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MARKSCHEME

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MATHEMATICS

Higher Level

Paper 1

17 pages

SECTION A

1. **EITHER**

$|x-1| > |2x-1| \Rightarrow (x-1)^2 > (2x-1)^2$ ***M1***

$x^2 - 2x + 1 > 4x^2 - 4x + 1$

$3x^2 - 2x < 0$ ***A1***

$0 < x < \frac{2}{3}$ ***A1A1*** ***N2***

Note: Award ***A1A0*** for incorrect inequality signs.

OR

$|x-1| > |2x-1|$

$x - 1 = 2x - 1$ $\quad$ $x - 1 = 1 - 2x$ ***M1A1***

$-x = 0$ $\quad$ $3x = 2$

$x = 0$ $\quad$ $x = \frac{2}{3}$

Note: Award ***M1*** for any attempt to find a critical value. If graphical methods are used, award ***M1*** for correct graphs, ***A1*** for correct values of x.

$0 < x < \frac{2}{3}$ ***A1A1*** ***N2***

Note: Award ***A1A0*** for incorrect inequality signs.

[4 marks]

2. $\det\begin{pmatrix} k & 1 & 1 \\ 0 & 2 & k-1 \\ k & 0 & k-2 \end{pmatrix} = k\begin{vmatrix} 2 & k-1 \\ 0 & k-2 \end{vmatrix} - \begin{vmatrix} 0 & k-1 \\ k & k-2 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ k & 0 \end{vmatrix}$ ***(M1)***

$= 2k(k-2) + k(k-1) - 2k$ ***A1***

Note: Allow expansion about any row or column.

$2k(k-2) + k(k-1) - 2k = 0$ ***M1***

$3k^2 - 7k = 0$

$k(3k-7) = 0$

$k = 0$ or $k = \frac{7}{3}$ ***A1A1*** ***N2***

[5 marks]

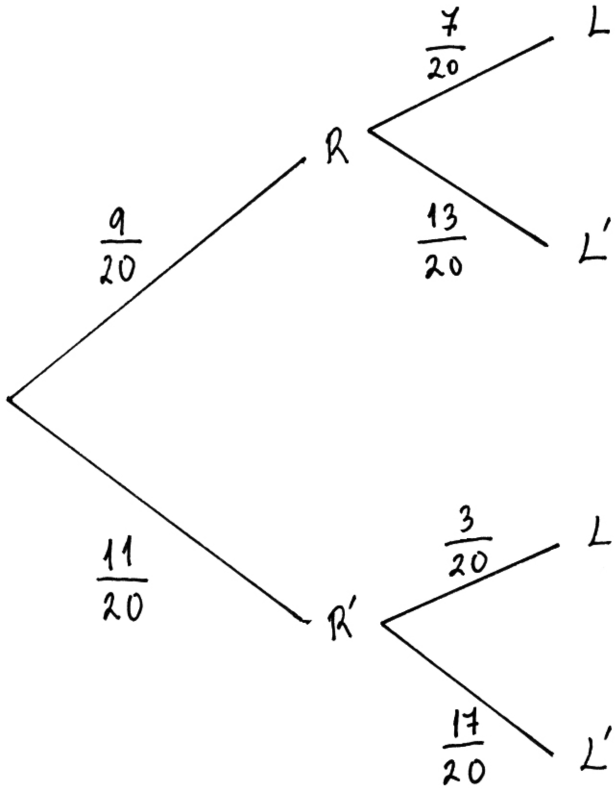
3. $\left(x^2 - \frac{2}{x}\right)^4 = (x^2)^4 + 4(x^2)^3\left(-\frac{2}{x}\right) + 6(x^2)^2\left(-\frac{2}{x}\right)^2 + 4(x^2)\left(-\frac{2}{x}\right)^3 + \left(-\frac{2}{x}\right)^4$ ***(M1)***

$= x^8 - 8x^5 + 24x^2 - \frac{32}{x} + \frac{16}{x^4}$ ***A3***

Note: Deduct one ***A*** mark for each incorrect or omitted term.

