Chapter 5

Sequences and series

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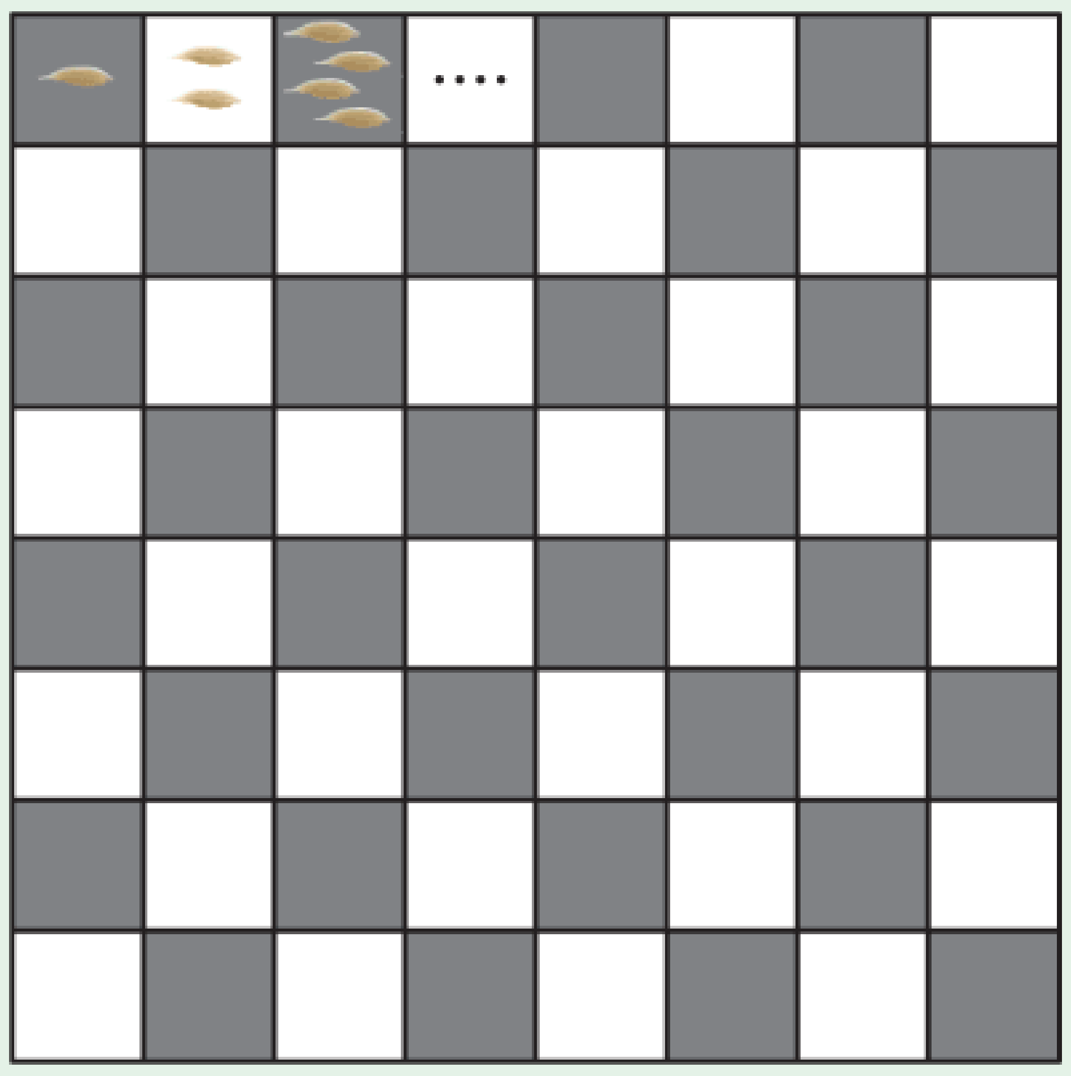
OPENING PROBLEM THE LEGEND OF SISSA IBN DAHIR

Around 1260 AD, the Kurdish historian Ibn Khallikān recorded the following story about Sissa ibn Dahir and a chess game against the Indian King Shihram.

> King Shihram was a tyrant king, and his subject Sissa ibn Dahir wanted to teach him how important all of his people were. He invented the game of chess for the king, and the king was greatly impressed. He insisted on Sissa ibn Dahir naming his reward, and the wise man asked for one grain of wheat for the first square, two grains of wheat for the second square, four grains of wheat for the third square, and so on, doubling the wheat on each successive square on the board.
>
> The king laughed at first and agreed, for there was so little grain on the first few squares. By halfway he was surprised at the amount of grain being paid, and soon he realised his great error: that he owed more grain than there was in the world.

Things to think about:

- **a** How can we describe the number of grains of wheat for each square?
- **b** What expression gives the number of grains of wheat for the nth square?
- **c** Find the total number of grains of wheat that the king owed.



To help understand problems like the **Opening Problem**, we need to study **sequences** and their sums which are called **series**.

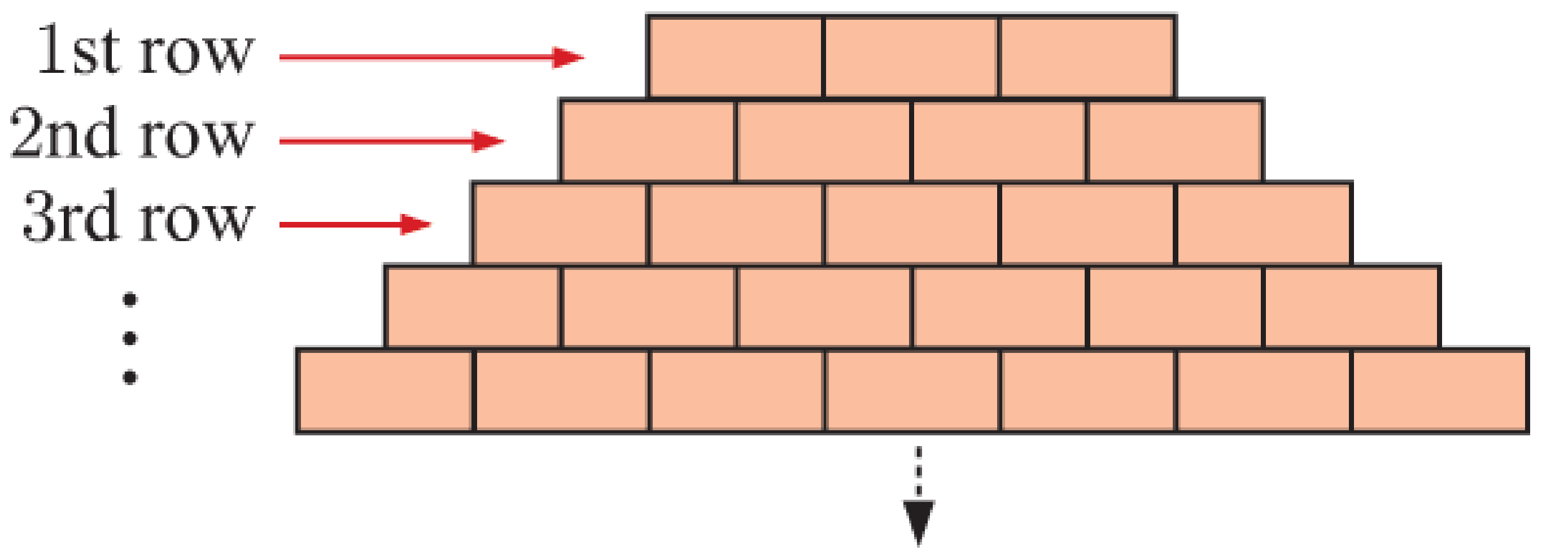
A NUMBER SEQUENCES

In mathematics it is important that we can:

