International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

MARKSCHEME

November 2010

MATHEMATICS

Higher Level

Paper 2

17 pages

SECTION A

1. (a) area $= \frac{1}{2} \times \text{BC} \times \text{AB} \times \sin B$ ***(M1)***

$\left(10 = \frac{1}{2} \times 5 \times 6 \times \sin B\right)$

$\sin \hat{B} = \frac{2}{3}$ ***A1***

(b) $\cos B = \pm \frac{\sqrt{5}}{3}$ $(= \pm 0.7453...)$ or $B = 41.8...$ and $138.1...$ ***(A1)***

$\text{AC}^2 = \text{BC}^2 + \text{AB}^2 - 2 \times \text{BC} \times \text{AB} \times \cos B$ ***(M1)***

$\text{AC} = \sqrt{5^2 + 6^2 - 2 \times 5 \times 6 \times 0.7453...}$ or $\sqrt{5^2 + 6^2 + 2 \times 5 \times 6 \times 0.7453...}$

$\text{AC} = 4.03$ or 10.28 ***A1A1***

[6 marks]

2. $\overline{m} = \frac{6.7 + 7.2 + ... + 7.3}{10} = 6.91$ ***(M1)A1***

$s_{n-1}^2 = \frac{1}{9}\left((6.7 - 6.91)^2 + ... + (7.3 - 6.91)^2\right)$ ***(M1)***

$= \frac{0.489}{9} = 0.0543$ (3 sf) ***A1***

Note: Award ***M1A0*** for 0.233.

[4 marks]

3. $X \sim \text{N}(\mu, \sigma^2)$

$\text{P}(X \le 5) = 0.670 \Leftrightarrow \frac{5 - \mu}{\sigma} = 0.4399\ldots$ ***M1A1***

$\text{P}(X > 7) = 0.124 \Leftrightarrow \frac{7 - \mu}{\sigma} = 1.155\ldots$ ***A1***

solve simultaneously

$\mu + 0.4399\sigma = 5$ and $\mu + 1.1552\sigma = 7$ ***M1***

$\mu = 3.77$ (3 sf) ***A1*** ***N3***

the expected weight loss is 3.77 kg

Note: Award ***A0*** for $\mu = 3.78$ (answer obtained due to early rounding).

[5 marks]

4. $x^3y^3 - xy = 0$

$3x^2y^3 + 3x^3y^2y' - y - xy' = 0$ ***M1A1A1***

Note: Award ***A1*** for correctly differentiating each term.

$x = 1$, $y = 1$ $\quad 3 + 3y' - 1 - y' = 0$

$2y' = -2$

$y' = -1$ ***(M1)A1***

gradient of normal $= 1$ ***(A1)***

equation of the normal $y - 1 = x - 1$ ***A1*** ***N2***

$y = x$

Note: Award ***A2R5*** for correct answer and correct justification.

[7 marks]

5. **EITHER**

$$\begin{cases} \ln\frac{x}{y} = 1 \\ \ln x^3 + \ln y^2 = 5 \end{cases} \Leftrightarrow \begin{cases} \ln x - \ln y = 1 \\ 3\ln x + 2\ln y = 5 \end{cases}$$ ***M1A1***

solve simultaneously ***M1***

$$\begin{cases} \ln x = \frac{7}{5} \\ \ln y = \frac{2}{5} \end{cases}$$

$x = e^{\frac{7}{5}}$ $(= 4.06)$ and $y = e^{\frac{2}{5}}$ $(= 1.49)$ ***A1A1***

OR

$\ln\frac{x}{y} = 1$

$\Rightarrow x = ey$ ***A1***

$\ln x^3 + \ln y^2 = 5$

$\ln x^3y^2 = 5$

$x^3y^2 = e^5$ ***M1***

$e^3y^5 = e^5$

$y^5 = e^2$ ***M1***

$y = e^{\frac{2}{5}}$, $x = e^{\frac{7}{5}}$ ***A1A1***

[5 marks]

6. **METHOD 1**

$1+\mathrm{i}$ is a zero $\Rightarrow$ $1-\mathrm{i}$ is a zero *(A1)*

$1-2\mathrm{i}$ is a zero $\Rightarrow$ $1+2\mathrm{i}$ is a zero *(A1)*

$\left(x-(1-\mathrm{i})\right)\left(x-(1+\mathrm{i})\right)=(x^2-2x+2)$ *(M1)A1*

$\left(x-(1-2\mathrm{i})\right)\left(x-(1+2\mathrm{i})\right)=(x^2-2x+5)$ *A1*

$p(x)=(x^2-2x+2)(x^2-2x+5)$ *M1*

$\quad =x^4-4x^3+11x^2-14x+10$ *A1*

$a=-4,\ b=11,\ c=-14,\ d=10$

[7 marks]