[4 marks]

4.



(A1)

$\mathrm{P}(R' \cap L) = \frac{11}{20} \times \frac{3}{20}$ ***A1***

$\mathrm{P}(L) = \frac{9}{20} \times \frac{7}{20} + \frac{11}{20} \times \frac{3}{20}$ ***A1***

$\mathrm{P}(R' \mid L) = \frac{\mathrm{P}(R' \cap L)}{\mathrm{P}(L)}$ ***(M1)***

$= \frac{33}{96} \left(= \frac{11}{32}\right)$ ***A1***

[5 marks]

5. **METHOD 1**

$5(2a+9d)=60$ (or $2a+9d=12$) ***M1A1***

$10(2a+19d)=320$ (or $2a+19d=32$) ***A1***

solve simultaneously to obtain ***M1***

$a=-3$, $d=2$ ***A1***

the 15^{th} term is $-3+14\times 2=25$ ***A1***

Note: ***FT*** the final ***A1*** on the values found in the penultimate line.

METHOD 2

with an AP the mean of an even number of consecutive terms equals the mean of the middle terms ***(M1)***

$\frac{a_{10}+a_{11}}{2}=16$ (or $a_{10}+a_{11}=32$) ***A1***

$\frac{a_5+a_6}{2}=6$ (or $a_5+a_6=12$) ***A1***

$a_{10}-a_5+a_{11}-a_6=20$ ***M1***

$5d+5d=20$

$d=2$ and $a=-3$ (or $a_5=5$ or $a_{10}=15$) ***A1***

the 15^{th} term is $-3+14\times 2=25$ (or $5+10\times 2=25$ or $15+5\times 2=25$) ***A1***

Note: ***FT*** the final ***A1*** on the values found in the penultimate line.

[6 marks]

6. METHOD 1

(a) $u_n = S_n - S_{n-1}$ *(M1)*

$= \frac{7^n - a^n}{7^n} - \frac{7^{n-1} - a^{n-1}}{7^{n-1}}$ ***A1***

(b) **EITHER**

$u_1 = 1 - \frac{a}{7}$ ***A1***

$u_2 = 1 - \frac{a^2}{7^2} - \left(1 - \frac{a}{7}\right)$ ***M1***

$= \frac{a}{7}\left(1 - \frac{a}{7}\right)$ ***A1***

common ratio $= \frac{a}{7}$ ***A1***

OR

$u_n = 1 - \left(\frac{a}{7}\right)^n - 1 + \left(\frac{a}{7}\right)^{n-1}$ ***M1***

$= \left(\frac{a}{7}\right)^{n-1}\left(1 - \frac{a}{7}\right)$

$= \frac{7-a}{7}\left(\frac{a}{7}\right)^{n-1}$ ***A1***

$u_1 = \frac{7-a}{7}$, common ratio $= \frac{a}{7}$ ***A1A1***

(c) (i) $0 < a < 7$ (accept $a < 7$) ***A1***

(ii) 1 ***A1***

[8 marks]

continued …

Question 6 continued

METHOD 2

(a) $u_n = br^{n-1} = \left(\frac{7-a}{7}\right)\left(\frac{a}{7}\right)^{n-1}$ ***A1A1***

(b) for a GP with first term b and common ratio r

$S_n = \frac{b(1-r^n)}{1-r} = \left(\frac{b}{1-r}\right) - \left(\frac{b}{1-r}\right)r^n$ ***M1***

as $S_n = \frac{7^n - a^n}{7^n} = 1 - \left(\frac{a}{7}\right)^n$

comparing both expressions ***M1***

$\frac{b}{1-r} = 1$ and $r = \frac{a}{7}$

$b = 1 - \frac{a}{7} = \frac{7-a}{7}$

$u_1 = b = \frac{7-a}{7}$, common ratio $= r = \frac{a}{7}$ ***A1A1***

Note: Award method marks if the expressions for b and r are deduced in part (a).

