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MARKSCHEME

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MATHEMATICS

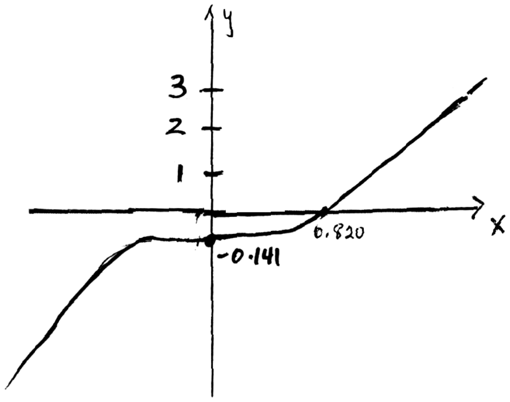
Higher Level

Paper 2

16 pages

SECTION A

1. (a)



A1A1A1

Note: Award ***A1*** for shape,
A1 for x-intercept is 0.820, accept sin(–3) or –sin(3)
A1 for y-intercept is -0.141.

(b) $A = \int_0^{0.8202} |x + \sin(x-3)| \, dx \approx 0.0816$ sq units *(M1)A1*

[5 marks]

2. $\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix}$ *(M1)*

$= a\begin{vmatrix} a & 1 \\ 1 & a \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & a \end{vmatrix} + \begin{vmatrix} 1 & a \\ 1 & 1 \end{vmatrix}$

$= a(a^2-1)-(a-1)+(1-a)$ *(A1)*

$= a^3 - 3a + 2$ ***A1***

set $a^3 - 3a + 2 = 0$ ***M1***

$\Rightarrow a = -2$; $a = 1$ ***A1A1***

hence the system has a unique solution for all reals such that

$a \neq -2$; $a \neq 1$ ***R1***

Note: Award ***R1*** for their values of a.

[7 marks]

3. (a) $m = \frac{300}{60} = 5$ *(A1)*

$P(X = 0) = 0.00674$ ***A1***

or e^{-5}

(b) $E(X) = 5 \times 2 = 10$ ***A1***

(c) $P(X > 10) = 1 - P(X \leq 10)$ *(M1)*

$= 0.417$ ***A1***

[5 marks]

4. (a) $\tan\left(\arctan\frac{1}{2}-\arctan\frac{1}{3}\right)=\tan(\arctan a)$ ***(M1)***

$a=0.14285\ldots=\frac{1}{7}$ ***(A1)A1***

(b) $\arctan\left(\frac{1}{7}\right)=\arcsin(x)\Rightarrow x=\sin\left(\arctan\frac{1}{7}\right)\approx 0.141$ ***(M1)A1***

Note: Accept exact value of $\left(\frac{1}{\sqrt{50}}\right)$.

[5 marks]

5. (a) $X\sim \mathrm{B}(5,\,0.1)$ ***(M1)***

$\mathrm{P}(X=2)=0.0729$ ***A1***

(b) $\mathrm{P}(X\geq 1)=1-\mathrm{P}(X=0)$ ***(M1)***

$0.9<1-\left(\frac{9}{10}\right)^n$ ***(M1)***

$n>\frac{\ln 0.1}{\ln 0.9}$

$n=22$ days ***A1***

[5 marks]

6. METHOD 1

$\arg(z_1 z_2) = \frac{5\pi}{6}$ (150°) ***(A1)***

$\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2}$ (90°) ***(A1)***

$\Rightarrow \arg(z_1) + \arg(z_2) = \frac{5\pi}{6}$; $\arg(z_1) - \arg(z_2) = \frac{\pi}{2}$ ***M1***

solving simultaneously

$\arg(z_1) = \frac{2\pi}{3}$ (120°) and $\arg(z_2) = \frac{\pi}{6}$ (30°) ***A1A1***

Note: Accept decimal approximations of the radian measures.

$|z_1 z_2| = 2 \Rightarrow |z_1||z_2| = 2$; $\left|\frac{z_1}{z_2}\right| = 2 \Rightarrow \frac{|z_1|}{|z_2|} = 2$ ***M1***

solving simultaneously

$|z_1| = 2$; $|z_2| = 1$ ***A1***

[7 marks]

METHOD 2

$z_1 = 2iz_2$ $\quad 2iz_2{}^2 = -\sqrt{3} + i$ ***(M1)***

$z_2{}^2 = \frac{-\sqrt{3} + i}{2i}$ ***A1***

$z_2 = \sqrt{\frac{-\sqrt{3} + i}{2i}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ or $e^{\frac{\pi}{6}i}$ ***(M1)(A1)***

(allow $0.866 + 0.5i$ or $\mathrm{e}^{0.524\mathrm{i}}$)

$z_1 = -1 + \sqrt{3}i$ or $2\mathrm{e}^{\frac{2\pi}{3}i}$ – (allow $-1+1.73\mathrm{i}$ or $2\mathrm{e}^{2.09\mathrm{i}}$) ***(A1)***

z_1 modulus = 2, argument = $\frac{2\pi}{3}$ ***A1***

z_2 modulus = 1, argument = $\frac{\pi}{6}$ ***A1***

Note: Accept degrees and decimal approximations to radian measure.

