M11/5/MATHL/HP2/ENG/TZ1/XX/M

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MARKSCHEME

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MATHEMATICS

Higher Level

Paper 2

20 pages

SECTION A

1. (a) (i) median $= 104$ grams *A1*

Note: Accept 105.

(ii) 30^{th} percentile $= 90$ grams *A1*

(b) $80 - 49$ *(M1)*
$= 31$ *A1*

Note: Accept answers 30 to 32.

[4 marks]

2. (a) $f'(x) = 3x^2 - 6x - 9 \ (= 0)$ *(M1)*
$(x+1)(x-3) = 0$
$x = -1;\ x = 3$
$(\max)(-1, 15);(\min)(3, -17)$ *A1A1*

Note: The coordinates need not be explicitly stated but the values need to be seen.

$y = -8x + 7$ *A1* *N2*

(b) $f''(x) = 6x - 6 = 0 \Rightarrow$ inflexion $(1, -1)$ *A1*
which lies on $y = -8x + 7$ *R1AG*

[6 marks]

3. **METHOD 1**

$\dfrac{\sin C}{7} = \dfrac{\sin 40}{5}$ *M1(A1)*

$\text{B}\hat{\text{C}}\text{D} = 64.14...^\circ$ *A1*
$\text{CD} = 2 \times 5 \cos 64.14...$ *M1*

Note: Also allow use of sine or cosine rule.

$\text{CD} = 4.36$ *A1*

[5 marks]

METHOD 2

let $\text{AC} = x$
cosine rule
$5^2 = 7^2 + x^2 - 2 \times 7 \times x \cos 40$ *M1A1*
$x^2 - 10.7...x + 24 = 0$

$x = \dfrac{10.7... \pm \sqrt{(10.7...)^2 - 4 \times 24}}{2}$ *(M1)*

$x = 7.54;\ 3.18$ *(A1)*
CD is the difference in these two values $= 4.36$ *A1*

Note: Other methods may be seen.

[5 marks]

4. (a) $f(a)=4a^3+2a^2-7a=-10$ ***M1***

$4a^3+2a^2-7a+10=0$

$(a+2)(4a^2-6a+5)=0$ or sketch or GDC ***(M1)***

$a=-2$ ***A1***

(b) substituting $a=-2$ into $f(x)$

$f(x)=4x^3-4x+14=0$ ***A1***

EITHER

graph showing unique solution which is indicated (must include max and min) ***R1***

OR

convincing argument that only one of the solutions is real ***R1***

$(-1.74,\ 0.868\pm 1.12\text{i})$

[5 marks]

5. (a) $2x^2+x-3=(2x+3)(x-1)$ ***A1***

Note: Accept $2\left(x+\frac{3}{2}\right)(x-1)$.

Note: Either of these may be seen in (b) and if so ***A1*** should be awarded.

(b) **EITHER**

$(2x^2+x-3)^8=(2x+3)^8(x-1)^8$ ***M1***

$=\left(3^8+8(3^7)(2x)+\ldots\right)\left((-1)^8+8(-1)^7(x)+\ldots\right)$ ***(A1)***

coefficient of $x=3^8\times 8\times(-1)^7+3^7\times 8\times 2\times(-1)^8$ ***M1***

$=-17\,496$ ***A1***

Note: Under ft, final ***A1*** can only be achieved for an integer answer.

OR

$(2x^2+x-3)^8=\left(3-(x-2x^2)\right)^8$ ***M1***

$=3^8+8\left(-(x-2x^2)(3^7)+\ldots\right)$ ***(A1)***

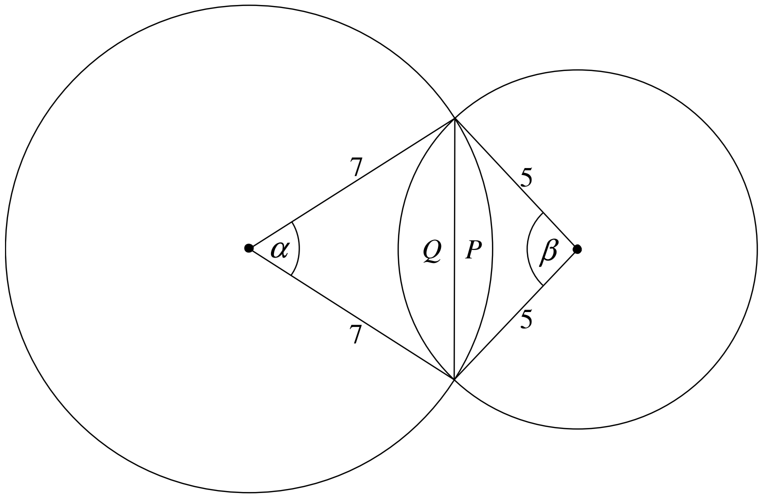
coefficient of $x=8\times(-1)\times 3^7$ ***M1***

$=-17\,496$ ***A1***

Note: Under ft, final ***A1*** can only be achieved for an integer answer.