- **recognise** a pattern in a set of numbers
- **describe** the pattern in words
- **continue** the pattern.

> A **number sequence** is an ordered list of numbers defined by a rule.
>
> The numbers in a sequence are called the **terms** of the sequence.

Consider the illustrated tower of bricks:



The first row has 3 bricks.
The second row has 4 bricks.
The third row has 5 bricks.
The fourth row has 6 bricks.

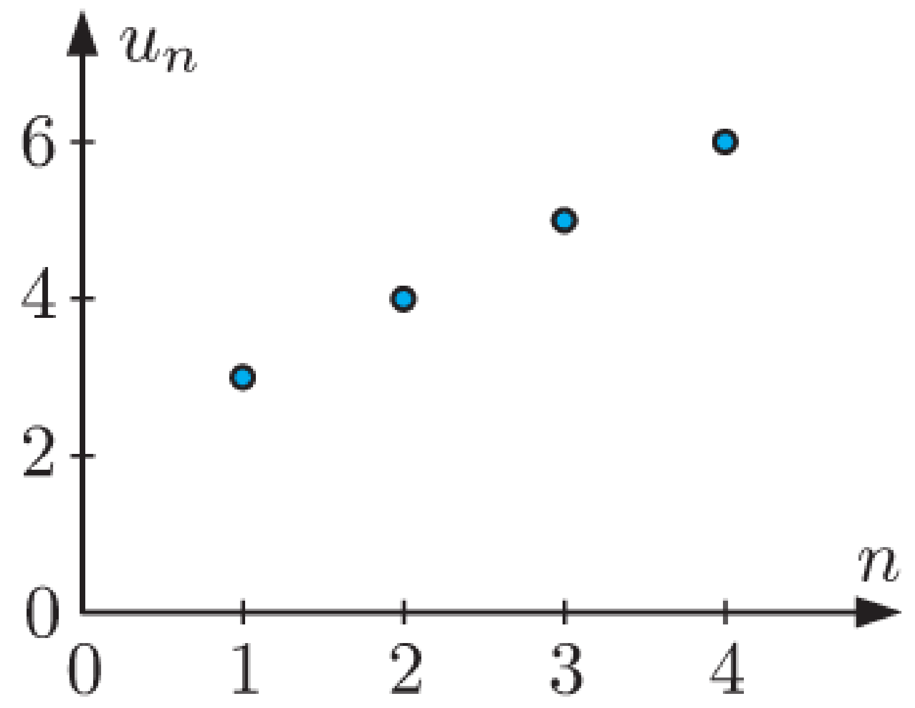
If we let u_n represent the number of bricks in the nth row, then $u_1 = 3$, $u_2 = 4$, $u_3 = 5$, and $u_4 = 6$.

The pattern could be continued forever, generating a **sequence** of numbers.

There are many ways to describe the sequence, including:

- **listing the terms**: 3, 4, 5, 6, 7, ...
- using **words**: "The sequence starts at 3, and increases by 1 each time".
- using the **explicit formula** $u_n = n + 2$ which gives the **nth term** or **general term** of the sequence in terms of n.
  We can use this formula to find, for example, the 20th term of the sequence, which is $u_{20} = 20 + 2 = 22$.
- a **graph** where each term of a sequence is represented by a dot. The dots *must not* be joined because n must be an integer.

The string of dots indicates that the pattern continues forever.



Example 1 **Self Tutor**

Describe the sequence: 14, 17, 20, 23, ... and write down the next two terms.

The sequence starts at 14, and each term is 3 more than the previous term.

The next two terms are 26 and 29.

THE GENERAL TERM OF A SEQUENCE

The **general term** or **nth term** of a sequence is represented by a symbol with a subscript, for example u_n, T_n, t_n, or A_n.

$\{u_n\}$ represents the sequence that can be generated by using u_n as the nth term.

Unless stated otherwise, we assume the first term of the sequence is u_1, and that the sequence is defined for $n \in \mathbb{Z}^+$. Sometimes we might choose for a sequence to start with u_0, particularly if n represents the number of time periods after the start of an experiment or investment.

You can use technology to help generate sequences from a formula.

GRAPHICS CALCULATOR INSTRUCTIONS

EXERCISE 5A

1 Write down the first four terms of the sequence if you start with:

a 4 and add 9 each time
b 45 and subtract 6 each time
c 2 and multiply by 3 each time
d 96 and divide by 2 each time.

2 The sequence of prime numbers is 2, 3, 5, 7, 11, 13, 17, 19, ... Write down the value of:

a u_2
b u_5
c u_{10}.

3 Consider the sequence 4, 7, 10, 13, 16, ...

a Describe the sequence in words. **b** Write down the values of u_1 and u_4.

c Assuming the pattern continues, find the value of u_8.

4 A sequence is defined by the explicit formula $u_n = 2n + 5$.

a Write down the first four terms of the sequence.

b Display these terms on a graph.

5 A sequence is defined by the explicit formula $u_n = 3n - 2$. Find:

a u_1 **b** u_5 **c** u_{27}.

6 Consider the number sequence -9, -6, -1, 6, 15, ...

a Which of these is the correct explicit formula for this sequence?

A $u_n = n - 10$ **B** $u_n = n^2 - 10$ **C** $u_n = n^3 - 10$

b Use the correct formula to find the 20th term of the sequence.

7 Write a description of the sequence and find the next 2 terms:

a 8, 16, 24, 32, ... **b** 2, 5, 8, 11, ... **c** 36, 31, 26, 21, ...

d 96, 89, 82, 75, ... **e** 1, 4, 16, 64, ... **f** 2, 6, 18, 54, ...

g 480, 240, 120, 60, ... **h** 243, 81, 27, 9, ... **i** 50 000, 10 000, 2000, 400, ...