[7 marks]

7. (a) for the series to have a finite sum, $\left|\frac{2x}{x+1}\right|<1$ ***R1***

(sketch from gdc or algebraic method) ***M1***

S_∞ exists when $-\frac{1}{3}<x<1$ ***A1A1***

Note: Award ***A1*** for bounds and ***A1*** for strict inequalities.

(b) $S_\infty=\dfrac{\frac{2x}{x+1}}{1-\frac{2x}{x+1}}=\dfrac{2x}{1-x}$ ***M1A1***

[6 marks]

8. (a) $y=\dfrac{1}{1+e^{-x}}$

$y(1+e^{-x})=1$ ***M1***

$1+e^{-x}=\frac{1}{y}\Rightarrow e^{-x}=\frac{1}{y}-1$ ***A1***

$\Rightarrow x=-\ln\left(\frac{1}{y}-1\right)$ ***A1***

$f^{-1}(x)=-\ln\left(\frac{1}{x}-1\right)\quad\left(=\ln\left(\frac{x}{1-x}\right)\right)$ ***A1***

domain: $0<x<1$ ***A1A1***

Note: Award ***A1*** for endpoints and ***A1*** for strict inequalities.

(b) 0.659 ***A1***

[7 marks]

9. $V=\frac{\pi}{3}r^2h$

$\frac{dV}{dt}=\frac{\pi}{3}\left[2rh\frac{dr}{dt}+r^2\frac{dh}{dt}\right]$ ***M1A1A1***

at the given instant

$\frac{dV}{dt}=\frac{\pi}{3}\left[2(40)(200)\left(-\frac{1}{2}\right)+40^2(3)\right]$ ***M1***

$=\frac{-3200\pi}{3}=-3351.03\ldots\approx-3350$ ***A1***

hence, the volume is decreasing (at approximately 3350 mm^3 per century) ***R1***

[6 marks]

10. METHOD 1

$\frac{2-\mathrm{i}}{1+\mathrm{i}}=\frac{1-3\mathrm{i}}{2}$ ***A1***

$\frac{6+8\mathrm{i}}{u+\mathrm{i}}\times\frac{u-\mathrm{i}}{u-\mathrm{i}}=\frac{6u+8+(8u-6)\mathrm{i}}{u^2+1}$ ***M1A1***

$\Rightarrow\frac{2-\mathrm{i}}{1+\mathrm{i}}-\frac{6+8u}{u+\mathrm{i}}=\frac{1}{2}-\frac{6u+8}{u^2+1}-\left(\frac{3}{2}+\frac{8u-6}{u^2+1}\right)\mathrm{i}$

$\operatorname{Im} z=\operatorname{Re} z$

$\Rightarrow\frac{1}{2}-\frac{6u+8}{u^2+1}=-\frac{3}{2}-\frac{8u-6}{u^2+1}$ ***A1***

(sketch from gdc, or algebraic method) ***(M1)***

$u=-3;\ u=2$ ***A1A1*** ***N2***

[7 marks]

METHOD 2

$\frac{2-\mathrm{i}}{1+\mathrm{i}}-\frac{6+8\mathrm{i}}{u+\mathrm{i}}=\frac{(2-\mathrm{i})(u+\mathrm{i})-(1+\mathrm{i})(6+8\mathrm{i})}{(u-1)+\mathrm{i}(u+1)}$ ***M1A1***

$=\frac{(2-\mathrm{i})(u+\mathrm{i})-(1+\mathrm{i})(6+8\mathrm{i})}{(u-1)+\mathrm{i}(u+1)}\cdot\frac{(u-1)-\mathrm{i}(u+1)}{(u-1)-\mathrm{i}(u+1)}$ ***M1***

$=\frac{u^2-12u-15+\mathrm{i}(-3u^2-16u+9)}{2(u^2+1)}$ ***A1***

$\operatorname{Re} z=\operatorname{Im} z\Rightarrow u^2-12u-15=-3u^2-16u+9$ ***M1***

$u=-3;\ u=2$ ***A1A1*** ***N2***

[7 marks]

SECTION B

11. (a) $X \sim \mathrm{N}(60.33, 1.95^2)$

$\mathrm{P}(X < x) = 0.2 \Rightarrow x = 58.69$ m ***(M1)A1***

[2 marks]

