N11/5/MATHL/HP1/ENG/TZ0/XX/M

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MARKSCHEME

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MATHEMATICS

Higher Level

Paper 1

16 pages

SECTION A

1. area of triangle $= \frac{1}{2}(2x)^2 \sin\frac{\pi}{3}$ ***(M1)***

$= x^2\sqrt{3}$ ***A1***

Note: A $0.5 \times$ base $\times$ height calculation is acceptable.

area of sector $= \frac{\theta}{2}r^2 = \frac{\pi}{6}r^2$ ***(M1)A1***

area of triangle is twice the area of the sector

$\Rightarrow 2\left(\frac{\pi}{6}r^2\right) = x^2\sqrt{3}$ ***M1***

$\Rightarrow r = x\sqrt{\frac{3\sqrt{3}}{\pi}}$ or equivalent ***A1***

[6 marks]

2. $\mathrm{i} = \cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}$ ***(A1)***

$z_1 = \mathrm{i}^{\frac{1}{3}} = \left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)^{\frac{1}{3}} = \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6} \quad \left(= \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right)$ ***M1A1***

$z_2 = \cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6} \quad \left(= -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right)$ ***(M1)A1***

$z_3 = \cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right) = -\mathrm{i}$ ***A1***

Note: Accept exponential and cis forms for intermediate results, but not the final roots.

Note: Accept the method based on expanding $(\mathrm{a}+\mathrm{b})^3$. ***M1*** for attempt, ***M1*** for equating real and imaginary parts, ***A1*** for finding $\mathrm{a} = 0$ and $\mathrm{b} = \frac{1}{2}$, then ***A1A1A1*** for the roots.

[6 marks]

3. tree diagram *(M1)*

$$P(I|D) = \frac{P(D|I) \times P(I)}{P(D)}$$ *(M1)*

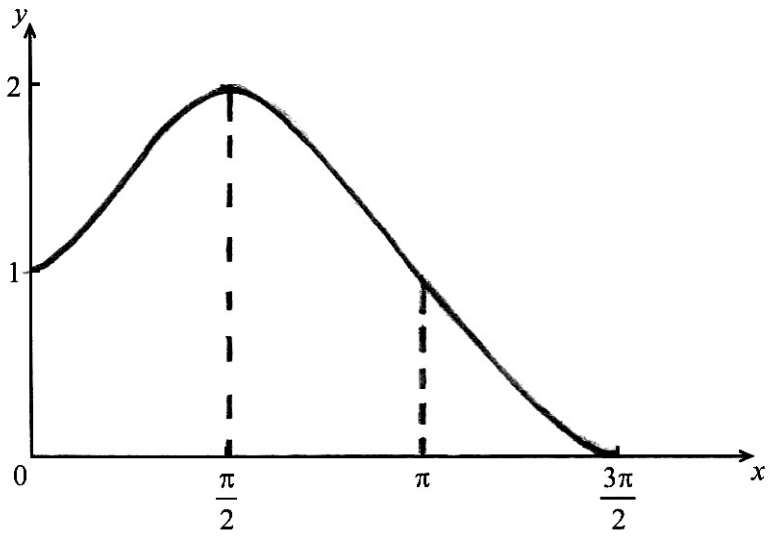
$$= \frac{0.1 \times 0.2}{0.1 \times 0.2 + 0.8 \times 0.75}$$ ***A1A1A1***

$$\left(= \frac{0.02}{0.62}\right) = \frac{1}{31}$$ ***A1***

Note: Alternative presentation of results: ***M1*** for labelled tree; ***A1*** for initial branching probabilities, 0.2 and 0.8; ***A1*** for at least the relevant second branching probabilities, 0.1 and 0.75; ***A1*** for the 'infected' end-point probabilities, 0.02 and 0.6; ***M1A1*** for the final conditional probability calculation.

