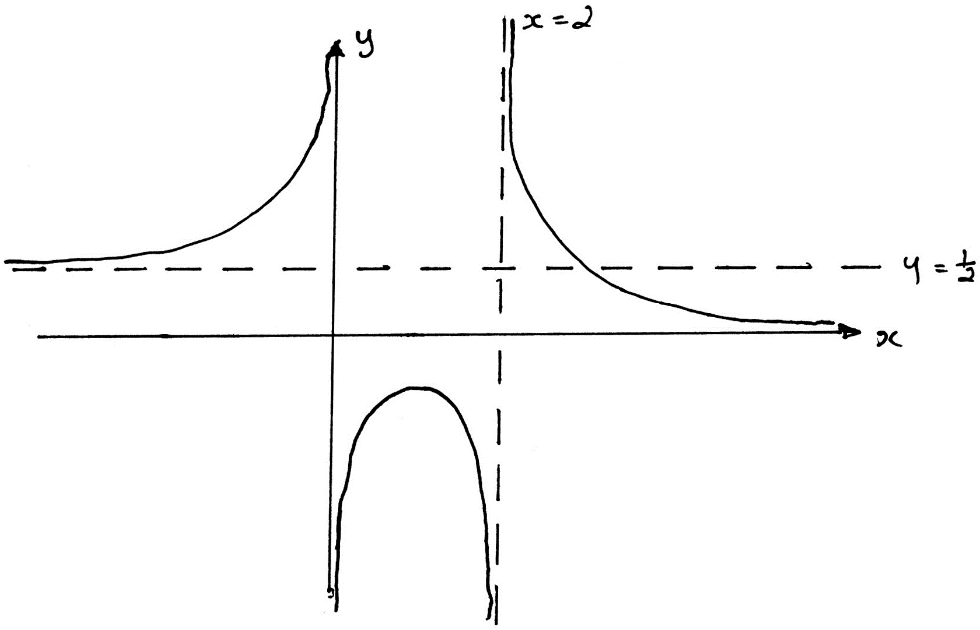
attempt to use scalar product or cosine rule ***M1***

$\cos A\hat{O}B = \frac{1+\sqrt{3}}{2\sqrt{2}}$ ***A1***

$A\hat{O}B = \frac{\pi}{12}$ ***A1***

[6 marks]

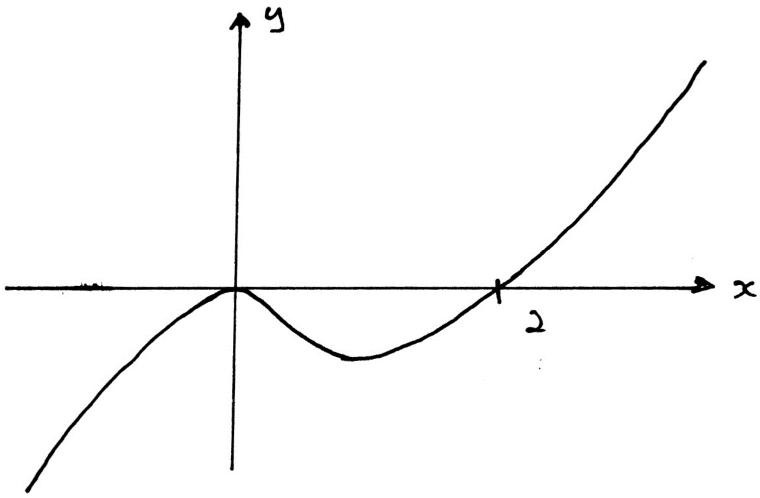
5. (a)



A3

Note:	Award ***A1*** for each correct branch with position of asymptotes clearly indicated. If $x = 2$ is not indicated, only penalise once.

(b)



A3

Note:	Award ***A1*** for behaviour at $x = 0$, ***A1*** for intercept at $x = 2$, ***A1*** for behaviour for large $\lvert x \rvert$.

[6 marks]

6. (a) $\overrightarrow{CB} = \boldsymbol{b} - \boldsymbol{c}$, $\overrightarrow{AC} = \boldsymbol{b} + \boldsymbol{c}$ ***A1A1***

Note: Condone absence of vector notation in (a).