METHOD 2

$p(1+\mathrm{i})=-4+(-2+2\mathrm{i})a+(2\mathrm{i})b+(1+\mathrm{i})c+d$ *M1*

$p(1+\mathrm{i})=0\Rightarrow\begin{cases}-4-2a+c+d=0\\ 2a+2b+c=0\end{cases}$ *M1A1A1*

$p(1-2\mathrm{i})=-7+24\mathrm{i}+(-11+2\mathrm{i})a+(-3-4\mathrm{i})b+(1-2\mathrm{i})c+d$

$p(1-2\mathrm{i})=0\Rightarrow\begin{cases}-7-11a-3b+c+d=0\\ 24+2a-4b-2c=0\end{cases}$ *A1*

$$\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}-2&0&1&1\\2&2&1&0\\-11&-3&1&1\\2&-4&-2&0\end{pmatrix}^{-1}\begin{pmatrix}4\\0\\7\\-24\end{pmatrix}=\begin{pmatrix}-4\\11\\-14\\10\end{pmatrix}$$ *M1A1*

$a=-4,\ b=11,\ c=-14,\ d=10$

[7 marks]

7. (a) $\mathrm{P}(X=1)+\mathrm{P}(X=3)=\mathrm{P}(X=0)+\mathrm{P}(X=2)$

$m\mathrm{e}^{-m}+\dfrac{m^3\mathrm{e}^{-m}}{6}=\mathrm{e}^{-m}+\dfrac{m^2\mathrm{e}^{-m}}{2}$ *(M1)(A1)*

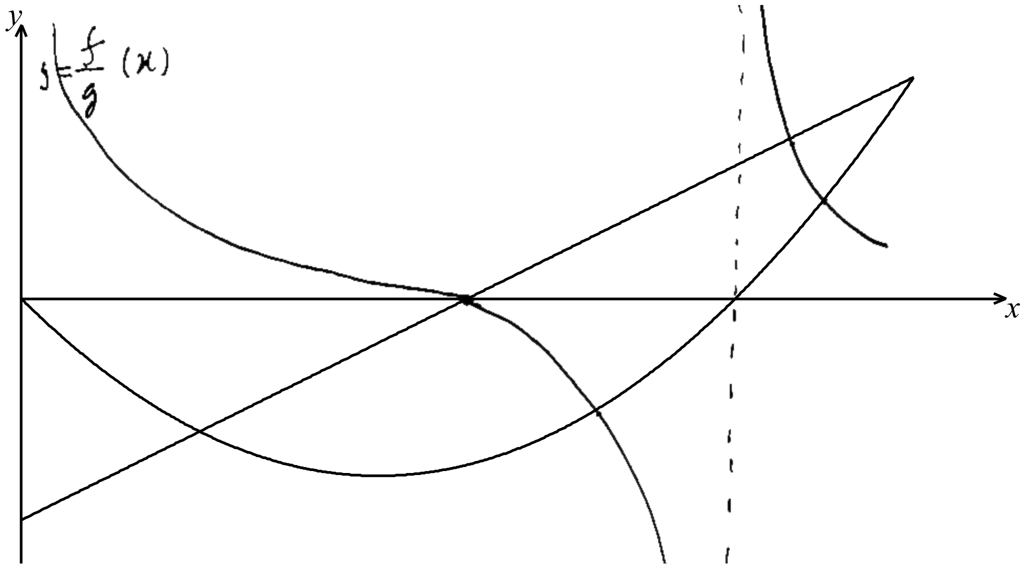
$m^3-3m^2+6m-6=0$ *(M1)*

$m=1.5961$ (4 decimal places) *A1*

(b) $m=1.5961\Rightarrow \mathrm{P}(1\le X\le 2)=m\mathrm{e}^{-m}+\dfrac{m^2\mathrm{e}^{-m}}{2}=0.582$ *(M1)A1*

[6 marks]

8.



correct concavities *A1A1*

Note: Award ***A1*** for concavity of each branch of the curve.

correct x-intercept of $\frac{f}{g}$ (which is EXACTLY the x-intercept of f) *A1*

correct vertical asymptotes of $\frac{f}{g}$ (which ONLY occur when x equals the x-intercepts of g) *A1A1*

