M11/5/MATHL/HP2/ENG/TZ2/XX/M

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MARKSCHEME

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MATHEMATICS

Higher Level

Paper 2

15 pages

SECTION A

1. area of triangle $\text{POQ} = \frac{1}{2}8^2 \sin 59°$ ***M1***

$= 27.43$ ***(A1)***

area of sector $= \pi 8^2 \frac{59}{360}$ ***M1***

$= 32.95$ ***(A1)***

area between arc and chord $= 32.95 - 27.43$

$= 5.52 \ (\text{cm}^2)$ ***A1***

[5 marks]

2. $u_4 = u_1 + 3d = 7$, $u_9 = u_1 + 8d = 22$ ***A1A1***

Note: 5d=15 gains both above marks

$u_1 = -2$, $d = 3$ ***A1***

$S_n = \frac{n}{2}\left(-4 + (n-1)3\right) > 10\,000$ ***M1***

$n = 83$ ***A1***

[5 marks]

3. (a) $a = 10\text{e}^{-0.2t}$ ***(M1)(A1)***

at $t = 10$, $a = 1.35 \ (\text{m s}^{-2})$ (accept $10e^{-2}$) ***A1***

(b) **METHOD 1**

$d = \int_0^{10} 50(1 - \text{e}^{-0.2t})\,\text{d}t$ ***(M1)***

$= 283.83...$ ***A1***

so distance above ground $= 1720$ (m) (3 sf) (accept 1716 (m)) ***A1***

METHOD 2

$\text{s} = \int 50(1 - \text{e}^{-0.2t})\,\text{d}t = 50t + 250e^{-0.2t} (+c)$ ***M1***

Taking $s = 0$ when $t = 0$ gives $c = -250$ ***M1***

So when $t = 10$, $s = 283.3...$

so distance above ground $= 1720$ (m)(3 sf) (accept 1716 (m)) ***A1***

[6 marks]

4. (a) $\det \boldsymbol{A} = \cos 2\theta \cos\theta + \sin 2\theta \sin\theta$ ***M1A1***