[6 marks]

4. (a)



A1

(b) $(1+\sin x)^2 = 1 + 2\sin x + \sin^2 x$

$= 1 + 2\sin x + \frac{1}{2}(1 - \cos 2x)$ ***A1***

$= \frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x$ ***AG***

continued …

Question 4 continued

(c) $V = \pi\int_0^{\frac{3\pi}{2}} (1+\sin x)^2 \mathrm{d}x$ ***(M1)***

$= \pi\int_0^{\frac{3\pi}{2}} \left(\frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x\right)\mathrm{d}x$

$= \pi\left[\frac{3}{2}x - 2\cos x - \frac{\sin 2x}{4}\right]_0^{\frac{3\pi}{2}}$ ***A1***

$= \frac{9\pi^2}{4} + 2\pi$ ***A1A1***

[6 marks]

5. $\mathrm{P}(A) = \frac{\pi}{25\pi} \times \frac{1}{2} = \frac{1}{50}$ ***(M1)A1***

$\mathrm{P}(B) = \frac{8\pi}{25\pi} \times \frac{1}{2} = \frac{4}{25}$ ***A1***

$\mathrm{P}(C) = \frac{16\pi}{25\pi} \times \frac{1}{2} = \frac{8}{25}$ ***A1***

Note:	The ***M1*** is for the use of 3 areas

$\mathrm{E}(X) = (0.5 \times 0) + \frac{1}{50} \times 10 + \frac{4}{25} \times 6 + \frac{8}{25} \times 3 = \frac{106}{50}\ (= 2.12)$ ***M1A1***

Note:	The final ***M1*** is available if the probabilities are incorrect but sum to 1 or

[6 marks]

6. proposition is true for $n=1$ since $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{(1-x)^2}$ ***M1***

$=\frac{1!}{(1-x)^2}$ ***A1***

Note: Must see the 1! for the ***A1***.

assume true for $n=k$, $k\in\mathbb{Z}^+$, *i.e.* $\frac{\mathrm{d}^k y}{\mathrm{d}x^k}=\frac{k!}{(1-x)^{k+1}}$ ***M1***

consider $\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}}=\frac{\mathrm{d}\left(\frac{\mathrm{d}^k y}{\mathrm{d}x^k}\right)}{\mathrm{d}x}$ ***(M1)***

$=(k+1)k!(1-x)^{-(k+1)-1}$ ***A1***

$=\frac{(k+1)!}{(1-x)^{k+2}}$ ***A1***

hence, P_{k+1} is true whenever P_k is true, and P_1 is true, and therefore the proposition is true for all positive integers ***R1***

Note: The final ***R1*** is only available if at least 4 of the previous marks have been awarded.

[7 marks]

7. to find the points of intersection of the two curves

$-x^2+2=x^3-x^2-bx+2$ ***M1***

$x^3-bx=x(x^2-b)=0$

$\Rightarrow x=0$; $x=\pm\sqrt{b}$ ***A1A1***

$A_1=\int_{-\sqrt{b}}^{0}\left[(x^3-x^2-bx+2)-(-x^2+2)\right]\mathrm{d}x\left(=\int_{-\sqrt{b}}^{0}(x^3-bx)\,\mathrm{d}x\right)$ ***M1***

$=\left[\frac{x^4}{4}-\frac{bx^2}{2}\right]_{-\sqrt{b}}^{0}$

$=-\left(\frac{(-\sqrt{b})^4}{4}-\frac{b(-\sqrt{b})^2}{2}\right)=-\frac{b^2}{4}+\frac{b^2}{2}=\frac{b^2}{4}$ ***A1***

$A_2=\int_{0}^{\sqrt{b}}\left[(-x^2+2)-(x^3-x^2-bx+2)\right]\mathrm{d}x$ ***M1***

continued …

Question 7 continued

$$= \int_0^{\sqrt{b}} (-x^3 + bx)\,\mathrm{d}x$$

$$= \left[-\frac{x^4}{4} + \frac{bx^2}{2}\right]_0^{\sqrt{b}} = \frac{b^2}{4}$$ ***A1***

therefore $A_1 = A_2 = \dfrac{b^2}{4}$ ***AG***

[7 marks]

8. (a) angle APB is a right angle

$\Rightarrow \cos\theta = \dfrac{\mathrm{AP}}{4} \Rightarrow \mathrm{AP} = 4\cos\theta$ ***A1***

Note:	Allow correct use of cosine rule.

arc $\mathrm{PB} = 2 \times 2\theta = 4\theta$ ***A1***

$t = \dfrac{\mathrm{AP}}{3} + \dfrac{\mathrm{PB}}{6}$ ***M1***

Note:	Allow use of their AP and their PB for the ***M1***.