8 Describe the sequence and write down the next 3 terms:

a 1, 4, 9, 16, ... **b** 1, 8, 27, 64, ... **c** 2, 6, 12, 20, ...

9 Find the next two terms of the sequence:

a 95, 91, 87, 83, ... **b** 5, 20, 80, 320, ... **c** 1, 16, 81, 256, ...

d 2, 3, 5, 7, 11, ... **e** 2, 4, 7, 11, ... **f** 9, 8, 10, 7, 11, ...

10 Evaluate the first *five* terms of the sequence:

a $\{2n\}$ **b** $\{2n - 3\}$ **c** $\{2n + 11\}$ **d** $\{4n - 3\}$

e $\{2^n\}$ **f** $\{6 \times (\frac{1}{2})^n\}$ **g** $\{(-2)^n\}$ **h** $\{15 - (-2)^n\}$

ACTIVITY 1 RECURRENCE FORMULAE

A **recurrence formula** describes the nth term of a sequence using a formula which involves the preceding terms.

Click on the icon to obtain this Activity.

RECURRENCE FORMULAE

B ARITHMETIC SEQUENCES

An **arithmetic sequence** is a sequence in which each term differs from the previous one by the same fixed number. We call this number the **common difference** d.

A sequence is arithmetic $\Leftrightarrow$ $\boldsymbol{u_{n+1} - u_n = d}$ for all $n \in \mathbb{Z}^+$.

An arithmetic sequence can also be referred to as an **arithmetic progression**.

For example:

- the tower of bricks in the previous Section forms an arithmetic sequence with common difference 1
- 2, 5, 8, 11, 14, ... is arithmetic with common difference 3 since
  $5 - 2 = 3$
  $8 - 5 = 3$
  $11 - 8 = 3$, and so on.
- 30, 25, 20, 15, 10, ... is arithmetic with common difference -5 since
  $25 - 30 = -5$
  $20 - 25 = -5$
  $15 - 20 = -5$, and so on.

The name "arithmetic" is given because the middle term of any three consecutive terms is the **arithmetic mean** of the terms on either side.

If the terms are a, b, c, then

$$\begin{aligned} b - a &= c - b \quad \{\text{equating common differences}\} \\ \therefore\ 2b &= a + c \\ \therefore\ b &= \frac{a+c}{2} \quad \text{which is the arithmetic mean.} \end{aligned}$$

THE GENERAL TERM FORMULA

If we know that a sequence is arithmetic, we can use a formula to find the value of any term of the sequence.

Suppose the first term of an arithmetic sequence is u_1 and the common difference is d.

Then $u_2 = u_1 + d$, $u_3 = u_1 + 2d$, $u_4 = u_1 + 3d$, and so on.

$$\text{Hence}\quad \underset{\text{term number}}{u_n} = u_1 + \underbrace{(n-1)}_{\substack{\text{the coefficient of } d \text{ is}\\ \text{one less than the term number}}} d$$

For an **arithmetic sequence** with **first term u_1** and **common difference d**, the **general term** or **nth term** is $\boldsymbol{u_n = u_1 + (n-1)d}$.

Example 2 **Self Tutor**

Consider the sequence 2, 9, 16, 23, 30, ...

a Show that the sequence is arithmetic.

b Find a formula for the general term u_n.

c Find the 100th term of the sequence.

d Is **i** 828 **ii** 2341 a term of the sequence?

a $9 - 2 = 7$, $16 - 9 = 7$, $23 - 16 = 7$, $30 - 23 = 7$

The difference between successive terms is constant.

$\therefore$ the sequence is arithmetic with $u_1 = 2$ and $d = 7$.

b
$$\begin{aligned} u_n &= u_1 + (n-1)d \\ \therefore\ u_n &= 2 + 7(n-1) \\ \therefore\ u_n &= 7n - 5 \end{aligned}$$

c
$$\begin{aligned} u_{100} &= 7(100) - 5 \\ &= 695 \end{aligned}$$

d **i**
$$\begin{aligned} \text{Let}\quad u_n &= 828 \\ \therefore\ 7n - 5 &= 828 \\ \therefore\ 7n &= 833 \\ \therefore\ n &= 119 \end{aligned}$$

$\therefore$ 828 is a term of the sequence, and in fact is the 119th term.

ii
$$\begin{aligned} \text{Let}\quad u_n &= 2341 \\ \therefore\ 7n - 5 &= 2341 \\ \therefore\ 7n &= 2346 \\ \therefore\ n &= 335\tfrac{1}{7} \end{aligned}$$

But n must be an integer, so 2341 is not a member of the sequence.

EXERCISE 5B.1

1 For each of these arithmetic sequences:

- **i** State u_1 and d.
- **ii** Find the formula for the general term u_n.
- **iii** Find the 15th term of the sequence.

a 19, 25, 31, 37, ... **b** 101, 97, 93, 89, ... **c** 8, $9\frac{1}{2}$, 11, $12\frac{1}{2}$, ...

d 31, 36, 41, 46, ... **e** 5, -3, -11, -19, ... **f** a, $a+d$, $a+2d$, $a+3d$, ...

2 Consider the sequence 6, 17, 28, 39, 50, ...

a Show that the sequence is arithmetic. **b** Find the formula for its general term.

c Find the 50th term. **d** Is 325 a member?

e Is 761 a member?

3 Consider the sequence 87, 83, 79, 75, 71, ...

a Show that the sequence is arithmetic. **b** Find the formula for its general term.

c Find the 40th term. **d** Which term of the sequence is -297?

4 A sequence is defined by $u_n = 3n - 2$.

a By finding $u_{n+1} - u_n$, prove that the sequence is arithmetic.

b Find u_1 and d. **c** Find the 57th term.

d What is the largest term of the sequence that is smaller than 450? Which term is this?

5 A sequence is defined by $u_n = \dfrac{71-7n}{2}$.

a Prove that the sequence is arithmetic. **b** Find u_1 and d. **c** Find u_{75}.

d For what values of n are the terms of the sequence less than -200?

6 An arithmetic sequence starts 23, 36, 49, 62, ... Find the first term of the sequence to exceed 100 000.

7 A sequence is defined by the formula $u_1 = -12$, $u_{n+1} = u_n + 7$, $n \geqslant 1$.

a Prove that the sequence is arithmetic. **b** Find the 200th term of the sequence.

c Is 1000 a member of the sequence?

Example 3 **Self Tutor**

Find k given that $3k+1$, k, and -3 are consecutive terms of an arithmetic sequence.