(b) $\overrightarrow{AC} \bullet \overrightarrow{CB} = (\boldsymbol{b} + \boldsymbol{c}) \bullet (\boldsymbol{b} - \boldsymbol{c})$ ***M1***

$= |\boldsymbol{b}|^2 - |\mathbf{c}|^2$ ***A1***

$= 0$ since $|\boldsymbol{b}| = |\mathbf{c}|$ ***R1***

Note: Only award the ***A1*** and ***R1*** if working indicates that they understand that they are working with vectors.

so $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{CB}$ *i.e.* $A\hat{C}B$ is a right angle ***AG***

[5 marks]

7. (a) area of AOP $= \frac{1}{2} r^2 \sin\theta$ ***A1***

(b) $TP = r\tan\theta$ ***(M1)***

area of POT $= \frac{1}{2} r (r\tan\theta)$

$= \frac{1}{2} r^2 \tan\theta$ ***A1***

(c) area of sector OAP $= \frac{1}{2} r^2 \theta$ ***A1***

area of triangle OAP < area of sector OAP < area of triangle POT ***R1***

$\frac{1}{2} r^2 \sin\theta < \frac{1}{2} r^2 \theta < \frac{1}{2} r^2 \tan\theta$

$\sin\theta < \theta < \tan\theta$ ***AG***

[5 marks]

8. $x = 2e^y - \frac{1}{e^y}$ ***M1***

> **Note:** The ***M1*** is for switching the variables and may be awarded at any stage in the process and is awarded independently. Further marks do not rely on this mark being gained.

$xe^y = 2e^{2y} - 1$

$2e^{2y} - xe^y - 1 = 0$ ***A1***

$e^y = \frac{x \pm \sqrt{x^2 + 8}}{4}$ ***M1A1***

$y = \ln\left(\frac{x \pm \sqrt{x^2 + 8}}{4}\right)$

therefore $h^{-1}(x) = \ln\left(\frac{x + \sqrt{x^2 + 8}}{4}\right)$ ***A1***

since ln is undefined for the second solution ***R1***

> **Note:** Accept $y = \ln\left(\frac{x + \sqrt{x^2 + 8}}{4}\right)$.

> **Note:** The ***R1*** may be gained by an appropriate comment earlier.

[6 marks]

9. (a) **METHOD 1**

$$\mathrm{P}(3 \text{ defective in first } 8) = \binom{8}{3} \times \frac{4}{15} \times \frac{3}{14} \times \frac{2}{13} \times \frac{11}{12} \times \frac{10}{11} \times \frac{9}{10} \times \frac{8}{9} \times \frac{7}{8}$$ ***M1A1A1***

Note:	Award ***M1*** for multiplication of probabilities with decreasing denominators. Award ***A1*** for multiplication of correct eight probabilities. Award ***A1*** for multiplying by $\binom{8}{3}$.

$$= \frac{56}{195}$$ ***A1***

METHOD 2

$$\mathrm{P}(3 \text{ defective DVD players from } 8) = \frac{\binom{4}{3}\binom{11}{5}}{\binom{15}{8}}$$ ***M1A1***

Note:	Award ***M1*** for an expression of this form containing three combinations.

$$= \frac{\frac{4!}{3!1!} \times \frac{11!}{5!6!}}{\frac{15!}{8!7!}}$$ ***M1***

$$= \frac{56}{195}$$ ***A1***

(b) $\mathrm{P}(9^{\text{th}} \text{ selected is } 4^{\text{th}} \text{ defective player} | 3 \text{ defective in first } 8) = \frac{1}{7}$ ***(A1)***

$\mathrm{P}(9^{\text{th}} \text{ selected is } 4^{\text{th}} \text{ defective player}) = \frac{56}{195} \times \frac{1}{7}$ ***M1***

$= \frac{8}{195}$ ***A1***

[7 marks]

10. (a) let the first three terms of the geometric sequence be given by u_1, u_1r, u_1r^2

$\therefore u_1 = a + 2d$, $u_1r = a + 3d$ and $u_1r^2 = a + 6d$ ***(M1)***

$\frac{a+6d}{a+3d} = \frac{a+3d}{a+2d}$ ***A1***

$a^2 + 8ad + 12d^2 = a^2 + 6ad + 9d^2$ ***A1***

$2a + 3d = 0$

$a = -\frac{3}{2}d$ ***AG***

(b) $u_1 = \frac{d}{2}$, $u_1r = \frac{3d}{2}$, $\left(u_1r^2 = \frac{9d}{2}\right)$ ***M1***

$r = 3$ ***A1***

geometric 4th term $u_1r^3 = \frac{27d}{2}$ ***A1***

arithmetic 16th term $a + 15d = -\frac{3}{2}d + 15d$ ***M1***

$= \frac{27d}{2}$ ***A1***

Note: Accept alternative methods.