(c) (i) $0 < a < 7$ (accept $a < 7$) ***A1***

(ii) 1 ***A1***

[8 marks]

7. (a) $\boldsymbol{a}=\begin{pmatrix}4\\-2\\-1\end{pmatrix}\perp$ to the plane $\boldsymbol{e}=\begin{pmatrix}-2\\1\\k\end{pmatrix}$ is parallel to the line ***(A1)(A1)***

Note: Award ***A1*** for each correct vector written down, even if not identified.

line $\perp$ plane $\Rightarrow$ $\boldsymbol{e}$ parallel to $\boldsymbol{a}$

since $\begin{pmatrix}4\\-2\\-1\end{pmatrix}=t\begin{pmatrix}-2\\1\\k\end{pmatrix}\Rightarrow k=\frac{1}{2}$ ***(M1)A1***

(b) $4(3-2\lambda)-2\lambda-\left(-1+\frac{1}{2}\lambda\right)=1$ ***(M1)(A1)***

Note: ***FT*** their value of k as far as possible.

$\lambda=\frac{8}{7}$ ***A1***

$\mathrm{P}\left(\frac{5}{7},\frac{8}{7},-\frac{3}{7}\right)$ ***A1***

[8 marks]

8. $(1+x^3)\frac{\mathrm{d}y}{\mathrm{d}x}=2x^2\tan y\Rightarrow\int\frac{\mathrm{d}y}{\tan y}=\int\frac{2x^2}{1+x^3}\mathrm{d}x$ ***M1***

$\int\frac{\cos y}{\sin y}\mathrm{d}y=\frac{2}{3}\int\frac{3x^2}{1+x^3}\mathrm{d}x$ ***(A1)(A1)***

$\ln\left|\sin y\right|=\frac{2}{3}\ln\left|1+x^3\right|+C$ ***A1A1***

Notes: Do not penalize omission of modulus signs. Do not penalize omission of constant at this stage.

EITHER

$\ln\left|\sin\frac{\pi}{2}\right|=\frac{2}{3}\ln\left|1\right|+C\Rightarrow C=0$ ***M1***

OR

$\left|\sin y\right|=A\left|1+x^3\right|^{\frac{2}{3}},\ A=\mathrm{e}^C$

$\left|\sin\frac{\pi}{2}\right|=A\left|1+0^3\right|^{\frac{2}{3}}\Rightarrow A=1$ ***M1***

THEN

$y=\arcsin\left((1+x^3)^{\frac{2}{3}}\right)$ ***A1***

Note: Award ***M0A0*** if constant omitted earlier.

[7 marks]

9. (a) $\frac{\pi}{4} - \arccos x \geq 0$

$\arccos x \leq \frac{\pi}{4}$ ***(M1)***

$x \geq \frac{\sqrt{2}}{2}$ $\left(\text{accept } x \geq \frac{1}{\sqrt{2}}\right)$ ***(A1)***

since $-1 \leq x \leq 1$ ***(M1)***

$\Rightarrow \frac{\sqrt{2}}{2} \leq x \leq 1$ $\left(\text{accept } \frac{1}{\sqrt{2}} \leq x \leq 1\right)$ ***A1***

Note: Penalize the use of $<$ instead of $\leq$ only once.