Since the terms are consecutive, $k-(3k+1) = -3-k$ {equating differences}

$$\begin{aligned} \therefore\ k-3k-1 &= -3-k \\ \therefore\ -2k-1 &= -3-k \\ \therefore\ -1+3 &= -k+2k \\ \therefore\ k &= 2 \end{aligned}$$

8 Find k given the consecutive arithmetic terms:

a 32, k, 3 **b** k, 7, 10 **c** k, $2k-1$, 13

d k, $2k+1$, $8-k$ **e** $2k+7$, $3k+5$, $5k-4$ **f** $2k+18$, $-2-k$, $2k+2$

g k, k^2, k^2+6 **h** 5, k, k^2-8

9 Suppose $10k + 1$, $2k$, and $4k^2 - 5$ are consecutive terms of an arithmetic sequence.

  a Find the possible values of k.

  b For each value of k, find the common difference of the sequence.

Example 4 **Self Tutor**

Find the general term u_n for an arithmetic sequence with $u_3 = 8$ and $u_8 = -17$.

$u_3 = 8$ $\quad\therefore\ u_1 + 2d = 8$ .... (1) $\quad$ {using $u_n = u_1 + (n-1)d$}

$u_8 = -17$ $\quad\therefore\ u_1 + 7d = -17$ .... (2)

We now solve (1) and (2) simultaneously:

$$\begin{aligned} -u_1 - 2d &= -8 && \text{\{multiplying both sides of (1) by } -1\text{\}} \\ u_1 + 7d &= -17 && \\ \hline \therefore\ 5d &= -25 && \text{\{adding the equations\}} \\ \therefore\ d &= -5 && \end{aligned}$$

So, in (1):

$$\begin{aligned} u_1 + 2(-5) &= 8 \\ \therefore\ u_1 - 10 &= 8 \\ \therefore\ u_1 &= 18 \\ \text{Now}\quad u_n &= u_1 + (n-1)d \\ \therefore\ u_n &= 18 - 5(n-1) \\ \therefore\ u_n &= 18 - 5n + 5 \\ \therefore\ u_n &= 23 - 5n \end{aligned}$$

Check:

$$\begin{aligned} u_3 &= 23 - 5(3) \\ &= 23 - 15 \\ &= 8 \ \checkmark \\ u_8 &= 23 - 5(8) \\ &= 23 - 40 \\ &= -17 \ \checkmark \end{aligned}$$

10 Find the general term u_n for an arithmetic sequence with:

  a $u_7 = 41$ and $u_{13} = 77$

  b $u_5 = -2$ and $u_{12} = -12\frac{1}{2}$

  c seventh term 1 and fifteenth term -39

  d eleventh and eighth terms being -16 and $-11\frac{1}{2}$ respectively.

11 Suppose a sequence $u_1, u_2, u_3, u_4, u_5, u_6, ...$ is arithmetic.

  a Show that $u_1 + u_2$, $u_3 + u_4$, $u_5 + u_6$, ... is also arithmetic.

  b Given that $u_2 = 5$ and $u_{10} = 61$, find the 30th term of the sequence in **a**.

Example 5 **Self Tutor**

Insert four numbers between 3 and 12 so that all six numbers are in arithmetic sequence.

Suppose the common difference is d.

$\therefore$ the numbers are 3, $3 + d$, $3 + 2d$, $3 + 3d$, $3 + 4d$, and 12

$$\begin{aligned} \therefore\ 3 + 5d &= 12 \\ \therefore\ 5d &= 9 \\ \therefore\ d &= \tfrac{9}{5} = 1.8 \end{aligned}$$

So, the sequence is 3, 4.8, 6.6, 8.4, 10.2, 12.

12 Insert three numbers between 5 and 10 so that all five numbers are in arithmetic sequence.

13 Insert six numbers between -1 and 32 so that all eight numbers are in arithmetic sequence.

14 **a** Insert three numbers between 50 and 44 so that all five numbers are in arithmetic sequence.
  b Assuming the sequence continues, find the first negative term of the sequence.

15 $\frac{1}{k}$, k, k^2+1 are respectively the 3rd, 4th, and 6th terms of an arithmetic sequence.

Given that $k \in \mathbb{Q}$, find: **a** k **b** the general term u_n.

16 The three numbers x, y, and z are such that $x > y > z > 0$.

Show that if $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$ are consecutive terms of an arithmetic sequence, then $x-z$, y, and $x-y+z$ could be the lengths of the sides of a right angled triangle.

17 It is known that there are infinitely many prime numbers. Is it possible to construct an infinite arithmetic sequence such that all of the terms are distinct primes? Explain your answer.

Example 6 **Self Tutor**

Ryan is a cartoonist. His comic strip has just been bought by a newspaper, so he sends them the 28 comic strips he has drawn so far. Each week after the first he sends 3 more comic strips to the newspaper.

a Find the total number of comic strips sent after 1, 2, 3, and 4 weeks.
b Show that the total number of comic strips sent after n weeks forms an arithmetic sequence.
c Find the number of comic strips sent after 15 weeks.
d When does Ryan send his 120th comic strip?

a *Week 1*: 28 comic strips
Week 2: $28+3=31$ comic strips
Week 3: $31+3=34$ comic strips
Week 4: $34+3=37$ comic strips

b Every week, Ryan sends 3 comic strips, so the difference between successive weeks is always 3. We have an arithmetic sequence with $u_1 = 28$ and common difference $d = 3$.

c $$\begin{aligned} u_n &= u_1 + (n-1)d \\ &= 28 + (n-1)\times 3 \\ &= 25 + 3n \end{aligned} \qquad \begin{aligned} \therefore\ u_{15} &= 25 + 3\times 15 \\ &= 70 \end{aligned}$$

After 15 weeks Ryan has sent 70 comic strips.

d We want to find n such that
$$\begin{aligned} u_n &= 120 \\ \therefore\ 25+3n &= 120 \\ \therefore\ 3n &= 95 \\ \therefore\ n &= 31\tfrac{2}{3} \end{aligned}$$

Ryan sends the 120th comic strip in the 32nd week.

18 A luxury car manufacturer sets up a factory for a new model vehicle. In the first month only 5 cars are made. After this, 13 cars are made every month.

a List the total number of cars that have been made by the end of each of the first six months.
b Explain why the total number of cars made after n months forms an arithmetic sequence.
c How many cars are made in the first year?
d How long is it until the 250th car is made?

19 At the start of the dry season, Yafiah's 3000 L water tank is full. She uses 183 L of water each week to water her garden.

- **a** Find the amount of water left in the tank after 1, 2, 3, and 4 weeks.
- **b** Explain why the amount of water left in the tank after n weeks forms an arithmetic sequence.
- **c** When will Yafiah's tank run out of water?

APPROXIMATIONS USING ARITHMETIC SEQUENCES

In the real-world questions we have just seen, exactly the same number of items is added or subtracted in each time period. We can therefore use an arithmetic sequence to model the number of items exactly.

Most real-world scenarios will not be this exact. Instead, random variation may give us a sequence where the difference between terms is *similar*, but not the same. In these cases we can use an arithmetic sequence as an *approximation*.