[8 marks]

SECTION B

11. (a) $\frac{dy}{dx} = 2x - \frac{1}{2}x^3$ ***A1***

$x\left(2 - \frac{1}{2}x^2\right) = 0$

$x = 0, \pm 2$

$\frac{dy}{dx} = 0$ at $\left(0, \frac{9}{8}\right), \left(-2, \frac{25}{8}\right), \left(2, \frac{25}{8}\right)$ ***A1A1A1***

Note: Award ***A2*** for all three x-values correct with errors/omissions in y-values.

[4 marks]

(b) at $x = 1$, gradient of tangent $= \frac{3}{2}$ ***(A1)***

Note: In the following, allow ***FT*** on incorrect gradient.

equation of tangent is $y - 2 = \frac{3}{2}(x-1)$ $\left(y = \frac{3}{2}x + \frac{1}{2}\right)$ ***(A1)***

meets x-axis when $y = 0$, $-2 = \frac{3}{2}(x-1)$ ***(M1)***

$x = -\frac{1}{3}$

coordinates of T are $\left(-\frac{1}{3}, 0\right)$ ***A1***

[4 marks]

(c) gradient of normal $= -\frac{2}{3}$ ***(A1)***

equation of normal is $y - 2 = -\frac{2}{3}(x-1)$ $\left(y = -\frac{2}{3}x + \frac{8}{3}\right)$ ***(M1)***

at $x = 0$, $y = \frac{8}{3}$ ***A1***

Note: In the following, allow FT on incorrect coordinates of T and N.

lengths of $\text{PN} = \sqrt{\frac{13}{9}}$, $\text{PT} = \sqrt{\frac{52}{9}}$ ***A1A1***

area of triangle PTN $= \frac{1}{2} \times \sqrt{\frac{13}{9}} \times \sqrt{\frac{52}{9}}$ ***M1***

$= \frac{13}{9}$ (or equivalent *e.g.* $\frac{\sqrt{676}}{18}$) ***A1***

[7 marks]

Total [15 marks]

12. (a) using the factor theorem $z+1$ is a factor *(M1)*

$z^3+1=(z+1)(z^2-z+1)$ ***A1***

[2 marks]

(b) (i) **METHOD 1**

$z^3=-1 \Rightarrow z^3+1=(z+1)(z^2-z+1)=0$ *(M1)*

solving $z^2-z+1=0$ ***M1***

$z=\frac{1\pm\sqrt{1-4}}{2}=\frac{1\pm i\sqrt{3}}{2}$ ***A1***

therefore one cube root of -1 is γ ***AG***

METHOD 2

$\gamma^2=\left(\frac{1+i\sqrt{3}}{2}\right)^2=\frac{-1+i\sqrt{3}}{2}$ ***M1A1***

$\gamma^3=\frac{-1+i\sqrt{3}}{2}\times\frac{1+i\sqrt{3}}{2}=\frac{-1-3}{4}$ ***A1***

$=-1$ ***AG***

METHOD 3

$\gamma=\frac{1+i\sqrt{3}}{2}=e^{i\frac{\pi}{3}}$ ***M1A1***

$\gamma^3=e^{i\pi}=-1$ ***A1***

(ii) **METHOD 1**

as γ is a root of $z^2-z+1=0$ then $\gamma^2-\gamma+1=0$ ***M1R1***

$\therefore \gamma^2=\gamma-1$ ***AG***

Note: Award ***M1*** for the use of $z^2-z+1=0$ in any way.
Award ***R1*** for a correct reasoned approach.

METHOD 2

$\gamma^2=\frac{-1+i\sqrt{3}}{2}$ ***M1***

$\gamma-1=\frac{1+i\sqrt{3}}{2}-1=\frac{-1+i\sqrt{3}}{2}$ ***A1***

continued …

Question 12 continued

(iii) **METHOD 1**

$(1-\gamma)^6 = (-\gamma^2)^6$ ***(M1)***

$= (\gamma)^{12}$ ***A1***

$= (\gamma^3)^4$ ***(M1)***

$= (-1)^4$

$= 1$ ***A1***

METHOD 2

$(1-\gamma)^6$

$= 1 - 6\gamma + 15\gamma^2 - 20\gamma^3 + 15\gamma^4 - 6\gamma^5 + \gamma^6$ ***M1A1***

Note:	Award ***M1*** for attempt at binomial expansion.

use of any previous result *e.g.* $= 1 - 6\gamma + 15\gamma^2 + 20 - 15\gamma + 6\gamma^2 + 1$ ***M1***

$=1$ ***A1***

Note:	As the question uses the word 'hence', other methods that do not use previous results are awarded no marks.