(b) $y = \sqrt{\frac{\pi}{4} - \arccos x} \Rightarrow x = \cos\left(\frac{\pi}{4} - y^2\right)$ ***M1A1***

$f^{-1} : x \to \cos\left(\frac{\pi}{4} - x^2\right)$ ***A1***

$0 \leq x \leq \sqrt{\frac{\pi}{4}}$ ***A1***

[8 marks]

10. **METHOD 1**

(a) $|\boldsymbol{a} - \boldsymbol{b}| = \sqrt{|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}||\boldsymbol{b}|\cos\alpha}$ ***M1***

$= \sqrt{2 - 2\cos\alpha}$ ***A1***

$|\boldsymbol{a} + \boldsymbol{b}| = \sqrt{|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}||\boldsymbol{b}|\cos(\pi - \alpha)}$

$= \sqrt{2 + 2\cos\alpha}$ ***A1***

Note: Accept the use of a, b for $|\boldsymbol{a}|, |\boldsymbol{b}|$.

(b) $\sqrt{2 + 2\cos\alpha} = 3\sqrt{2 - 2\cos\alpha}$ ***M1***

$\cos\alpha = \frac{4}{5}$ ***A1***

METHOD 2

(a) $|\boldsymbol{a} - \boldsymbol{b}| = 2\sin\frac{\alpha}{2}$ ***M1A1***

$|\boldsymbol{a} + \boldsymbol{b}| = 2\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) = 2\cos\frac{\alpha}{2}$ ***A1***

Note: Accept the use of a, b for $|\boldsymbol{a}|, |\boldsymbol{b}|$.

(b) $2\cos\frac{\alpha}{2} = 6\sin\frac{\alpha}{2}$

$\tan\frac{\alpha}{2} = \frac{1}{3} \Rightarrow \cos^2\frac{\alpha}{2} = \frac{9}{10}$ ***M1***

$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 = \frac{4}{5}$ ***A1***

[5 marks]

SECTION B

11. (a) **METHOD 1**

$\frac{z+\mathrm{i}}{z+2}=\mathrm{i}$

$z+\mathrm{i}=\mathrm{i}z+2\mathrm{i}$ ***M1***

$(1-\mathrm{i})z=\mathrm{i}$ ***A1***

$z=\frac{\mathrm{i}}{1-\mathrm{i}}$ ***A1***

EITHER

$z=\frac{\mathrm{cis}\left(\frac{\pi}{2}\right)}{\sqrt{2}\,\mathrm{cis}\left(\frac{3\pi}{4}\right)}$ ***M1***

$z=\frac{\sqrt{2}}{2}\mathrm{cis}\left(\frac{3\pi}{4}\right)$ $\left(\text{or } \frac{1}{\sqrt{2}}\mathrm{cis}\left(\frac{3\pi}{4}\right)\right)$ ***A1A1***

OR

$z=\frac{-1+\mathrm{i}}{2}$ $\left(=-\frac{1}{2}+\frac{1}{2}\mathrm{i}\right)$ ***M1***

$z=\frac{\sqrt{2}}{2}\mathrm{cis}\left(\frac{3\pi}{4}\right)$ $\left(\text{or } \frac{1}{\sqrt{2}}\mathrm{cis}\left(\frac{3\pi}{4}\right)\right)$ ***A1A1***

[6 marks]

METHOD 2

$\mathrm{i}=\frac{x+\mathrm{i}(y+1)}{x+2+\mathrm{i}y}$ ***M1***

$x+\mathrm{i}(y+1)=-y+\mathrm{i}(x+2)$ ***A1***

$x=-y;\ x+2=y+1$ ***A1***

solving, $x=-\frac{1}{2};\ y=\frac{1}{2}$ ***A1***

$z=-\frac{1}{2}+\frac{1}{2}\mathrm{i}$

$z=\frac{\sqrt{2}}{2}\mathrm{cis}\left(\frac{3\pi}{4}\right)$ $\left(\text{or } \frac{1}{\sqrt{2}}\mathrm{cis}\left(\frac{3\pi}{4}\right)\right)$ ***A1A1***

> **Note:** Award ***A1*** fort the correct modulus and ***A1*** for the correct argument, but the final answer must be in the form $r\,\mathrm{cis}\,\theta$. Accept 135° for the argument.

[6 marks]

continued ...