For example, the table below shows the total mass of people in a lift as they walk in:

n (people)	1	2	3	4	5	6
Mass (kg)	86.2	147.5	210.1	298.4	385.0	459.8

No two people will have exactly the same mass, so the total mass of people will not form an exact arithmetic sequence. However, since the *average* mass of the people is $\frac{459.8}{6} \approx 76.6$ kg, a reasonable model for the total mass would be the arithmetic sequence $u_n = 76.6n$.

EXERCISE 5B.2

1 Halina is measuring the mass of oranges on a scale. When there are 8 oranges, the total mass is 1.126 kg.

- **a** Find the average mass of the oranges on the scale.
- **b** Hence write an arithmetic sequence for u_n, the approximate total mass when n oranges have been placed on the scale.

2 Nadir has placed an empty egg carton on a set of scales. Its mass is 32 g. When the carton is filled with 12 eggs, the total mass of eggs and carton is 743 g.

- **a** Find the average mass of the eggs in the carton.
- **b** Hence write an arithmetic sequence for u_n, the approximate total mass when n eggs have been added to the carton.
- **c** For what values of n is your model valid?

3 A farmer has 580 square bales of hay in his shed with total mass 9850 kg.
The farm has 8 yards of animals. Each day, the farmer feeds out 2 bales of hay to each yard.

- **a** Write an arithmetic sequence for the number of bales of hay remaining after n days.
- **b** Write an arithmetic sequence which *approximates* the mass of hay remaining after n days.

4 Valéria joins a social networking website. After 1 week she has 34 online friends, and after 9 weeks she has 80 online friends.

- **a** Find the average number of online friends Valéria has made each week from week 1 to week 9.
- **b** Assuming that her total number of online friends after n weeks forms an arithmetic sequence, find a model which approximates the number of online friends after n weeks.

c Do you think it is a problem that the common difference in your model is not an integer? Explain your answer.

d Use your model to predict how many online friends Valéria will have after 20 weeks.

5 A wedding reception venue advertises all-inclusive venue hire and catering costs of €6950 for 50 guests or €11 950 for 100 guests.
Assume that the cost of venue hire and catering for n guests forms an arithmetic sequence.

a Write a formula for the general term u_n of the sequence.

b Explain the significance of:

i the common difference **ii** the constant term.

c Estimate the cost of venue hire and catering for a reception with 85 guests.

C GEOMETRIC SEQUENCES

A **geometric sequence** is a sequence in which each term can be obtained from the previous one by multiplying by the same non-zero number. We call this number the **common ratio** r.

A sequence is geometric $\Leftrightarrow$ $\boldsymbol{\frac{u_{n+1}}{u_n} = r}$ for all $n \in \mathbb{Z}^+$.

A geometric sequence can also be referred to as a **geometric progression**.

For example:

- 2, 10, 50, 250, ... is a geometric sequence as each term can be obtained by multiplying the previous term by 5.
  The common ratio is 5 since $\frac{10}{2} = \frac{50}{10} = \frac{250}{50} = 5$.
- 2, -10, 50, -250, ... is a geometric sequence with common ratio -5.

The name "geometric" is given because the middle term of any three consecutive terms is the **geometric mean** of the terms on either side.

If the terms are a, b, c then $\frac{b}{a} = \frac{c}{b}$ {common ratio}

$\therefore\ b^2 = ac$

$\therefore\ b = \pm\sqrt{ac}$ where $\sqrt{ac}$ is the geometric mean.

THE GENERAL TERM FORMULA

Suppose the first term of a geometric sequence is u_1 and the common ratio is r.

Then $u_2 = u_1 r$, $u_3 = u_1 r^2$, $u_4 = u_1 r^3$, and so on.

Hence $u_n = u_1 r^{n-1}$

term number

The power of r is one less than the term number.

For a **geometric sequence** with **first term** $\boldsymbol{u_1}$ and **common ratio** $\boldsymbol{r}$, the **general term** or $\boldsymbol{n}$**th term** is $\boldsymbol{u_n = u_1 r^{n-1}}$.

Example 7 **Self Tutor**

Consider the sequence $8, 4, 2, 1, \frac{1}{2}, ...$

a Show that the sequence is geometric. **b** Find the general term u_n.

c Hence find the 12th term as a fraction.

a $\frac{4}{8} = \frac{2}{4} = \frac{1}{2} = \frac{\frac{1}{2}}{1}$, so consecutive terms have the common ratio $\frac{1}{2}$.

$\therefore$ the sequence is geometric with $u_1 = 8$ and $r = \frac{1}{2}$.

b $u_n = u_1 r^{n-1}$

$\therefore \ u_n = 8\left(\frac{1}{2}\right)^{n-1}$ *or* $u_n = 2^3 \times 2^{-(n-1)}$

$= 2^3 \times 2^{1-n}$

$= 2^{4-n}$

c $u_{12} = 8 \times \left(\frac{1}{2}\right)^{11}$

$= \frac{1}{256}$

EXERCISE 5C

1 For each of these geometric sequences:

i State u_1 and r. **ii** Find the formula for the general term u_n.

iii Find the 9th term of the sequence.

a $3, 6, 12, 24, ...$ **b** $2, 10, 50, ...$ **c** $512, 256, 128, ...$

d $1, 3, 9, 27, ...$ **e** $12, 18, 27, ...$ **f** $\frac{1}{16}, -\frac{1}{8}, \frac{1}{4}, -\frac{1}{2}, ...$

2 **a** Show that the sequence $5, 10, 20, 40, ...$ is geometric.

b Find u_n, and hence find the 15th term.

3 **a** Show that the sequence $12, -6, 3, -\frac{3}{2}, ...$ is geometric.

b Find u_n, and hence write the 13th term as a rational number.

4 **a** Show that the sequence $8, -6, 4.5, -3.375, ...$ is geometric.

b Hence find the 10th term as a decimal.

5 **a** Show that the sequence $8, 4\sqrt{2}, 4, 2\sqrt{2}, ...$ is geometric.

b Hence show that the general term of the sequence is $u_n = 2^{\frac{7}{2}-\frac{1}{2}n}$.

Example 8 **Self Tutor**

$k-1$, $2k$, and $21-k$ are consecutive terms of a geometric sequence. Find k.