(b) $z = -0.8416\ldots$ ***(A1)***

$-0.8416 = \dfrac{56.52 - 59.39}{\sigma}$ ***(M1)***

$\sigma \approx 3.41$ ***A1***

[3 marks]

(c) Jan $X \sim \mathrm{N}(60.33, 1.95^2)$; Sia $X \sim \mathrm{N}(59.50, 3.00^2)$

(i) Jan: $\mathrm{P}(X > 65) \approx 0.00831$ ***(M1)A1***

Sia: $\mathrm{P}(Y > 65) \approx 0.0334$ ***A1***

Sia is more likely to qualify ***R1***

Note: Only award ***R1*** if ***(M1)*** has been awarded.

(ii) Jan: $\mathrm{P}(X \geq 1) = 1 - \mathrm{P}(X = 0)$ ***(M1)***

$= 1 - (1 - 0.00831\ldots)^3 \approx 0.0247$ ***(M1)A1***

Sia: $\mathrm{P}(Y \geq 1) = 1 - \mathrm{P}(Y = 0) = 1 - (1 - 0.0334\ldots)^3 \approx 0.0968$ ***A1***

Note: Accept 0.0240 and 0.0969.

hence, $\mathrm{P}(X \geq 1 \text{ and } Y \geq 1) = 0.0247 \times 0.0968 = 0.00239$ ***(M1)A1***

[10 marks]

Total [15 marks]

12. (a) $S_{2n} = \frac{2n}{2}\left(2(8) + (2n-1)\frac{1}{4}\right)$ ***(M1)***

$= n\left(16 + \frac{2n-1}{4}\right)$ ***A1***

$S_{3n} = \frac{3n}{2}\left(2 \times 8 + (3n-1)\frac{1}{4}\right)$ ***(M1)***

$= \frac{3n}{2}\left(16 + \frac{3n-1}{4}\right)$ ***A1***

$S_{2n} = S_{3n} - S_{2n} \Rightarrow 2S_{2n} = S_{3n}$ ***M1***

solve $2S_{2n} = S_{3n}$

$\Rightarrow 2n\left(16 + \frac{2n-1}{4}\right) = \frac{3n}{2}\left(16 + \frac{3n-1}{4}\right)$ ***A1***

$\left(\Rightarrow 2\left(16 + \frac{2n-1}{4}\right) = \frac{3}{2}\left(16 + \frac{3n-1}{4}\right)\right)$

(gdc or algebraic solution) ***(M1)***

$n = 63$ ***A2***

[9 marks]

(b) $(a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_4)^2 + \dots$

$= (a_1 - a_1 r)^2 + (a_1 r - a_1 r^2)^2 + (a_1 r^2 - a_1 r^3) + \dots$ ***M1A1***

$= \left[a_1(1-r)\right]^2 + \left[a_1 r(1-r)\right]^2 + \left[a_1 r^2(1-r)\right]^2 + \dots + \left[a_1 r^{n-1}(1-r)\right]^2$ ***(A1)***

Note: This ***A1*** is for the expression for the last term.

$= a_1^{\,2}(1-r)^2 + a_1^{\,2}r^2(1-r)^2 + a_1^{\,2}r^4(1-r)^2 + \dots + a_1^{\,2}r^{2n-2}(1-r)^2$ ***A1***

$= a_1^{\,2}(1-r)^2(1 + r^2 + r^4 + \dots + r^{2n-2})$ ***A1***

$= a_1^{\,2}(1-r)^2\left(\frac{1-r^{2n}}{1-r^2}\right)$ ***M1A1***

$= \frac{a_1^{\,2}(1-r)(1-r^{2n})}{1+r}$ ***AG***

[7 marks]

Total [16 marks]

13. (a) **METHOD 1**

solving simultaneously (gdc) ***(M1)***

$x = 1 + 2z$; $y = -1 - 5z$ ***A1A1***

$$L: \boldsymbol{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$$ ***A1A1A1***

Note: 1^{st} ***A1*** is for $\boldsymbol{r} =$.

[6 marks]

METHOD 2

direction of line $= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix}$ (last two rows swapped) ***M1***

$= 2\boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k}$ ***A1***

putting $z = 0$, a point on the line satisfies $2x + y = 1$, $3x + y = 2$ ***M1***

i.e. $(1, -1, 0)$ ***A1***

the equation of the line is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$$ ***A1A1***

Note: Award ***A0A1*** if $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is missing.

