88107201

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 1

Thursday 4 November 2010 (afternoon)

2 hours

Candidate session number

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INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your session number on each answer sheet, and attach them to this examination paper and your cover sheet using the tag provided.
- At the end of the examination, indicate the number of sheets used in the appropriate box on your cover sheet.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** the questions in the spaces provided. Working may be continued below the lines, if necessary.*

1. *[Maximum mark: 4]*

Find the set of values of x for which $|x-1|>|2x-1|$.

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2. *[Maximum mark: 5]*

Consider the matrix $\begin{pmatrix} k & 1 & 1 \\ 0 & 2 & k-1 \\ k & 0 & k-2 \end{pmatrix}$.

Find all possible values of k for which the matrix is singular.

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3. *[Maximum mark: 4]*

Expand and simplify $\left(x^2 - \dfrac{2}{x}\right)^4$.

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4. *[Maximum mark: 5]*

Jenny goes to school by bus every day. When it is not raining, the probability that the bus is late is $\frac{3}{20}$. When it is raining, the probability that the bus is late is $\frac{7}{20}$. The probability that it rains on a particular day is $\frac{9}{20}$. On one particular day the bus is late. Find the probability that it is not raining on that day.

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5. *[Maximum mark: 6]*

The mean of the first ten terms of an arithmetic sequence is 6. The mean of the first twenty terms of the arithmetic sequence is 16. Find the value of the 15^{th} term of the sequence.

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6. *[Maximum mark: 8]*

The sum, S_n, of the first n terms of a geometric sequence, whose n^{th} term is u_n, is given by

$$S_n = \frac{7^n - a^n}{7^n}, \text{ where } a > 0.$$

(a) Find an expression for u_n. *[2 marks]*

(b) Find the first term and common ratio of the sequence. *[4 marks]*

(c) Consider the sum to infinity of the sequence.

(i) Determine the values of a such that the sum to infinity exists.

(ii) Find the sum to infinity when it exists. *[2 marks]*

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7. *[Maximum mark: 8]*

Consider the plane with equation $4x - 2y - z = 1$ and the line given by the parametric equations

$$x = 3 - 2\lambda$$
$$y = (2k - 1) + \lambda$$
$$z = -1 + k\lambda .$$

Given that the line is perpendicular to the plane, find

(a) the value of k; *[4 marks]*

(b) the coordinates of the point of intersection of the line and the plane. *[4 marks]*

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8. *[Maximum mark: 7]*

Find y in terms of x, given that $(1+x^3)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x^2 \tan y$ and $y = \frac{\pi}{2}$ when $x = 0$.

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9. *[Maximum mark: 8]*

Consider the function $f : x \rightarrow \sqrt{\frac{\pi}{4} - \arccos x}$.

(a) Find the largest possible domain of f . *[4 marks]*

(b) Determine an expression for the inverse function, f^{-1}, and write down its domain. *[4 marks]*

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10. *[Maximum mark: 5]*

Let α be the angle between the unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, where $0 \le \alpha \le \pi$.

(a) Express $|\boldsymbol{a}-\boldsymbol{b}|$ and $|\boldsymbol{a}+\boldsymbol{b}|$ in terms of α. *[3 marks]*

(b) **Hence** determine the value of $\cos\alpha$ for which $|\boldsymbol{a}+\boldsymbol{b}| = 3|\boldsymbol{a}-\boldsymbol{b}|$. *[2 marks]*

*Do **NOT** write solutions on this page. Any working on this page will **NOT** be marked.*

SECTION B

*Answer **all** the questions on the answer sheets provided. Please start each question on a new page.*

11. *[Maximum mark: 19]*

Consider the complex number $\omega = \dfrac{z+\mathrm{i}}{z+2}$, where $z = x + \mathrm{i}y$ and $\mathrm{i} = \sqrt{-1}$.

(a) If $\omega = \mathrm{i}$, determine z in the form $z = r\,\mathrm{cis}\,\theta$. *[6 marks]*

(b) Prove that $\omega = \dfrac{(x^2+2x+y^2+y)+\mathrm{i}(x+2y+2)}{(x+2)^2+y^2}$. *[3 marks]*

(c) **Hence** show that when $\mathrm{Re}(\omega) = 1$ the points $(x,\ y)$ lie on a straight line, l_1, and write down its gradient. *[4 marks]*

(d) Given $\arg(z) = \arg(\omega) = \dfrac{\pi}{4}$, find $|z|$. *[6 marks]*

12. *[Maximum mark: 18]*

(a) A particle P moves in a straight line with displacement relative to origin given by

$$s = 2\sin(\pi t) + \sin(2\pi t),\ t \geq 0,$$

where t is the time in seconds and the displacement is measured in centimetres.

(i) Write down the period of the function s.

(ii) Find expressions for the velocity, v, and the acceleration, a, of P.

(iii) Determine all the solutions of the equation $v = 0$ for $0 \leq t \leq 4$. *[10 marks]*

(b) Consider the function

$$f(x) = A\sin(ax) + B\sin(bx),\ A,\ a,\ B,\ b,\ x \in \mathbb{R}.$$

Use mathematical induction to prove that the $(2n)^{\text{th}}$ derivative of f is given by $f^{(2n)}(x) = (-1)^n \left(Aa^{2n}\sin(ax) + Bb^{2n}\sin(bx)\right)$, for all $n \in \mathbb{Z}^+$. *[8 marks]*

*Do **NOT** write solutions on this page. Any working on this page will **NOT** be marked.*

13. *[Maximum mark: 23]*

Consider the curve $y = xe^x$ and the line $y = kx$, $k \in \mathbb{R}$.

(a) Let $k = 0$.

(i) Show that the curve and the line intersect once.

(ii) Find the angle between the tangent to the curve and the line at the point of intersection. *[5 marks]*

(b) Let $k = 1$. Show that the line is a tangent to the curve. *[3 marks]*

(c) (i) Find the values of k for which the curve $y = xe^x$ and the line $y = kx$ meet in two distinct points.

(ii) Write down the coordinates of the points of intersection.

(iii) Write down an integral representing the area of the region A enclosed by the curve and the line.

(iv) **Hence**, given that $0 < k < 1$, show that $A < 1$. *[15 marks]*