Since the terms are geometric, $\frac{2k}{k-1} = \frac{21-k}{2k}$ {equating the common ratio r}

$\therefore \ 4k^2 = (21-k)(k-1)$

$\therefore \ 4k^2 = 21k - 21 - k^2 + k$

$\therefore \ 5k^2 - 22k + 21 = 0$

$\therefore \ (5k-7)(k-3) = 0$

$\therefore \ k = \frac{7}{5}$ or 3

Check: If $k = \frac{7}{5}$ the terms are: $\frac{2}{5}, \frac{14}{5}, \frac{98}{5}$. ✓ $\{r = 7\}$

If $k = 3$ the terms are: $2, 6, 18$. ✓ $\{r = 3\}$

6 Find k given that the following are consecutive terms of a geometric sequence:

- **a** $k,\ 3k,\ 54$
- **b** $1000,\ 4k,\ k$
- **c** $7,\ k,\ 28$
- **d** $18,\ k,\ \frac{2}{9}$
- **e** $k,\ 12,\ \frac{k}{9}$
- **f** $k,\ 20,\ \frac{25}{4}k$
- **g** $k,\ 3k,\ 20-k$
- **h** $k,\ k+8,\ 9k$

7 The first three terms of a geometric sequence are $k-1$, 6, and $3k$.

- **a** Find the possible values of k.
- **b** For each value of k, find the next term in the sequence.

Example 9

Self Tutor

A geometric sequence has $u_2 = -6$ and $u_5 = 162$. Find its general term.

$$\begin{aligned} & u_2 = u_1 r = -6 \quad .... (1) \\ \text{and} \quad & u_5 = u_1 r^4 = 162 \quad .... (2) \\ \text{Now} \quad & \frac{u_1 r^4}{u_1 r} = \frac{162}{-6} \qquad \{(2) \div (1)\} \\ \therefore \quad & r^3 = -27 \\ \therefore \quad & r = -3 \\ \text{Using (1),} \quad & u_1(-3) = -6 \\ \therefore \quad & u_1 = 2 \\ \text{Thus} \quad & u_n = 2 \times (-3)^{n-1} \end{aligned}$$

8 Find the general term u_n of the geometric sequence which has:

- **a** $u_4 = 24$ and $u_7 = 192$
- **b** $u_3 = 8$ and $u_6 = -1$
- **c** $u_7 = 24$ and $u_{15} = 384$
- **d** $u_3 = 5$ and $u_7 = \frac{5}{4}$

9 A geometric sequence has $u_3 = 80$ and $u_6 = 270$.

- **a** Find the first term and common ratio of the sequence.
- **b** Find the 10th term of the sequence.

Example 10

Self Tutor

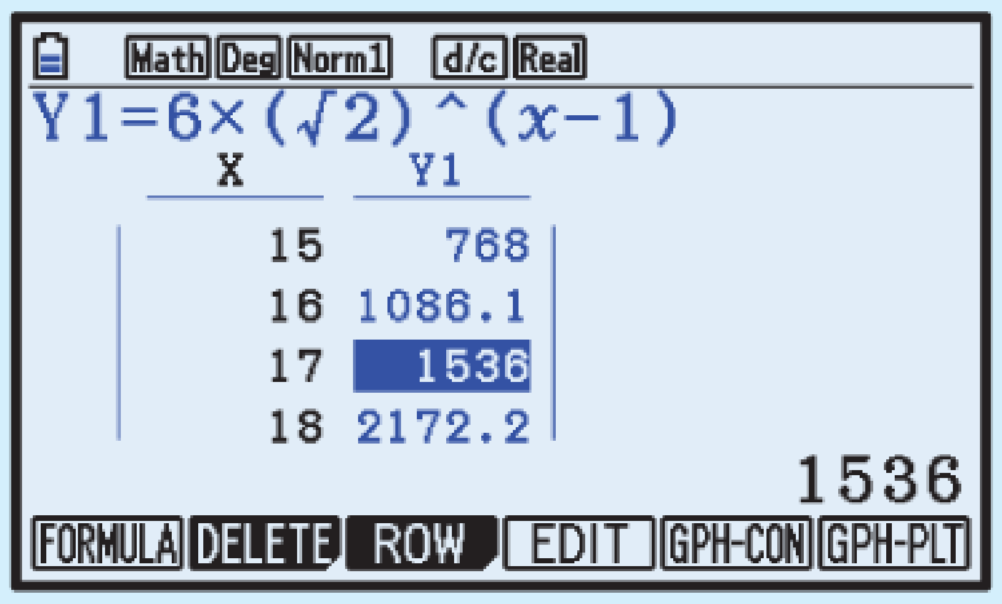
Find the first term of the sequence $6,\ 6\sqrt{2},\ 12,\ 12\sqrt{2},\ ....$ which exceeds 1400.

The sequence is geometric with $u_1 = 6$ and $r = \sqrt{2}$

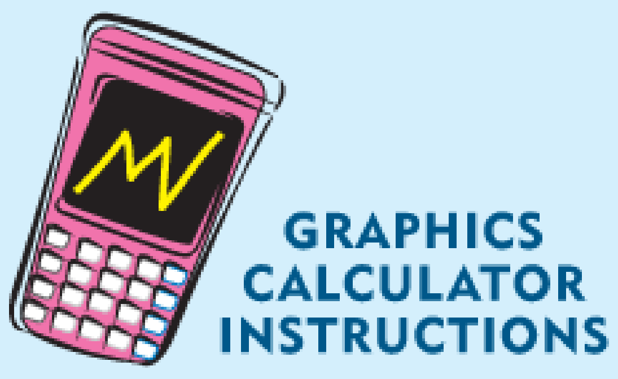
$$\therefore \quad u_n = 6 \times (\sqrt{2})^{n-1}$$

We need to find n such that $u_n > 1400$.

Using a graphics calculator with $\text{Y}_1 = 6 \times (\sqrt{2})\text{\^{}}(\text{X}-1)$, we view a table of values:



```
Y1=6×(√2)^(x−1)
      X      Y1
     15     768
     16  1086.1
     17    1536
     18  2172.2
                1536
```



The first term to exceed 1400 is $u_{17} = 1536$.

10 **a** Find the first term of the sequence 2, 6, 18, 54, ... which exceeds 10 000.

b Find the first term of the sequence $4, 4\sqrt{3}, 12, 12\sqrt{3}, ...$ which exceeds 4800.

c Find the first term of the sequence 12, 6, 3, 1.5, ... which is less than 0.0001 .

d Find the first term of the sequence $5, -\frac{15}{2}, \frac{45}{4}, -\frac{135}{8}, ...$ which is less than -100.

11 What is the maximum number of distinct prime numbers that can occur in a geometric sequence? Give an example of such a sequence.

12 A geometric sequence with common ratio r and an arithmetic sequence with common difference d have the same first two terms. The third terms of the geometric and arithmetic sequences are in the ratio 2 : 1.

a Find the possible values of r.

b For each value of r, find the ratio of the fourth terms of the sequences.

D GROWTH AND DECAY

We now turn our attention to applications of geometric sequences, the first of which being growth and decay problems. Typically, geometric sequences are observed when a quantity increases or decreases by a fixed percentage of its size each time period.

For most real-world situations, we have an **initial condition** corresponding to time zero. This may be an initial investment or an initial population. We therefore allow a "zeroeth" term u_0 to begin the sequence.