[5 marks]

9. (a) $\cos\theta = \dfrac{\boldsymbol{ab}}{|\boldsymbol{a}||\boldsymbol{b}|} = \dfrac{\sin 2\alpha\cos\alpha + \sin\alpha\cos 2\alpha - 1}{\sqrt{2}\times\sqrt{2}} \quad \left(= \dfrac{\sin 3\alpha - 1}{2}\right)$ ***M1A1***

(b) $\boldsymbol{a}\perp\boldsymbol{b} \Rightarrow \cos\theta = 0$ ***M1***

$\sin 2\alpha\cos\alpha + \sin\alpha\cos 2\alpha - 1 = 0$

$\alpha = 0.524 \quad \left(= \dfrac{\pi}{6}\right)$ ***A1***

(c) **METHOD 1**

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \sin 2\alpha & -\cos 2\alpha & 1 \\ \cos\alpha & -\sin\alpha & -1 \end{vmatrix}$$ ***(M1)***

assuming $\alpha = \dfrac{7\pi}{6}$

Note: Allow substitution at any stage.

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 1 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & -1 \end{vmatrix}$$ ***A1***

$$= \boldsymbol{i}\left(\frac{1}{2} - \frac{1}{2}\right) - \boldsymbol{j}\left(-\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) + \boldsymbol{k}\left(\frac{\sqrt{3}}{2}\times\frac{1}{2} - \frac{1}{2}\times\frac{\sqrt{3}}{2}\right)$$

$= 0$ ***A1***

a and ***b*** are parallel ***R1***

Note: Accept decimal equivalents.

METHOD 2

from (a) $\cos\theta = -1$ (and $\sin\theta = 0$) ***M1A1***

$\boldsymbol{a}\times\boldsymbol{b} = 0$ ***A1***

a and ***b*** are parallel ***R1***

[8 marks]

10. **EITHER**

$y=\frac{1}{1-x}\Rightarrow y'=\frac{1}{(1-x)^2}$ ***M1A1***

solve simultaneously ***M1***

$\frac{1}{1-x}=m(x-m)$ and $\frac{1}{(1-x)^2}=m$

$\frac{1}{1-x}=\frac{1}{(1-x)^2}\left(x-\frac{1}{(1-x)^2}\right)$ ***A1***

Note: Accept equivalent forms.

$(1-x)^3-x(1-x)^2+1=0,\ x\neq 1$

$x=1.65729...\Rightarrow y=\frac{1}{1-1.65729...}=-1.521379...$

tangency point $(1.66,\ -1.52)$ ***A1A1***

$m=(-1.52137...)^2=2.31$ ***A1***

OR

$(1-x)\,y=1$

$m(1-x)(x-m)=1$ ***M1***

$m(x-x^2-m+mx)=1$

$mx^2-x(m+m^2)+(m^2+1)=0$ ***A1***

$b^2-4ac=0$ ***(M1)***

$(m+m^2)^2-4m(m^2+1)=0$

$m=2.31$ ***A1***

substituting $m=2.31...$ into $mx^2-x(m+m^2)+(m^2+1)=0$ ***(M1)***

$x=1.66$ ***A1***

$y=\frac{1}{1-1.65729}=-1.52$ ***A1***

tangency point $(1.66,\ -1.52)$

[7 marks]

SECTION B

11.

+	1	2	3
1	2	3	4
2	3	4	5
3	4	5	6

(a) let T be Tim's score

(i) $\mathrm{P}(T=6)=\frac{1}{9}$ $(=0.111 \text{ 3 sf})$ ***A1***

(ii) $\mathrm{P}(T\geq 3)=1-\mathrm{P}(T\leq 2)=1-\frac{1}{9}=\frac{8}{9}$ $(=0.889 \text{ 3 sf})$ ***(M1)A1***

[3 marks]

(b) let B be Bill's score

(i) $\mathrm{P}(T=6 \text{ and } B=6)=\frac{1}{9}\times\frac{1}{9}=\frac{1}{81}$ $(=0.012 \text{ 3 sf})$ ***(M1)A1***

(ii) $\mathrm{P}(B=T)=\mathrm{P}(2)\mathrm{P}(2)+\mathrm{P}(3)\mathrm{P}(3)+\ldots+\mathrm{P}(6)\mathrm{P}(6)$

$=\frac{1}{9}\times\frac{1}{9}+\frac{2}{9}\times\frac{2}{9}+\frac{3}{9}\times\frac{3}{9}+\frac{2}{9}\times\frac{2}{9}+\frac{1}{9}\times\frac{1}{9}$ ***M1***