$= \cos(2\theta - \theta)$ ***A1***

Note: Allow use of double angle formulae if they lead to the correct answer

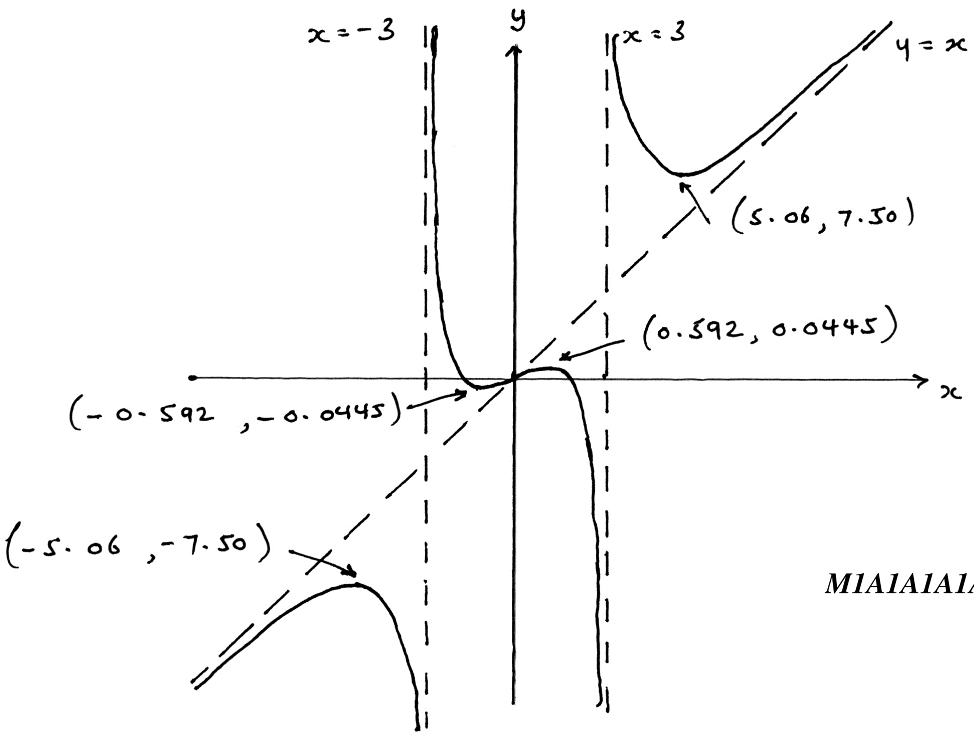
$= \cos\theta$ ***AG***

(b) $\cos^2\theta = \sin\theta$ ***A1***

$\theta = 0.666,\ 2.48$ ***A1A1***

[6 marks]

5.



M1A1A1A1A1A1A1

Note: Award ***A1*** for both vertical asymptotes correct, ***M1*** for recognizing that there are two turning points near the origin, ***A1*** for both turning points near the origin correct, (only this ***A*** mark is dependent on the ***M*** mark) ***A1*** for the other pair of turning points correct, ***A1*** for correct positioning of the oblique asymptote, ***A1*** for correct equation of the oblique asymptote, ***A1*** for correct asymptotic behaviour in all sections.

[7 marks]

6. (a) $\mathrm{P}(x < 1.4) = 0.691$ (accept 0.692) ***A1***

(b) **METHOD 1**

$Y \sim \mathrm{B}(6,\ 0.3085...)$ ***(M1)***

$\mathrm{P}(Y \geq 4) = 1 - \mathrm{P}(Y \leq 3)$ ***(M1)***

$= 0.0775$ (accept 0.0778 if 3sf approximation from (a) used) ***A1***

METHOD 2

$X \sim \mathrm{B}(6,\ 0.6914...)$ ***(M1)***

$\mathrm{P}(X \leq 2)$ ***(M1)***

$= 0.0775$ (accept 0.0778 if 3sf approximation from (a) used) ***A1***

(c) $\mathrm{P}(x < 1 \mid x < 1.4) = \dfrac{\mathrm{P}(x < 1)}{\mathrm{P}(x < 1.4)}$ ***M1***

$= \dfrac{0.06680...}{0.6914...}$

$= 0.0966$ (accept 0.0967) ***A1***

[6 marks]

7. (a) $x^3 + 1 = \dfrac{1}{x^3 + 1}$

$(-1.26,\ -1)$ $\left(= \left(-\sqrt[3]{2},\ -1\right)\right)$ ***A1***

(b) $f'(-1.259...) = 4.762...$ $(3 \times 2^{\frac{2}{3}})$ ***A1***

$g'(-1.259...) = -4.762...$ $(-3 \times 2^{\frac{2}{3}})$ ***A1***

required angle $= 2\arctan\left(\dfrac{1}{4.762...}\right)$ ***M1***

$= 0.414$ (accept $23.7°$) ***A1***

Note: Accept alternative methods including finding the obtuse angle first.

[5 marks]

8. let the length of one side of the triangle be x
consider the triangle consisting of a side of the triangle and two radii

EITHER

$x^2 = r^2 + r^2 - 2r^2 \cos 120°$ ***M1***

$= 3r^2$

OR

$x = 2r \cos 30°$ ***M1***

THEN

$x = r\sqrt{3}$ ***A1***

so perimeter $= 3\sqrt{3}\, r$ ***A1***

now consider the area of the triangle

area $= 3 \times \frac{1}{2} r^2 \sin 120°$ ***M1***

$= 3 \times \frac{\sqrt{3}}{4} r^2$ ***A1***

$\frac{P}{A} = \frac{3\sqrt{3}\, r}{\frac{3\sqrt{3}}{4} r^2}$

$= \frac{4}{r}$ ***A1***

Note: Accept alternative methods

[6 marks]

9. let $x =$ distance from observer to rocket
let $h =$ the height of the rocket above the ground