Question 11 continued

(b) substituting $z = x + \mathrm{i}y$ to obtain $w = \frac{x + (y+1)\mathrm{i}}{(x+2) + y\mathrm{i}}$ ***(A1)***

use of $(x+2) - y\mathrm{i}$ to rationalize the denominator ***M1***

$$\omega = \frac{x(x+2) + y(y+1) + \mathrm{i}\left(-xy + (y+1)(x+2)\right)}{(x+2)^2 + y^2}$$ ***A1***

$$= \frac{(x^2 + 2x + y^2 + y) + \mathrm{i}(x + 2y + 2)}{(x+2)^2 + y^2}$$ ***AG***

[3 marks]

(c) $\mathrm{Re}\,\omega = \frac{x^2 + 2x + y^2 + y}{(x+2)^2 + y^2} = 1$ ***M1***

$\Rightarrow x^2 + 2x + y^2 + y = x^2 + 4x + 4 + y^2$ ***A1***

$\Rightarrow y = 2x + 4$ ***A1***

which has gradient $m = 2$ ***A1***

[4 marks]

(d) **EITHER**

$\arg(z) = \frac{\pi}{4} \Rightarrow x = y$ (and $x,\ y > 0$) ***(A1)***

$$\omega = \frac{2x^2 + 3x}{(x+2)^2 + x^2} + \frac{\mathrm{i}(3x+2)}{(x+2)^2 + x^2}$$

if $\arg(\omega) = \theta \Rightarrow \tan\theta = \frac{3x+2}{2x^2 + 3x}$ ***(M1)***

$\frac{3x+2}{2x^2+3x} = 1$ ***M1A1***

OR

$\arg(z) = \frac{\pi}{4} \Rightarrow x = y$ (and $x,\ y > 0$) ***A1***

$\arg(w) = \frac{\pi}{4} \Rightarrow x^2 + 2x + y^2 + y = x + 2y + 2$ ***M1***

solve simultaneously ***M1***

$x^2 + 2x + x^2 + x = x + 2x + 2$ (or equivalent) ***A1***

THEN

$x^2 = 1$

$x = 1$ (as $x > 0$) ***A1***

Note: Award ***A0*** for $x = \pm 1$.

$|z| = \sqrt{2}$ ***A1***

Note: Allow ***FT*** from incorrect values of x.

[6 marks]

Total [19 marks]

12. (a) (i) the period is 2 ***A1***

(ii) $v = \frac{\mathrm{d}s}{\mathrm{d}t} = 2\pi\cos(\pi t) + 2\pi\cos(2\pi t)$ ***(M1)A1***

$a = \frac{\mathrm{d}v}{\mathrm{d}t} = -2\pi^2\sin(\pi t) - 4\pi^2\sin(2\pi t)$ ***(M1)A1***

(iii) $v = 0$

$2\pi\left(\cos(\pi t) + \cos(2\pi t)\right) = 0$

EITHER

$\cos(\pi t) + 2\cos^2(\pi t) - 1 = 0$ ***M1***

$\left(2\cos(\pi t) - 1\right)\left(\cos(\pi t) + 1\right) = 0$ ***(A1)***

$\cos(\pi t) = \frac{1}{2}$ or $\cos(\pi t) = -1$ ***A1***

$t = \frac{1}{3},\ t = 1$ ***A1***

$t = \frac{5}{3},\ t = \frac{7}{3},\ t = \frac{11}{3},\ t = 3$ ***A1***

OR

$2\cos\left(\frac{\pi t}{2}\right)\cos\left(\frac{3\pi t}{2}\right) = 0$ ***M1***

$\cos\left(\frac{\pi t}{2}\right) = 0$ or $\cos\left(\frac{3\pi t}{2}\right) = 0$ ***A1A1***