$=\frac{19}{81}$ $(=0.235 \text{ 3 sf})$ ***A1***

[4 marks]

(c) (i) **EITHER**

$\mathrm{P}(X\leq 2)=\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}$ ***M1A1***

because $\mathrm{P}(X\leq 2)=\mathrm{P}\big((a,b,c,d)\,\big|\,a,b,c,d=1,2\big)$ ***R1***

or equivalent

$\mathrm{P}(X\leq 2)=\frac{16}{81}$ ***AG***

OR

there are sixteen possible permutations which are

Combinations	Number
1111	1
1112	4
1122	6
1222	4
2222	1

M1A1

Note: This information may be presented in a variety of forms.

$\mathrm{P}(X\leq 2)=\frac{1+4+6+4+1}{81}$ ***A1***

$=\frac{16}{81}$ ***AG***

continued …

Question 11 continued

(ii)

x	1	2	3
$\mathrm{P}(X=x)$	$\frac{1}{81}$	$\frac{15}{81}$	$\frac{65}{81}$

A1A1

(iii) $\mathrm{E}(X)=\sum_{x=1}^{3} x\mathrm{P}(X=x)$ ***(M1)***

$=\frac{1}{81}+\frac{30}{81}+\frac{195}{81}$

$=\frac{226}{81}$ (2.79 to 3 sf) ***A1***

$\mathrm{E}(X^2)=\sum_{x=1}^{3} x^2\mathrm{P}(X=x)$

$=\frac{1}{81}+\frac{60}{81}+\frac{585}{81}$

$=\frac{646}{81}$ (7.98 to 3 sf) ***A1***

$\mathrm{Var}(X)=\mathrm{E}(X^2)-\left(\mathrm{E}(X)\right)^2$ ***(M1)***

$=0.191$ (3 sf) ***A1***

Note: Award ***M1A0*** for answers obtained using rounded values $\left(e.g.\ \mathrm{Var}(X)=0.196\right)$.

[10 marks]

(d)

Combinations	**Number**
3311	6
3221	12

$\mathrm{P}\left(\text{total is } 8 \cap (X=3)\right)=\frac{18}{81}$ ***M1A1***

since $\mathrm{P}(X=3)=\frac{65}{81}$

$\mathrm{P}\left(\text{total is } 8 \mid (X=3)\right)=\frac{\mathrm{P}\left((\text{total is } 8)\cap(X=3)\right)}{\mathrm{P}(X=3)}$ ***M1***

$=\frac{18}{65}$ $(=0.277)$ ***A1***

[4 marks]

Total [21 marks]

12.

(a) $\vec{\text{OM}} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\vec{\text{ON}} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ and $\vec{\text{OP}} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ ***A1A1A1***

[3 marks]

(b) $\vec{\text{MP}} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$ and $\vec{\text{MN}} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$ ***A1A1***

$\vec{\text{MP}} \times \vec{\text{MN}} = \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ ***(M1)A1***

[4 marks]

(c) (i) area of MNP $= \frac{1}{2}\left|\vec{\text{MP}} \times \vec{\text{MN}}\right|$ ***M1***

$= \frac{1}{2}\left|\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right|$

$= \frac{\sqrt{3}}{2}$ ***A1***

(ii) $\vec{\text{OA}} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\vec{\text{OG}} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$

$\vec{\text{AG}} = \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$ ***A1***

since $\vec{\text{AG}} = 2(\vec{\text{MP}} \times \vec{\text{MN}})$ AG is perpendicular to MNP ***R1***

(iii) $\boldsymbol{r} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ ***M1A1***

$\boldsymbol{r} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 3$ (accept $-x + y + z = 3$) ***A1***

[7 marks]

continued ...