METHOD 1

$\frac{dh}{dt} = 300$ when $h = 800$ ***A1***

$x = \sqrt{h^2 + 360\,000} = (h^2 + 360\,000)^{\frac{1}{2}}$ ***M1***

$\frac{dx}{dh} = \frac{h}{\sqrt{h^2 + 360\,000}}$ ***A1***

when $h = 800$

$\frac{dx}{dt} = \frac{dx}{dh} \times \frac{dh}{dt}$ ***M1***

$= \frac{300h}{\sqrt{h^2 + 360\,000}}$ ***A1***

$= 240\ (\text{m s}^{-1})$ ***A1***

[6 marks]

METHOD 2

$h^2 + 600^2 = x^2$ ***M1***

$2h = 2x\frac{dx}{dh}$ ***A1***

$\frac{dx}{dh} = \frac{h}{x}$

$= \frac{800}{1000}\ \left(= \frac{4}{5}\right)$ ***A1***

$\frac{dh}{dt} = 300$ ***A1***

$\frac{dx}{dt} = \frac{dx}{dh} \times \frac{dh}{dt}$ ***M1***

$= \frac{4}{5} \times 300$

$= 240\ (\text{m s}^{-1})$ ***A1***

[6 marks]

METHOD 3

$x^2 = 600^2 + h^2$ ***M1***

$2x\frac{dx}{dt} = 2h\frac{dh}{dt}$ ***A1A1***

when $h = 800$, $x = 1000$

$\frac{dx}{dt} = \frac{800}{1000} \times \frac{dh}{dt}$ ***M1A1***

$= 240\ \text{m s}^{-1}$ ***A1***

continued …

Question 9 continued

METHOD 4

Distance between the observer and the rocket = $(600^2 + 800^2)^{\frac{1}{2}} = 1000$ ***M1A1***

Component of the velocity in the line of sight = $\sin\theta \times 300$
(where θ = angle of elevation) ***M1A1***

$\sin\theta = \frac{800}{1000}$ ***A1***

component = 240 (ms^{-1}) ***A1***

[6 marks]

10. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$

$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{y}}} = -\sqrt{\frac{y}{x}}$ ***A1***

Note: Accept $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$ from making *y* the subject of the equation, and all correct subsequent working

therefore the gradient at the point P is given by

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\sqrt{\frac{q}{p}}$ ***A1***

equation of tangent is $y - q = -\sqrt{\frac{q}{p}}(x - p)$ ***M1***

$\left(y = -\sqrt{\frac{q}{p}}x + q + \sqrt{q}\sqrt{p}\right)$

x-intercept: $y = 0,\ n = \frac{q\sqrt{p}}{\sqrt{q}} + p = \sqrt{q}\sqrt{p} + p$ ***A1***

y-intercept: $x = 0,\ m = \sqrt{q}\sqrt{p} + q$ ***A1***

$n + m = \sqrt{q}\sqrt{p} + p + \sqrt{q}\sqrt{p} + q$ ***M1***

$= 2\sqrt{q}\sqrt{p} + p + q$

$= \left(\sqrt{p} + \sqrt{q}\right)^2$ ***A1***

$= a$ ***AG***

[8 marks]

SECTION B

11. (a) $\vec{PQ} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}, \vec{SR} = \begin{pmatrix} 0-x \\ 5-y \\ 1-z \end{pmatrix}$ ***(M1)***

point S $= (1, 6, -2)$ ***A1***

[2 marks]

(b) $\vec{PQ} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} \quad \vec{PS} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$ ***A1***

$\vec{PQ} \times \vec{PS} = \begin{pmatrix} -13 \\ 7 \\ -2 \end{pmatrix}$

$m = -2$ ***A1***

[2 marks]

(c) area of parallelogram PQRS $= \left| \vec{PQ} \times \vec{PS} \right| = \sqrt{(-13)^2 + 7^2 + (-2)^2}$ ***M1***

$= \sqrt{222} = 14.9$ ***A1***

[2 marks]

(d) equation of plane is $-13x + 7y - 2z = d$ ***M1A1***

substituting any of the points given gives $d = 33$

$-13x + 7y - 2z = 33$ ***A1***

[3 marks]

(e) equation of line is $\boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -13 \\ 7 \\ -2 \end{pmatrix}$ ***A1***

Note: To get the ***A1*** must have $\boldsymbol{r} =$ or equivalent.