Typically, geometric sequences are useful for short term population growth models, or longer term models of radioactive decay.

For situations where there is **growth** or **decay** in a geometric sequence:

$$u_n = u_0 \times r^n$$

where $u_0 =$ initial amount
$r =$ growth multiplier for each time period
$n =$ number of time periods
$u_n =$ amount after n time periods.

Example 11 **Self Tutor**

The initial population of rabbits on a farm was 50. The population increased by 7% each week.

a How many rabbits were present after:

i 15 weeks **ii** 30 weeks?

b How long will it take for the population to reach 500?

There is a fixed percentage increase each week, so the population forms a geometric sequence.

$u_0 = 50$ and $r = 1.07$

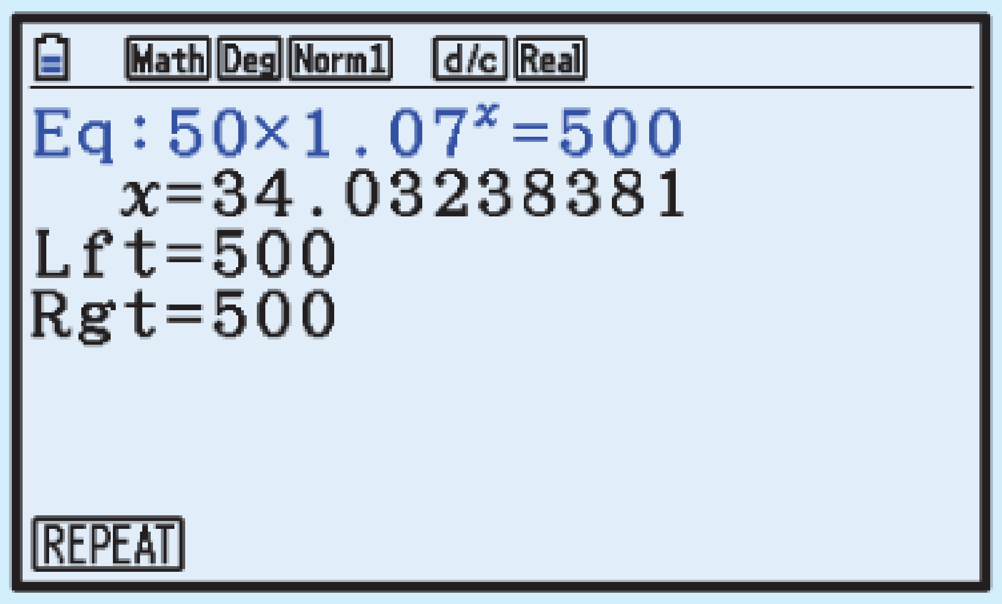
$\therefore$ the population after n weeks is $u_n = 50 \times 1.07^n$.

a **i** $u_{15} = 50 \times (1.07)^{15} \approx 137.95$
There were 138 rabbits.

ii $u_{30} = 50 \times (1.07)^{30} \approx 380.61$
There were 381 rabbits.

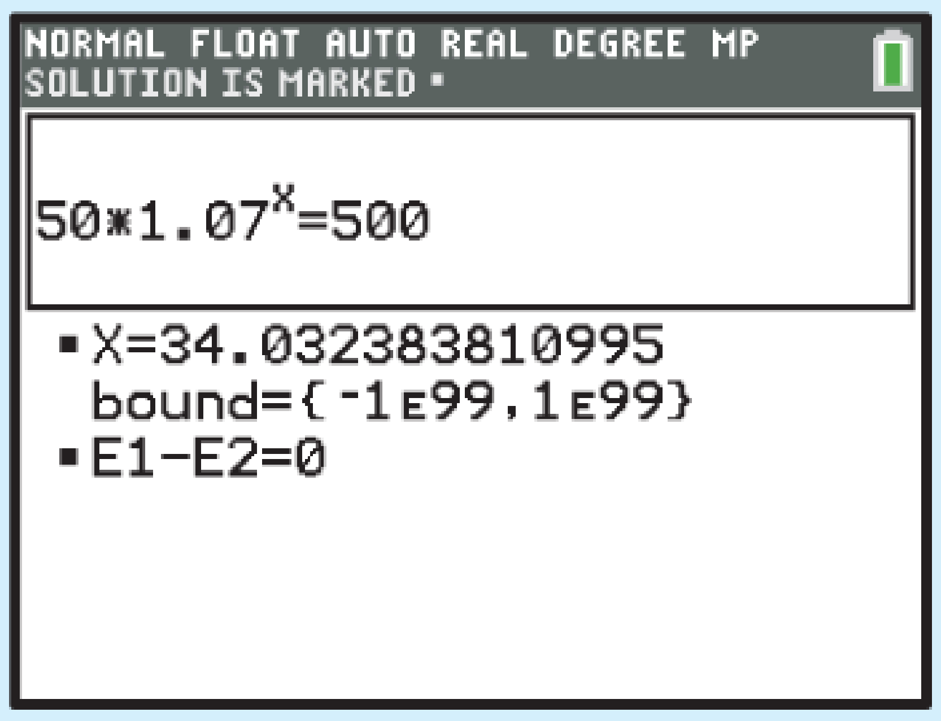
b We need to find when $50 \times (1.07)^n = 500$.

Casio fx-CG50



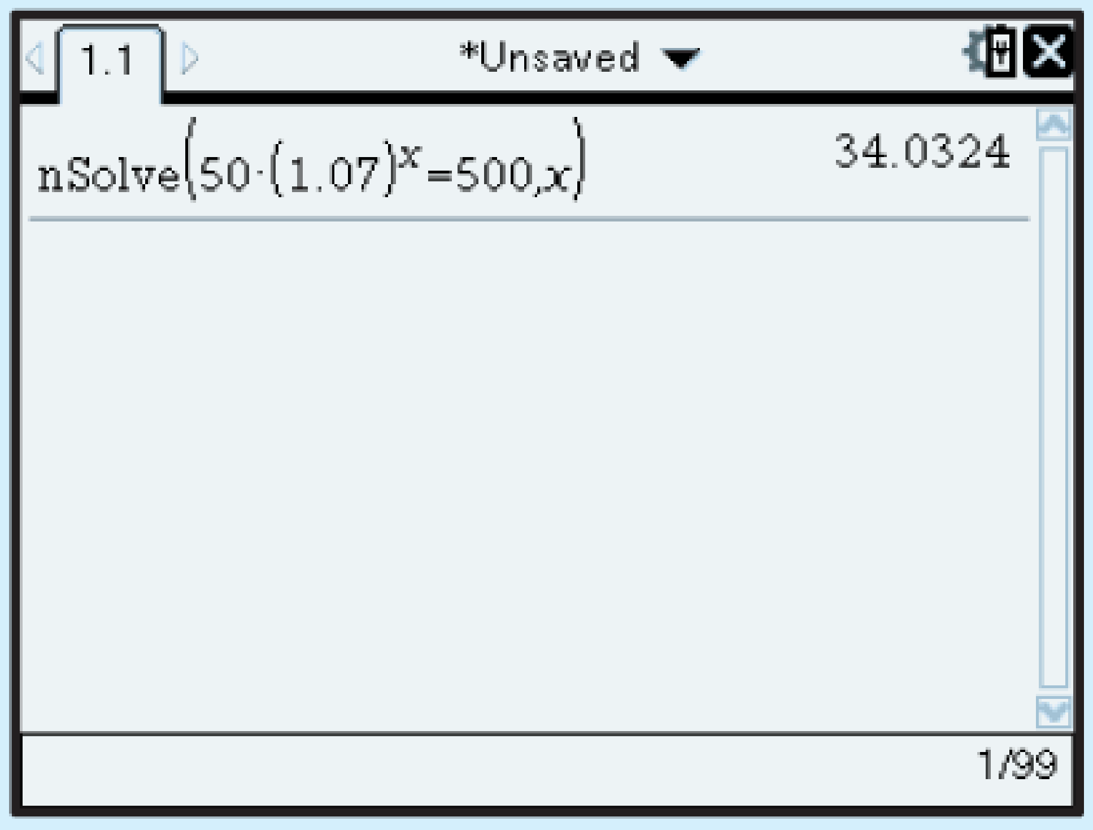
```
Eq:50×1.07^x=500
  x=34.03238381
Lft=500
Rgt=500
```