[9 marks]

continued …

Question 12 continued

(c) **METHOD 1**

$$\boldsymbol{A}^2 = \begin{pmatrix} \gamma & 1 \\ 0 & \frac{1}{\gamma} \end{pmatrix}\begin{pmatrix} \gamma & 1 \\ 0 & \frac{1}{\gamma} \end{pmatrix} = \begin{pmatrix} \gamma^2 & \gamma + \frac{1}{\gamma} \\ 0 & \frac{1}{\gamma^2} \end{pmatrix}$$ ***A1***

$$\boldsymbol{A}^2 - \boldsymbol{A} + \boldsymbol{I} = \begin{pmatrix} \gamma^2 - \gamma + 1 & \gamma + \frac{1}{\gamma} - 1 \\ 0 & \frac{1}{\gamma^2} - \frac{1}{\gamma} + 1 \end{pmatrix}$$ ***M1***

from part (b)

$\gamma^2 - \gamma + 1 = 0$

$\gamma + \frac{1}{\gamma} - 1 = \frac{1}{\gamma}(\gamma^2 - \gamma + 1) = 0$ ***A1***

$\frac{1}{\gamma^2} - \frac{1}{\gamma} + 1 = \frac{1}{\gamma^2}(\gamma^2 - \gamma + 1) = 0$ ***A1***

hence $\boldsymbol{A}^2 - \boldsymbol{A} + \boldsymbol{I} = \boldsymbol{0}$ ***AG***

METHOD 2

$$\boldsymbol{A}^2 = \begin{pmatrix} \frac{-1 + i\sqrt{3}}{2} & 1 \\ 0 & \frac{-1 - i\sqrt{3}}{2} \end{pmatrix}$$ ***A1A1A1***

Note: Award 1 mark for each of the non-zero elements expressed in this form.

verifying $\boldsymbol{A}^2 - \boldsymbol{A} + \boldsymbol{I} = \boldsymbol{0}$ ***M1AG***

[4 marks]

continued …

Question 12 continued

(d) (i) $\boldsymbol{A}^2 = \boldsymbol{A} - \boldsymbol{I}$

$\Rightarrow \boldsymbol{A}^3 = \boldsymbol{A}^2 - \boldsymbol{A}$ ***M1A1***

$= \boldsymbol{A} - \boldsymbol{I} - \boldsymbol{A}$ ***A1***

$= -\boldsymbol{I}$ ***AG***

Note: Allow other valid methods.

(ii) $\boldsymbol{I} = \boldsymbol{A} - \boldsymbol{A}^2$

$\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} - \boldsymbol{A}^{-1}\boldsymbol{A}^2$ ***M1A1***

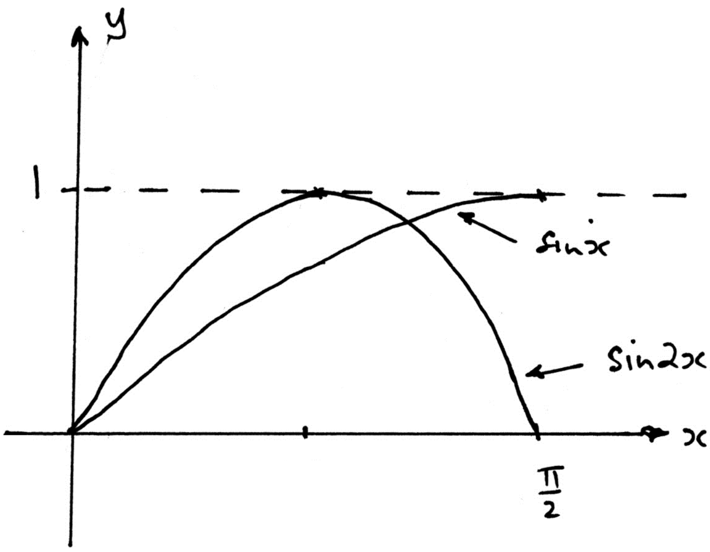
$\Rightarrow \boldsymbol{A}^{-1} = \boldsymbol{I} - \boldsymbol{A}$ ***AG***

Note: Allow other valid methods.