$\Rightarrow t = \dfrac{4\cos\theta}{3} + \dfrac{4\theta}{6} = \dfrac{4\cos\theta}{3} + \dfrac{2\theta}{3} = \dfrac{2}{3}(2\cos\theta + \theta)$ ***AG***

(b) $\dfrac{\mathrm{d}t}{\mathrm{d}\theta} = \dfrac{2}{3}(-2\sin\theta + 1)$ ***A1***

$\dfrac{2}{3}(-2\sin\theta + 1) = 0 \Rightarrow \sin\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{6}$ (or 30 degrees) ***A1***

(c) $\dfrac{\mathrm{d}^2 t}{\mathrm{d}\theta^2} = -\dfrac{4}{3}\cos\theta < 0 \left(\text{at } \theta = \dfrac{\pi}{6}\right)$ ***M1***

$\Rightarrow t$ is maximized at $\theta = \dfrac{\pi}{6}$ ***R1***

time needed to walk along arc AB is $\dfrac{2\pi}{6}$ (≈ 1 hour)

time needed to row from A to B is $\dfrac{4}{3}$ (≈ 1.33 hour)

hence, time is minimized in walking from A to B ***R1***

[8 marks]

9. (a) for the equation to have real roots

$(y-1)^2 - 4y(y-1) \geq 0$ ***M1***

$\Rightarrow 3y^2 - 2y - 1 \leq 0$

(by sign diagram, or algebraic method) ***M1***

$-\frac{1}{3} \leq y \leq 1$ ***A1A1***

> **Note:** Award first ***A1*** for $-\frac{1}{3}$ and 1, and second ***A1*** for inequalities. These are independent marks.

(b) $f : x \to \frac{x+1}{x^2+x+1} \Rightarrow x+1 = yx^2 + yx + y$ ***(M1)***

$\Rightarrow 0 = yx^2 + (y-1)x + (y-1)$ ***A1***

hence, from (a) range is $-\frac{1}{3} \leq y \leq 1$ ***A1***

(c) a value for y would lead to 2 values for x from (a) ***R1***

> **Note:** Do not award ***R1*** if (b) has not been tackled.

[8 marks]

SECTION B

10. (a) $k\int_0^{\frac{\pi}{2}} \sin x \, dx = 1$ ***M1***

$k[-\cos x]_0^{\frac{\pi}{2}} = 1$

$k = 1$ ***A1***

[2 marks]

(b) $E(X) = \int_0^{\frac{\pi}{2}} x \sin x \, dx$ ***M1***

integration by parts ***M1***

$[-x\cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \, dx$ ***A1A1***

$= 1$ ***A1***

[5 marks]

(c) $\int_0^{M} \sin x \, dx = \frac{1}{2}$ ***M1***

$[-\cos x]_0^{M} = \frac{1}{2}$ ***A1***

$\cos M = \frac{1}{2}$

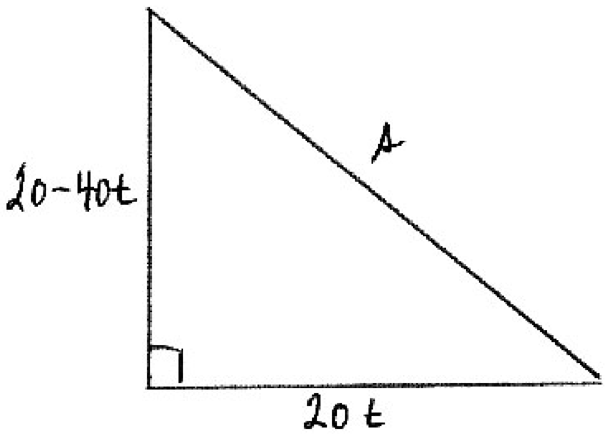
$M = \frac{\pi}{3}$ ***A1***

Note: accept $\arccos \frac{1}{2}$

[3 marks]

Total [10 marks]

11. (a)