[5 marks]

6.



$\alpha = 2\arcsin\left(\frac{4.5}{7}\right)$ $(\Rightarrow \alpha = 1.396\ldots = 80.010°\ldots)$ ***M1(A1)***

$\beta = 2\arcsin\left(\frac{4.5}{5}\right)$ $(\Rightarrow \beta = 2.239\ldots = 128.31°\ldots)$ ***(A1)***

Note: Allow use of cosine rule.

area $P = \frac{1}{2} \times 7^2 \times (\alpha - \sin\alpha) = 10.08...$ ***M1(A1)***

area $Q = \frac{1}{2} \times 5^2 \times (\beta - \sin\beta) = 18.18...$ ***(A1)***

Note: The ***M1*** is for an attempt at area of sector minus area of triangle.

Note: The use of degrees correctly converted is acceptable.

area $= 28.3(\text{cm}^2)$ ***A1***

[7 marks]

7. (a) $k\int_{-2}^{0}(x+2)^2\,dx + \int_{0}^{\frac{4}{3}} k\,dx = 1$ ***M1***

$\frac{8k}{3} + \frac{4k}{3} = 1$

$k = \frac{1}{4}$ ***A1***

Note: Only ft on positive values of *k*.

continued …

Question 7 continued

(b) (i) $\mathrm{E}(X) = \frac{1}{4}\int_{-2}^{0} x(x+2)^2\,\mathrm{d}x + \frac{1}{4}\int_{0}^{\frac{4}{3}} x\,\mathrm{d}x$ ***M1***

$= \frac{1}{4} \times \frac{-4}{3} + \frac{2}{9}$

$= -\frac{1}{9}$ (-0.111) ***A1***

(ii) median given by a such that $\mathrm{P}(X < a) = 0.5$

$\frac{1}{4}\int_{-2}^{a} (x+2)^2\,\mathrm{d}x = 0.5$ ***M1***

$\left[\frac{(x+2)^3}{3}\right]_{-2}^{a} = 2$ ***(A1)***

$(a+2)^3 - 0 = 6$

$a = \sqrt[3]{6} - 2$ $(= -0.183)$ ***A1***

[7 marks]

8. (a) equation of line in graph $a = -\frac{25}{60}t + 15$ ***A1***

$$\left(a = -\frac{5}{12}t + 15\right)$$

(b) $\frac{dv}{dt} = -\frac{5}{12}t + 15$ ***(M1)***

$v = -\frac{5}{24}t^2 + 15t + c$ ***(A1)***

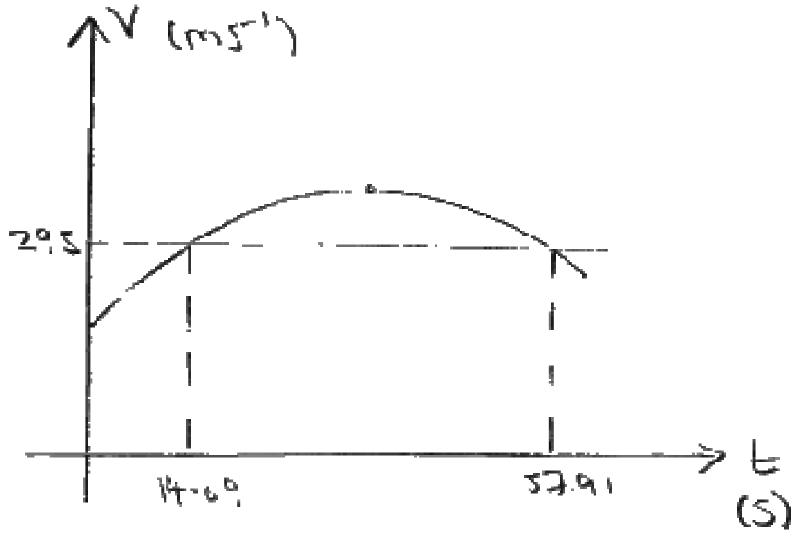
when $t = 0$, $v = 125 \text{ m s}^{-1}$