$t = \frac{1}{3}, 1$ ***A1***

$t = \frac{5}{3}, \frac{7}{3}, 3, \frac{11}{3}$ ***A1***

[10 marks]

continued …

Question 12 continued

(b) $P(n):\ f^{(2n)}(x) = (-1)^n \left(Aa^{2n}\sin(ax) + Bb^{2n}\sin(bx) \right)$

$P(1):\ f''(x) = \left(Aa\cos(ax) + Bb\cos(bx) \right)'$ ***M1***

$= -Aa^2\sin(ax) - Bb^2\sin(bx)$

$= -1\left(Aa^2\sin(ax) + Bb^2\sin(bx) \right)$ ***A1***

$\therefore P(1)$ true

assume that

$P(k): f^{(2k)}(x) = (-1)^k \left(Aa^{2k}\sin(ax) + Bb^{2k}\sin(bx) \right)$ is true ***M1***

consider $P(k+1)$

$f^{(2k+1)}(x) = (-1)^k \left(Aa^{2k+1}\cos(ax) + Bb^{2k+1}\cos(bx) \right)$ ***M1A1***

$f^{(2k+2)}(x) = (-1)^k \left(-Aa^{2k+2}\sin(ax) - Bb^{2k+2}\sin(bx) \right)$ ***A1***

$= (-1)^{k+1} \left(Aa^{2k+2}\sin(ax) + Bb^{2k+2}\sin(bx) \right)$ ***A1***

$P(k)$ true implies $P(k+1)$ true, $P(1)$ true so $P(n)$ true $\forall n \in \mathbb{Z}^+$ ***R1***

Note: Award the final ***R1*** only if the previous three ***M*** marks have been awarded.

[8 marks]

Total [18 marks]

13. (a) (i) $xe^x = 0 \Rightarrow x = 0$ *A1*

so, they intersect only once at $(0, 0)$

(ii) $y' = e^x + xe^x = (1+x)e^x$ *M1A1*

$y'(0) = 1$ *A1*

$\theta = \arctan 1 = \frac{\pi}{4}$ $(\theta = 45^\circ)$ *A1*

[5 marks]

(b) when $k = 1$, $y = x$

$xe^x = x \Rightarrow x(e^x - 1) = 0$ *M1*

$\Rightarrow x = 0$ *A1*

$y'(0) = 1$ which equals the gradient of the line $y = x$ *R1*

so, the line is tangent to the curve at origin *AG*

Note: Award full credit to candidates who note that the equation $x(e^x - 1) = 0$ has a double root $x = 0$ so $y = x$ is a tangent.

[3 marks]

(c) (i) $xe^x = kx \Rightarrow x(e^x - k) = 0$ *M1*

$\Rightarrow x = 0$ or $x = \ln k$ *A1*

$k > 0$ and $k \neq 1$ *A1*

(ii) $(0, 0)$ and $(\ln k, k \ln k)$ *A1A1*

(iii) $A = \left| \int_0^{\ln k} kx - xe^x \, dx \right|$ *M1A1*

Note: Do not penalize the omission of absolute value.

(iv) attempt at integration by parts to find $\int xe^x \, dx$ *M1*

$\int xe^x \, dx = xe^x - \int e^x \, dx = e^x(x-1)$ *A1*

as $0 < k < 1 \Rightarrow \ln k < 0$ *R1*

$A = \int_{\ln k}^{0} kx - xe^x \, dx = \left[\frac{k}{2}x^2 - (x-1)e^x \right]_{\ln k}^{0}$ *A1*

$= 1 - \left(\frac{k}{2}(\ln k)^2 - (\ln k - 1)k \right)$ *A1*

$= 1 - \frac{k}{2}\left((\ln k)^2 - 2\ln k + 2 \right)$

$= 1 - \frac{k}{2}\left((\ln k - 1)^2 + 1 \right)$ *M1A1*

since $\frac{k}{2}\left((\ln k - 1)^2 + 1 \right) > 0$ *R1*

$A < 1$ *AG*

[15 marks]

Total [23 marks]