Question 12 continued

(d) $\boldsymbol{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$ ***A1***

$\begin{pmatrix} 2-2\lambda \\ 2\lambda \\ 2\lambda \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 3$ ***M1A1***

$-2 + 2\lambda + 2\lambda + 2\lambda = 3$

$\lambda = \frac{5}{6}$ ***A1***

$\boldsymbol{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \frac{5}{6} \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix}$ ***M1***

coordinates of point $\left(\frac{1}{3}, \frac{5}{3}, \frac{5}{3}\right)$ ***A1***

[6 marks]

Total [20 marks]

13. (a) $f'(x)=\dfrac{b\mathrm{e}^x(a\mathrm{e}^x+b)-a\mathrm{e}^x(a+b\mathrm{e}^x)}{(a\mathrm{e}^x+b)^2}$ ***M1A1***

$=\dfrac{ab\mathrm{e}^{2x}+b^2\mathrm{e}^x-a^2\mathrm{e}^x-ab\mathrm{e}^{2x}}{(a\mathrm{e}^x+b)^2}$ ***A1***

$=\dfrac{(b^2-a^2)\mathrm{e}^x}{(a\mathrm{e}^x+b)^2}$ ***AG***

[3 marks]

(b) **EITHER**

$f'(x)=0\Rightarrow(b^2-a^2)\mathrm{e}^x=0\Rightarrow b=\pm a$ or $\mathrm{e}^x=0$ ***A1***

which is impossible as $0<b<a$ and $\mathrm{e}^x>0$ for all $x\in\mathbb{R}$ ***R1***

OR

$f'(x)<0$ for all $x\in\mathbb{R}$ since $0<b<a$ and $\mathrm{e}^x>0$ for all $x\in\mathbb{R}$ ***A1R1***

OR

$f'(x)$ cannot be equal to zero because e^x is never equal to zero ***A1R1***

[2 marks]

(c) **EITHER**

$f''(x)=\dfrac{(b^2-a^2)\mathrm{e}^x(a\mathrm{e}^x+b)^2-2a\mathrm{e}^x(a\mathrm{e}^x+b)(b^2-a^2)\mathrm{e}^x}{(a\mathrm{e}^x+b)^4}$ ***M1A1A1***

Note: Award ***A1*** for each term in the numerator.

$=\dfrac{(b^2-a^2)\mathrm{e}^x(a\mathrm{e}^x+b-2a\mathrm{e}^x)}{(a\mathrm{e}^x+b)^3}$

$=\dfrac{(b^2-a^2)(b-a\mathrm{e}^x)\mathrm{e}^x}{(a\mathrm{e}^x+b)^3}$

OR

$f'(x)=(b^2-a^2)\mathrm{e}^x(a\mathrm{e}^x+b)^{-2}$

$f''(x)=(b^2-a^2)\mathrm{e}^x(a\mathrm{e}^x+b)^{-2}+(b^2-a^2)\mathrm{e}^x(-2a\mathrm{e}^x)(a\mathrm{e}^x+b)^{-3}$ ***M1A1A1***

Note: Award ***A1*** for each term.

$=(b^2-a^2)\mathrm{e}^x(a\mathrm{e}^x+b)^{-3}\left((a\mathrm{e}^x+b)-2a\mathrm{e}^x\right)$

$=(b^2-a^2)\mathrm{e}^x(a\mathrm{e}^x+b)^{-3}(b-a\mathrm{e}^x)$

THEN

$f''(x)=0\Rightarrow b-a\mathrm{e}^x=0\Rightarrow x=\ln\dfrac{b}{a}$ ***M1A1***

$f\left(\ln\dfrac{b}{a}\right)=\dfrac{a^2+b^2}{2ab}$ ***A1***

coordinates are $\left(\ln\dfrac{b}{a},\dfrac{a^2+b^2}{2ab}\right)$

[6 marks]

continued ...

Question 13 continued

(d) $\lim_{x \to -\infty} f(x) = \frac{a}{b} \Rightarrow y = \frac{a}{b}$ horizontal asymptote ***A1***

$\lim_{x \to +\infty} f(x) = \frac{b}{a} \Rightarrow y = \frac{b}{a}$ horizontal asymptote ***A1***

$0 < b < a \Rightarrow a\mathrm{e}^x + b > 0$ for all $x \in \mathbb{R}$ (accept $a\mathrm{e}^x + b \neq 0$)
so no vertical asymptotes ***R1***

Note: Statement on vertical asymptote must be seen for ***R1***.

[3 marks]

(e) $y = \frac{4 + \mathrm{e}^x}{4\mathrm{e}^x + 1}$

$y = \frac{1}{2} \Leftrightarrow x = \ln\frac{7}{2}$ (or 1.25 to 3 sf) ***(M1)(A1)***

$V = \pi \int_0^{\ln\frac{7}{2}} \left(\left(\frac{4 + \mathrm{e}^x}{4\mathrm{e}^x + 1} \right)^2 - \frac{1}{4} \right) \mathrm{d}x$ ***(M1)A1***

$= 1.09$ (3 sf) ***A1*** ***N4***

[5 marks]

Total [19 marks]