$v = -\frac{5}{24}t^2 + 15t + 125$ ***A1***

from graph or by finding time when $a = 0$

maximum $= 395 \text{ m s}^{-1}$ ***A1***

(c) **EITHER**



graph drawn and intersection with $v = 295 \text{ m s}^{-1}$ ***(M1)(A1)***

$t = 57.91 - 14.09 = 43.8$ ***A1***

OR

$295 = -\frac{5}{24}t^2 + 15t + 125 \Rightarrow t = 57.91...; 14.09...$ ***(M1)(A1)***

$t = 57.91\ldots - 14.09\ldots = 43.8$ $\left(8\sqrt{30}\right)$ ***A1***

[8 marks]

9. $\log_{x+1} y = 2$

$\log_{y+1} x = \frac{1}{4}$

so $(x+1)^2 = y$ ***A1***

$(y+1)^{\frac{1}{4}} = x$ ***A1***

EITHER

$x^4 - 1 = (x+1)^2$ ***M1***

$x = -1$, not possible ***R1***

$x = 1.70,\ y = 7.27$ ***A1A1***

OR

$(x^2 + 2x + 2)^{\frac{1}{4}} - x = 0$ ***M1***

attempt to solve or graph of LHS ***M1***

$x = 1.70,\ y = 7.27$ ***A1A1***

[6 marks]

10. METHOD 1

equation of journey of ship S_1

$$\boldsymbol{r}_1 = t\begin{pmatrix}10\\20\end{pmatrix}$$

equation of journey of speedboat S_2 ,setting off k minutes later

$$\boldsymbol{r}_2 = \begin{pmatrix}70\\30\end{pmatrix} + (t-k)\begin{pmatrix}-60\\30\end{pmatrix}$$ ***M1A1A1***

Note:	Award ***M1*** for perpendicular direction, ***A1*** for speed, ***A1*** for change in parameter (*e.g.* by using $t-k$ or T, k being the time difference between the departure of the ships).

solve $t\begin{pmatrix}10\\20\end{pmatrix} = \begin{pmatrix}70\\30\end{pmatrix} + (t-k)\begin{pmatrix}-60\\30\end{pmatrix}$ ***(M1)***

Note:	***M*** mark is for equating their two expressions.

$10t = 70 - 60t + 60k$

$20t = 30 + 30t - 30k$ ***M1***

Note:	***M*** mark is for obtaining two equations involving two different parameters.

$7t - 6k = 7$

$-t + 3k = 3$

$k = \frac{28}{15}$ ***A1***

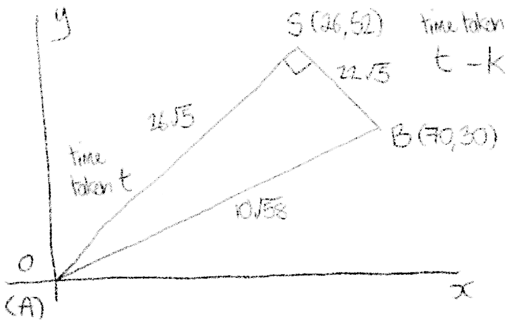
latest time is 11:52 ***A1***

[7 marks]

continued …

Question 10 continued

METHOD 2



$SB = 22\sqrt{5}$ ***M1A1***

(by perpendicular distance)

$SA = 26\sqrt{5}$ ***M1A1***

(by Pythagoras or coordinates)

$t = \frac{26\sqrt{5}}{10\sqrt{5}}$ ***A1***

$t - k = \frac{22\sqrt{5}}{30\sqrt{5}}$ ***A1***

$k = \frac{28}{15}$ leading to latest time 11:52 ***A1***

[7 marks]

SECTION B

11. (a) $\begin{pmatrix} 0 & 2 & 1 \\ -1 & 1 & 3 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ k \end{pmatrix}$

$\begin{vmatrix} 0 & 2 & 1 \\ -1 & 1 & 3 \\ -2 & 1 & 2 \end{vmatrix} = 0 - 2(-2+6) + (-1+2) = -7$ ***M1A1***

since determinant $\neq 0 \Rightarrow$ unique solution to the system ***R1***
planes intersect in a point ***AG***

> **Note:** For any method, including row reduction, leading to the explicit solution $\left(\frac{6-5k}{7}, \frac{10+k}{7}, \frac{1-2k}{7}\right)$, award ***M1*** for an attempt at a correct method ***A1*** for two correct coordinates and ***A1*** for a third correct coordinate.