[1 mark]

(f) $169\lambda + 49\lambda + 4\lambda = 33$ ***M1***

$\lambda = \frac{33}{222} \quad (= 0.149...)$ ***A1***

closest point is $\left(-\frac{143}{74}, \frac{77}{74}, -\frac{11}{37} \right) \quad \left(= (-1.93, 1.04, -0.297)\right)$ ***A1***

[3 marks]

(g) angle between planes is the same as the angle between the normals ***(R1)***

$\cos\theta = \frac{-13\times1 + 7\times-2 - 2\times1}{\sqrt{222}\times\sqrt{6}}$ ***M1A1***

$\theta = 143^\circ$ (accept $\theta = 37.4^\circ$ or 2.49 radians or 0.652 radians) ***A1***

[4 marks]

Total [17 marks]

12. (a) $\mathrm{P}(x=0)=0.607$ ***A1***

[1 mark]

(b) **EITHER**

Using $X \sim \mathrm{Po}(3)$ ***(M1)***

OR

Using $(0.6065...)^6$ ***(M1)***

THEN

$\mathrm{P}(X=0)=0.0498$ ***A1***

[2 marks]

(c) $X \sim \mathrm{Po}(0.5t)$ ***(M1)***

$\mathrm{P}(x \geq 1)=1-\mathrm{P}(x=0)$ ***(M1)***

$\mathrm{P}(x=0)<0.01$ ***A1***

$\mathrm{e}^{-0.5t}<0.01$ ***A1***

$-0.5t<\ln(0.01)$ ***(M1)***

$t>9.21$ months

therefore 10 months ***A1N4***

> **Note:** Full marks can be awarded for answers obtained directly from GDC if a systematic method is used and clearly shown.

[6 marks]

(d) (i) P(1 or 2 accidents) = 0.37908… ***A1***

$\mathrm{E}(B)=1000\times 0.60653...+500\times 0.37908...$ ***M1A1***

$=\$796$ (accept \$797 or \$796.07) ***A1***

(ii) $\mathrm{P}(2000)=\mathrm{P}(1000, 1000, 0)+\mathrm{P}(1000, 0, 1000)+\mathrm{P}(0, 1000, 1000)+$
$\mathrm{P}(1000, 500, 500)+\mathrm{P}(500, 1000, 500)+\mathrm{P}(500, 500, 1000)$ ***(M1)(A1)***

> **Note:** Award ***M1*** for noting that 2000 can be written both as $2\times 1000+1\times 0$ and $2\times 500+1\times 1000$.

$=3(0.6065...)^2(0.01437...)+3(0.3790...)^2(0.6065...)$ ***M1A1***

$=0.277$ (accept 0.278) ***A1***

[9 marks]

Total [18 marks]

13. Part A

prove that $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+...+n\left(\frac{1}{2}\right)^{n-1}=4-\frac{n+2}{2^{n-1}}$

for $n=1$

$\text{LHS}=1,\ \text{RHS}=4-\frac{1+2}{2^0}=4-3=1$

so true for $n=1$ ***R1***

assume true for $n=k$ ***M1***

so $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+...+k\left(\frac{1}{2}\right)^{k-1}=4-\frac{k+2}{2^{k-1}}$

now for $n=k+1$

LHS: $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)^3+...+k\left(\frac{1}{2}\right)^{k-1}+(k+1)\left(\frac{1}{2}\right)^k$ ***A1***

$=4-\frac{k+2}{2^{k-1}}+(k+1)\left(\frac{1}{2}\right)^k$ ***M1A1***

$=4-\frac{2(k+2)}{2^k}+\frac{k+1}{2^k}$ (or equivalent) ***A1***

$=4-\frac{(k+1)+2}{2^{(k+1)-1}}$ (accept $4-\frac{k+3}{2^k}$) ***A1***

Therefore if it is true for $n=k$ it is true for $n=k+1$. It has been shown to be true for $n=1$ so it is true for all n ($\in \mathbb{Z}^+$). ***R1***

Note: To obtain the final ***R*** mark, a reasonable attempt at induction must have been made.