(M1)

$s^2 = (20t)^2 + (20 - 40t)^2$ ***M1***

$s^2 = 2000t^2 - 1600t + 400$ ***A1***

to minimize s it is enough to minimize s^2

$f'(t) = 4000t - 1600$ ***A1***

setting $f'(t)$ equal to 0 ***M1***

$4000t - 1600 = 0 \Rightarrow t = \frac{2}{5}$ or 24 minutes ***A1***

$f''(t) = 4000 > 0$ ***M1***

$\Rightarrow$ at $t = \frac{2}{5}$, $f(t)$ is minimized

hence, the ships are closest at 12:24 ***A1***

Note: accept solution based on s.

[8 marks]

(b) $f\left(\frac{2}{5}\right) = \sqrt{80}$ ***M1A1***

since $\sqrt{80} < 9$, the captains can see one another ***R1***

[3 marks]

Total [11 marks]

12. (a) (i) $|\boldsymbol{a}-\boldsymbol{b}|=|\boldsymbol{a}+\boldsymbol{b}|$

$\Rightarrow (\boldsymbol{a}-\boldsymbol{b})\bullet(\boldsymbol{a}-\boldsymbol{b})=(\boldsymbol{a}+\boldsymbol{b})\bullet(\boldsymbol{a}+\boldsymbol{b})$ ***(M1)***

$\Rightarrow |\boldsymbol{a}|^2-2\boldsymbol{a}\bullet\boldsymbol{b}+|\boldsymbol{b}|^2=|\boldsymbol{a}|^2+2\boldsymbol{a}\bullet\boldsymbol{b}+|\boldsymbol{b}|^2$ ***A1***

$\Rightarrow 4\boldsymbol{a}\bullet\boldsymbol{b}=0\Rightarrow \boldsymbol{a}\bullet\boldsymbol{b}=0$ ***A1***

therefore $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular ***R1***

Note: Allow use of 2-d components.

Note: Do not condone sloppy vector notation, so we must see something to the effect that $|\boldsymbol{c}|^2=\boldsymbol{c}.\boldsymbol{c}$ is clearly being used for the ***M1***.

Note: Allow a correct geometric argument, for example that the diagonals of a parallelogram have the same length only if it is a rectangle.

(ii) $|\boldsymbol{a}\times\boldsymbol{b}|^2=(|\boldsymbol{a}||\boldsymbol{b}|\sin\theta)^2=|\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\theta$ ***M1A1***

$|\boldsymbol{a}|^2|\boldsymbol{b}|^2-(\boldsymbol{a}\bullet\boldsymbol{b})^2=|\boldsymbol{a}|^2|\boldsymbol{b}|^2-|\boldsymbol{a}|^2|\boldsymbol{b}|^2\cos^2\theta$ ***M1***

$=|\boldsymbol{a}|^2|\boldsymbol{b}|^2(1-\cos^2\theta)$ ***A1***

$=|\boldsymbol{a}|^2|\boldsymbol{b}|^2\sin^2\theta$

$\Rightarrow |\boldsymbol{a}\times\boldsymbol{b}|^2=|\boldsymbol{a}|^2|\boldsymbol{b}|^2-(\boldsymbol{a}\bullet\boldsymbol{b})^2$ ***AG***

[8 marks]

(b) (i) area of triangle $=\frac{1}{2}|\overrightarrow{\mathrm{AB}}\times\overrightarrow{\mathrm{AC}}|$ ***(M1)***

$=\frac{1}{2}|(\boldsymbol{b}-\boldsymbol{a})\times(\boldsymbol{c}-\boldsymbol{a})|$ ***A1***

$=\frac{1}{2}|\boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{b}\times-\boldsymbol{a}+-\boldsymbol{a}\times\boldsymbol{c}+-\boldsymbol{a}\times-\boldsymbol{a}|$ ***A1***

$\boldsymbol{b}\times-\boldsymbol{a}=\boldsymbol{a}\times\boldsymbol{b}$; $\boldsymbol{c}\times\boldsymbol{a}=-\boldsymbol{a}\times\boldsymbol{c}$; $-\boldsymbol{a}\times-\boldsymbol{a}=0$ ***M1***

hence, area of triangle is $\frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{c}\times\boldsymbol{a}|$ ***AG***