[3 marks]

(b) $\begin{vmatrix} a & 2 & 1 \\ -1 & a+1 & 3 \\ -2 & 1 & a+2 \end{vmatrix} = a\big((a+1)(a+2)-3\big) - 2\big(-1(a+2)+6\big) + \big(-1+2(a+1)\big)$ ***M1(A1)***

planes not meeting in a point $\Rightarrow$ no unique solution *i.e.* determinant $= 0$ ***(M1)***

$a(a^2+3a-1)+(2a-8)+(2a+1)=0$
$a^3+3a^2+3a-7=0$ ***A1***
$a=1$ ***A1***

[5 marks]

continued …

Question 11 continued

(c) $\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 4 & 4 \\ -2 & 1 & 3 & k \end{pmatrix} r_1 + r_2$ ***M1***

$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 4 & 4 \\ 0 & 5 & 5 & 6+k \end{pmatrix} 2r_1 + r_3$ ***(A1)***

$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 4 & 4 \\ 0 & 0 & 0 & 4+4k \end{pmatrix} 4r_3 - 5r_2$ ***(A1)***

for an infinite number of solutions to exist, $4 + 4k = 0 \Rightarrow k = -1$ ***A1***

$x + 2y + z = 3$
$y + z = 1$ ***M1***

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ ***A1***

Note: Accept methods involving elimination.

Note: Accept any equivalent form *e.g.* $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ or $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Award ***A0*** if $\begin{pmatrix} x \\ y \\ z \end{pmatrix} =$ or $\boldsymbol{r} =$ is absent.

[6 marks]

Total [14 marks]

12. (a) $\mathrm{P}(X < 30) = 0.4$

$\mathrm{P}(X < 55) = 0.9$

or relevant sketch ***(M1)***

given $Z = \dfrac{X - \mu}{\sigma}$

$\mathrm{P}(Z < z) = 0.4 \Rightarrow \dfrac{30 - \mu}{\sigma} = -0.253...$ ***(A1)***

$\mathrm{P}(Z < z) = 0.9 \Rightarrow \dfrac{55 - \mu}{\sigma} = 1.28...$ ***(A1)***

$\mu = 30 + (0.253...) \times \sigma = 55 - (1.28...) \times \sigma$ ***M1***

$\sigma = 16.3$, $\mu = 34.1$ ***A1***

Note: Accept 16 and 34.

Note: Working with 830 and 855 will only gain the two ***M*** marks.

[5 marks]

(b) $X \sim \mathrm{N}(34.12\ldots, 16.28\ldots^2)$

late to school $\Rightarrow X > 60$

$\mathrm{P}(X > 60) = 0.056...$ ***(A1)***

number of students late $= 0.0560\ldots \times 1200$ ***(M1)***

$= 67$ (to nearest integer) ***A1***

Note: Accept 62 for use of 34 and 16.

[3 marks]

(c) $\mathrm{P}(X > 60 \mid X > 30) = \dfrac{\mathrm{P}(X > 60)}{\mathrm{P}(X > 30)}$ ***M1***

$= 0.0935$ (accept anything between 0.093 and 0.094) ***A1***

Note: If 34 and 16 are used 0.0870 is obtained. This should be accepted.

[2 marks]

(d) let L be the random variable of the number of students who leave school in a 30 minute interval

since $24 \times 30 = 720$ ***A1***

$L \sim \mathrm{Po}(720)$

$\mathrm{P}(L \geq 700) = 1 - \mathrm{P}(L \leq 699)$ ***(M1)***

$= 0.777$ ***A1***

Note: Award ***M1A0*** for $\mathrm{P}(L > 700) = 1 - \mathrm{P}(L \leq 700)$ (this leads to 0.765).

[3 marks]

continued …

Question 12 continued

(e) (i) $Y \sim \mathrm{B}(200,\ 0.7767\ldots)$ ***(M1)***

$\mathrm{E}(Y) = 200 \times 0.7767\ldots = 155$ ***A1***

Note: On ft, use of 0.765 will lead to 153.

(ii) $\mathrm{P}(Y > 150) = 1 - \mathrm{P}(Y \le 150)$ ***(M1)***

$= 0.797$ ***A1***

Note: Accept 0.799 from using rounded answer.