[8 marks]

Part B

(a) **METHOD 1**

$\int e^{2x}\sin x\,dx=-\cos xe^{2x}+\int 2e^{2x}\cos x\,dx$ ***M1A1A1***

$=-\cos xe^{2x}+2e^{2x}\sin x-\int 4e^{2x}\sin x\,dx$ ***A1A1***

$5\int e^{2x}\sin x\,dx=-\cos xe^{2x}+2e^{2x}\sin x$ ***M1***

$\int e^{2x}\sin x\,dx=\frac{1}{5}e^{2x}(2\sin x-\cos x)+C$ ***AG***

METHOD 2

$\int \sin xe^{2x}\,dx=\frac{\sin xe^{2x}}{2}-\int \cos x\frac{e^{2x}}{2}dx$ ***M1A1A1***

$=\frac{\sin xe^{2x}}{2}-\cos x\frac{e^{2x}}{4}-\int \sin x\frac{e^{2x}}{4}dx$ ***A1A1***

$\frac{5}{4}\int e^{2x}\sin x\,dx=\frac{e^{2x}\sin x}{2}-\frac{\cos xe^{2x}}{4}$ ***M1***

$\int e^{2x}\sin x\,dx=\frac{1}{5}e^{2x}(2\sin x-\cos x)+C$ ***AG***

[6 marks]

continued …

Question 13 continued

(b) $\int \frac{dy}{\sqrt{1-y^2}} = \int e^{2x} \sin x \, dx$ ***M1A1***

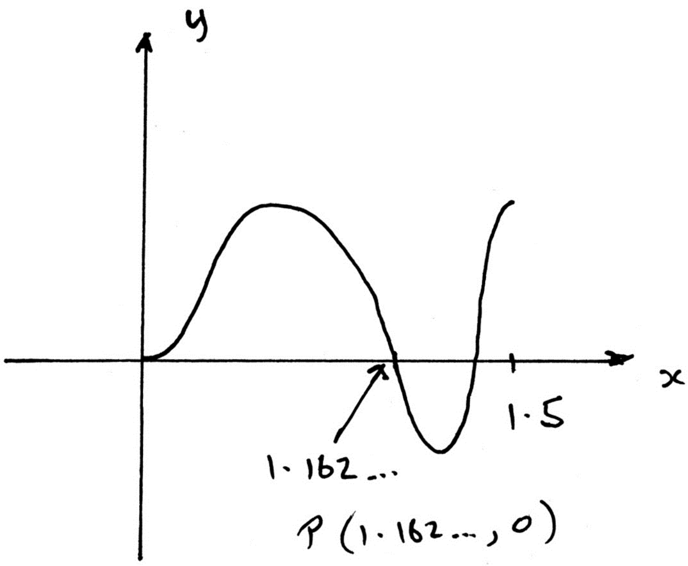
$\arcsin y = \frac{1}{5} e^{2x} (2\sin x - \cos x)(+C)$ ***A1***

when $x = 0,\ y = 0 \Rightarrow C = \frac{1}{5}$ ***M1***

$y = \sin\left(\frac{1}{5} e^{2x} (2\sin x - \cos x) + \frac{1}{5}\right)$ ***A1***

[5 marks]

(c) (i)



A1

P is $(1.16,\ 0)$ ***A1***

Note: Award ***A1*** for 1.16 seen anywhere, ***A1*** for complete sketch.

Note: Allow FT on their answer from (b)

(ii) $V = \int_0^{1.162...} \pi y^2 \, dx$ ***M1A1***

$= 1.05$ ***A2***

Note: Allow FT on their answers from (b) and (c)(i).

[6 marks]

Total [25 marks]