(ii) D is the foot of the perpendicular from B to AC

area of triangle $\mathrm{ABC}=\frac{1}{2}|\overrightarrow{\mathrm{AC}}||\overrightarrow{\mathrm{BD}}|$ ***A1***

therefore

$\frac{1}{2}|\overrightarrow{\mathrm{AC}}||\overrightarrow{\mathrm{BD}}|=\frac{1}{2}|\overrightarrow{\mathrm{AB}}\times\overrightarrow{\mathrm{AC}}|$ ***M1***

hence, $|\overrightarrow{\mathrm{BD}}|=\frac{|\overrightarrow{\mathrm{AB}}\times\overrightarrow{\mathrm{AC}}|}{|\overrightarrow{\mathrm{AC}}|}$ ***A1***

$=\frac{|\boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{c}\times\boldsymbol{a}|}{|\boldsymbol{c}-\boldsymbol{a}|}$ ***AG***

[7 marks]

Total [15 marks]

13. (a) $\frac{dy}{dx} = \frac{e}{\ln e}(2+2) = 4e$ ***A1***

at $(2, e)$ the tangent line is $y - e = 4e(x-2)$ ***M1***

hence $y = 4ex - 7e$ ***A1***

[3 marks]

(b) $\frac{dy}{dx} = \frac{y}{\ln y}(x+2) \Rightarrow \frac{\ln y}{y}dy = (x+2)dx$ ***M1***

$\int \frac{\ln y}{y}dy = \int (x+2)dx$

using substitution $u = \ln y$; $du = \frac{1}{y}dy$ ***(M1)(A1)***

$\Rightarrow \int \frac{\ln y}{y}dy = \int u\,du = \frac{1}{2}u^2$ ***(A1)***

$\Rightarrow \frac{(\ln y)^2}{2} = \frac{x^2}{2} + 2x + c$ ***A1A1***

at $(2, e)$, $\frac{(\ln e)^2}{2} = 6 + c$ ***M1***

$\Rightarrow c = -\frac{11}{2}$ ***A1***

$\Rightarrow \frac{(\ln y)^2}{2} = \frac{x^2}{2} + 2x - \frac{11}{2} \Rightarrow (\ln y)^2 = x^2 + 4x - 11$

$\ln y = \pm\sqrt{x^2+4x-11} \Rightarrow y = e^{\pm\sqrt{x^2+4x-11}}$ ***M1A1***

since $y > 1$, $f(x) = e^{\sqrt{x^2+4x-11}}$ ***R1***

[11 marks]

Note: ***M1*** for attempt to make y the subject.

(c) **EITHER**

$x^2 + 4x - 11 > 0$ ***A1***

using the quadratic formula ***M1***

critical values are $\frac{-4 \pm \sqrt{60}}{2}$ $\left(= -2 \pm \sqrt{15}\right)$ ***A1***

using a sign diagram or algebraic solution ***M1***

$x < -2 - \sqrt{15}$; $x > -2 + \sqrt{15}$ ***A1A1***

OR

$x^2 + 4x - 11 > 0$ ***A1***

by methods of completing the square ***M1***

$(x+2)^2 > 15$ ***A1***

$\Rightarrow x + 2 < -\sqrt{15}$ or $x + 2 > \sqrt{15}$ ***(M1)***

$x < -2 - \sqrt{15}$; $x > -2 + \sqrt{15}$ ***A1A1***

[6 marks]

continued …

Question 13 continued

(d) $f(x) = f'(x) \Rightarrow f(x) = \dfrac{f(x)}{\ln f(x)}(x+2)$ ***M1***

$\Rightarrow \ln\left(f(x)\right) = x+2 \quad \left(\Rightarrow x+2 = \sqrt{x^2+4x-11}\right)$ ***A1***

$\Rightarrow (x+2)^2 = x^2+4x-11 \Rightarrow x^2+4x+4 = x^2+4x-11$ ***A1***

$\Rightarrow 4 = -11$, hence $f(x) \neq f'(x)$ ***R1AG***

[4 marks]

Total [24 marks]