[6 marks]

(b) $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$ ***M1***

$= 6\boldsymbol{i} - 12\boldsymbol{k}$ ***A1***

hence, $\boldsymbol{n} = \boldsymbol{i} - 2\boldsymbol{k}$

$\boldsymbol{n} \bullet \boldsymbol{a} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1$ ***M1A1***

therefore $\boldsymbol{r} \bullet \boldsymbol{n} = \boldsymbol{a} \bullet \boldsymbol{n} \Rightarrow x - 2z = 1$ ***AG***

[4 marks]

continued …

Question 13 continued

(c) **METHOD 1**

$P = (-2, 4, 1)$, $Q = (x, y, z)$

$\overrightarrow{PQ} = \begin{pmatrix} x+2 \\ y-4 \\ z-1 \end{pmatrix}$ ***A1***

$\overrightarrow{PQ}$ is perpendicular to $3x + y - z = 2$

$\Rightarrow \overrightarrow{PQ}$ is parallel to $3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ ***R1***

$\Rightarrow x + 2 = 3t$; $y - 4 = t$; $z - 1 = -t$ ***A1***

$1 - z = t \Rightarrow x + 2 = 3 - 3z \Rightarrow x + 3z = 1$ ***A1***

solving simultaneously $x + 3z = 1$; $x - 2z = 1$ ***M1***

$5z = 0 \Rightarrow z = 0$; $x = 1$, $y = 5$ ***A1***

hence, $Q = (1, 5, 0)$

[6 marks]

METHOD 2

Line passing through PQ has equation

$\mathbf{r} = \begin{matrix} -2 \\ 4 \\ 1 \end{matrix} + t \begin{matrix} 3 \\ 1 \\ -1 \end{matrix}$ ***M1A1***

Meets π_3 when:

$-2 + 3t - 2(1 - t) = 1$ ***M1A1***

$t = 1$ ***A1***

Q has coordinates (1, 5, 0) ***A1***

[6 marks]

Total [16 marks]

14. (a) $\left|e^{i\theta}\right| \ (=|\cos\theta + i\sin\theta|) = \sqrt{\cos^2\theta + \sin^2\theta} = 1$ ***M1AG***

[1 mark]

(b) $z = \frac{1}{3}e^{i\theta}$ ***A1***

$|z| = \left|\frac{1}{3}e^{i\theta}\right| = \frac{1}{3}$ ***A1AG***

[2 marks]

(c) $S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}e^{i\theta}}$ ***(M1)A1***

[2 marks]

(d) **EITHER**

$S_\infty = \frac{1}{1-\frac{1}{3}\cos\theta - \frac{1}{3}i\sin\theta}$ ***A1***

$= \frac{1-\frac{1}{3}\cos\theta + \frac{1}{3}i\sin\theta}{\left(1-\frac{1}{3}\cos\theta - \frac{1}{3}i\sin\theta\right)\left(1-\frac{1}{3}\cos\theta + \frac{1}{3}i\sin\theta\right)}$ ***M1A1***

$= \frac{1-\frac{1}{3}\cos\theta + \frac{1}{3}i\sin\theta}{\left(1-\frac{1}{3}\cos\theta\right)^2 + \frac{1}{9}\sin^2\theta}$ ***A1***

$= \frac{1-\frac{1}{3}\cos\theta + \frac{1}{3}i\sin\theta}{1-\frac{2}{3}\cos\theta + \frac{1}{9}}$ ***A1***

continued …

Question 14 continued

OR

$$S_\infty = \frac{1}{1-\frac{1}{3}e^{i\theta}}$$

$$= \frac{1-\frac{1}{3}e^{-i\theta}}{\left(1-\frac{1}{3}e^{i\theta}\right)\left(1-\frac{1}{3}e^{-i\theta}\right)}$$ ***M1A1***

$$= \frac{1-\frac{1}{3}e^{-i\theta}}{1-\frac{1}{3}(e^{i\theta}+e^{-i\theta})+\frac{1}{9}}$$ ***A1***

$$= \frac{1-\frac{1}{3}e^{-i\theta}}{\frac{10}{9}-\frac{2}{3}\cos\theta}$$ ***A1***

$$= \frac{1-\frac{1}{3}(\cos\theta - i\sin\theta)}{\frac{10}{9}-\frac{2}{3}\cos\theta}$$ ***A1***

THEN

taking imaginary parts on both sides

$$\frac{1}{3}\sin\theta + \frac{1}{9}\sin 2\theta + \ldots = \frac{\frac{1}{3}\sin\theta}{\frac{10}{9}-\frac{2}{3}\cos\theta}$$ ***M1A1A1***

$$= \frac{\sin\theta}{\frac{10}{9}-\frac{2}{3}\cos\theta}$$

$$\Rightarrow \sin\theta + \frac{1}{3}\sin 2\theta + \ldots = \frac{9\sin\theta}{10-6\cos\theta}$$ ***AG***

[8 marks]

Total ***[13 marks]***