Note: On ft, use of 0.765 will lead to 0.666.

[4 marks]

Total [17 marks]

13. (a) $\boldsymbol{A}^2 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & \cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta - \cos\theta\sin\theta & -\sin^2\theta + \cos^2\theta \end{pmatrix}$ ***M1(A1)***

$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{pmatrix}$ ***A1***

$= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}$ ***AG***

[3 marks]

continued …

Question 13 continued

(b) let $\mathrm{P}(n)$ be the proposition that $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$

for all $n \in \mathbb{Z}^+$

$\mathrm{P}(1)$ is true **A1**

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^1 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

assume $\mathrm{P}(k)$ to be true **A1**

Note:	Must see the word 'true' or equivalent, that makes clear an assumption is being made that $\mathrm{P}(k)$ is true.

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^k = \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix}$$

consider $\mathrm{P}(k+1)$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{k+1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^k \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$ ***(M1)***

$$= \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$ **A1**

$$= \begin{pmatrix} \cos k\theta\cos\theta - \sin k\theta\sin\theta & \cos k\theta\sin\theta + \sin k\theta\cos\theta \\ -\sin k\theta\cos\theta - \cos k\theta\sin\theta & -\sin k\theta\sin\theta + \cos k\theta\cos\theta \end{pmatrix}$$ **A1**

$$= \begin{pmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{pmatrix}$$ **A1**

if $\mathrm{P}(k)$ is true then $\mathrm{P}(k+1)$ is true and since $\mathrm{P}(1)$ is true then $\mathrm{P}(n)$ is true for all $n \in \mathbb{Z}^+$ **R1**

Note:	The final ***R1*** can only be gained if the ***M1*** has been gained.

[7 marks]

continued …

Question 13 continued

(c) **EITHER**

$$A^{-1} = \begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{pmatrix} \text{ from formula}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ ***A1***

$$A^{-1}A = AA^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ ***M1***

Note: Accept either just $A^{-1}A$ or just AA^{-1}.

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ ***A1***

$\therefore A^{-1}$ is inverse of A

OR

$$A^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ ***M1***

$$A^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ ***A1***

putting $n = -1$ in formula gives inverse ***A1***

[3 marks]

Total [13 marks]

14. (a) volume $= \pi\int_0^h x^2\,dy$ ***(M1)***

$\pi\int_0^h y\,dy$ ***M1***

$= \pi\left[\frac{y^2}{2}\right]_0^h = \frac{\pi h^2}{2}$ ***A1***

[3 marks]

(b) $\frac{dV}{dt} = -3 \times \text{surface area}$ ***A1***

surface area $= \pi x^2$ ***(M1)***

$= \pi h$ ***A1***

since $V = \frac{\pi h^2}{2} \Rightarrow h = \sqrt{\frac{2V}{\pi}}$ ***M1A1***

$\frac{dV}{dt} = -3\pi\sqrt{\frac{2V}{\pi}}$ ***A1***

$\frac{dV}{dt} = -3\sqrt{2\pi V}$ ***AG***

Note: Assuming that $\frac{dh}{dt} = -3$ without justification gains no marks.

[6 marks]

continued …

Question 14 continued

(c) $V_0 = 5000\pi \ (= 15700 \text{ cm}^3)$ ***A1***

$\frac{dV}{dt} = -3\sqrt{2\pi V}$

attempting to separate variables ***M1***

EITHER

$\int \frac{dV}{\sqrt{V}} = -3\sqrt{2\pi}\int dt$ ***A1***

$2\sqrt{V} = -3\sqrt{2\pi}t + c$ ***A1***

$c = 2\sqrt{5000\pi}$ ***A1***

$V = 0$ ***M1***

$\Rightarrow t = \frac{2}{3}\sqrt{\frac{5000\pi}{2\pi}} = 33\tfrac{1}{3}$ hours ***A1***

OR

$\int_{5000\pi}^{0} \frac{dV}{\sqrt{V}} = -3\sqrt{2\pi}\int_0^T dt$ ***M1A1A1***

Note: Award ***M1*** for attempt to use definite integrals, ***A1*** for correct limits and ***A1*** for correct integrands.

$\left[2\sqrt{V}\right]_{5000\pi}^{0} = -3\sqrt{2\pi}T$ ***A1***

$T = \frac{2}{3}\sqrt{\frac{5000\pi}{2\pi}} = 33\tfrac{1}{3}$ hours ***A1***

[7 marks]

Total [16 marks]