[5 marks]

Total [20 marks]

13. (a) (i)



A2

Note: Award ***A1*** for correct $\sin x$, ***A1*** for correct $\sin 2x$.

Note: Award ***A1A0*** for two correct shapes with $\frac{\pi}{2}$ and/or 1 missing.

Note: Condone graph outside the domain.

(ii) $\sin 2x = \sin x,\ 0 \le x \le \frac{\pi}{2}$

$2\sin x \cos x - \sin x = 0$ ***M1***

$\sin x(2\cos x - 1) = 0$

$x = 0, \frac{\pi}{3}$ ***A1A1*** ***N1N1***

continued …

Question 13 continued

(iii) $\text{area} = \int_0^{\frac{\pi}{3}} (\sin 2x - \sin x)\,\mathrm{d}x$ ***M1***

Note: Award ***M1*** for an integral that contains limits, not necessarily correct, with $\sin x$ and $\sin 2x$ subtracted in either order.

$= \left[-\frac{1}{2}\cos 2x + \cos x\right]_0^{\frac{\pi}{3}}$ ***A1***

$= \left(-\frac{1}{2}\cos\frac{2\pi}{3} + \cos\frac{\pi}{3}\right) - \left(-\frac{1}{2}\cos 0 + \cos 0\right)$ ***(M1)***

$= \frac{3}{4} - \frac{1}{2}$

$= \frac{1}{4}$ ***A1***

[9 marks]

(b) $\int_0^1 \sqrt{\frac{x}{4-x}}\,\mathrm{d}x = \int_0^{\frac{\pi}{6}} \sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}} \times 8\sin\theta\cos\theta\,\mathrm{d}\theta$ ***M1A1A1***

Note: Award ***M1*** for substitution and reasonable attempt at finding expression for $\mathrm{d}x$ in terms of $\mathrm{d}\theta$, first ***A1*** for correct limits, second ***A1*** for correct substitution for $\mathrm{d}x$.

$\int_0^{\frac{\pi}{6}} 8\sin^2\theta\,\mathrm{d}\theta$ ***A1***

$\int_0^{\frac{\pi}{6}} 4 - 4\cos 2\theta\,\mathrm{d}\theta$ ***M1***

$= [4\theta - 2\sin 2\theta]_0^{\frac{\pi}{6}}$ ***A1***

$= \left(\frac{2\pi}{3} - 2\sin\frac{\pi}{3}\right) - 0$ ***(M1)***

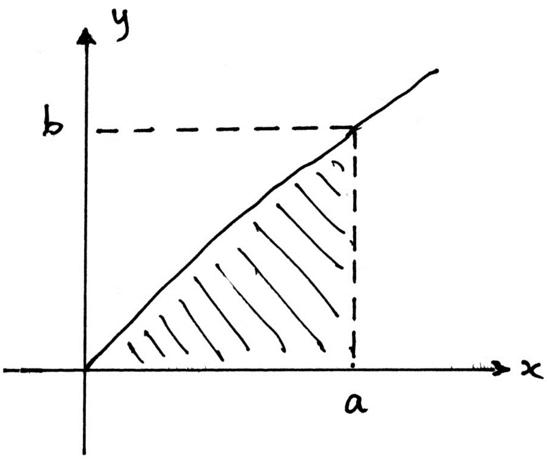
$= \frac{2\pi}{3} - \sqrt{3}$ ***A1***

[8 marks]

continued …

Question 13 continued

(c) (i)



M1

from the diagram above

the shaded area $= \int_0^a f(x)\,\mathrm{d}x = ab - \int_0^b f^{-1}(y)\,\mathrm{d}y$ **R1**

$= ab - \int_0^b f^{-1}(x)\,\mathrm{d}x$ **AG**

(ii) $f(x) = \arcsin\frac{x}{4} \Rightarrow f^{-1}(x) = 4\sin x$ **A1**

$\int_0^2 \arcsin\left(\frac{x}{4}\right)\mathrm{d}x = \frac{\pi}{3} - \int_0^{\frac{\pi}{6}} 4\sin x\,\mathrm{d}x$ **M1A1A1**

Note: Award ***A1*** for the limit $\frac{\pi}{6}$ seen anywhere, ***A1*** for all else correct.

$= \frac{\pi}{3} - \left[-4\cos x\right]_0^{\frac{\pi}{6}}$ **A1**

$= \frac{\pi}{3} - 4 + 2\sqrt{3}$ **A1**

Note: Award no marks for methods using integration by parts.

[8 marks]

Total [25 marks]