TI-84 Plus CE



```
50*1.07^X=500
▪X=34.032383810995
 bound={-1E99,1E99}
▪E1-E2=0
```

TI-*n*spire



```
nSolve(50·(1.07)^x=500,x)        34.0324
```

So, it will take approximately 34.0 weeks.

EXERCISE 5D

1 A nest of ants initially contains 500 individuals. The population is increasing by 12% each week.

a How many ants will there be after:

i 10 weeks **ii** 20 weeks?

b How many weeks will it take for the ant population to reach 2000?

2 The animal *Eraticus* is endangered. Since 2005 there has only been one colony remaining, and in 2005 the population of that colony was 555. The population has been steadily decreasing by 4.5% per year.

a Estimate the population in the year 2020.

b In what year do we expect the population to have declined to 50?

3 A herd of 32 deer is to be left unchecked in a new sanctuary. It is estimated that the size of the herd will increase each year by 18%.

a Estimate the size of the herd after:

i 5 years **ii** 10 years.

b How long will it take for the herd size to reach 5000?

4 An endangered species of marsupial has a population of 178. However, with a successful breeding program it is expected to increase by 32% each year.

a Find the expected population size after:

i 10 years **ii** 25 years.

b Estimate how long it will take for the population to reach 10 000.

5 Each year, a physicist measures the remaining radioactivity in a sample. She finds that it reduces by 18% each year. If there was 1.52 g of radioactive material left after 4 years:

a Find the initial quantity of radioactive material.

b How many *more* years will it take for the amount of radioactive material to reduce to 0.2 g?

6 Each year, Maria's salary is increased by 2.3%. She has been working for her company for 10 years, and she currently earns €49 852 per annum.

a What was Maria's salary when she joined the company?

b If she stays with the company for another 4 years, what will her salary be?

E FINANCIAL MATHEMATICS

At some stage in life, most people need to either **invest** or **borrow** money. It is very important that potential investors and borrowers understand these procedures so they can make the right decisions according to their circumstances and their goals.

When money is lent, the person lending the money is known as the **lender**, and the person receiving the money is known as the **borrower**. The amount borrowed is called the **principal**.

The lender usually charges a fee called **interest** to the borrower. This fee represents the cost of using the other person's money. The borrower must repay the principal borrowed as well as the interest charged for using that money.

The rate at which interest is charged is usually expressed as a percentage of the principal. This percentage is known as the **interest rate**, and it is an important factor when deciding where to invest your money and where to borrow money from.

The total amount of interest charged on a loan depends on the principal, the time the money is borrowed for, and the interest rate.

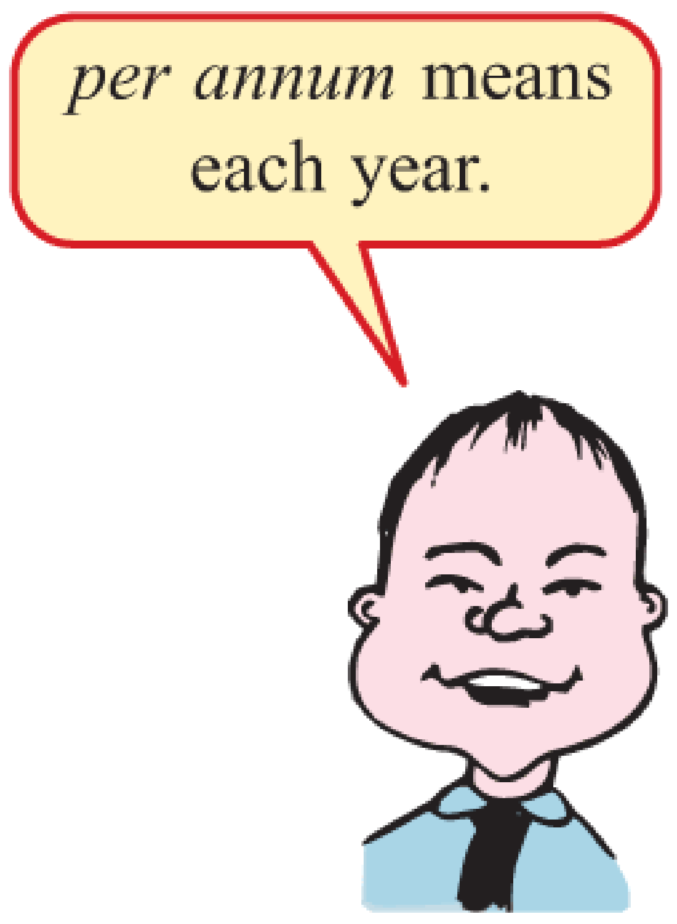
COMPOUND INTEREST

When money is deposited in a bank, it will usually earn **compound interest**.

After a certain amount of time called the **period**, the bank pays interest, which is calculated as a percentage of the money already in the account.

It is called *compound* interest because the interest generated in one period will itself earn more interest in the next period.

For example, suppose you invest \$1000 in the bank. The account pays an interest rate of 4% per annum (p.a.). The interest is added to your investment each year, so at the end of each year you will have $100\% + 4\% = 104\%$ of the value at its start. This corresponds to a *multiplier* of 1.04.



After one year your investment is worth $\$1000 \times 1.04 = \$1040$.

After two years it is worth

$$\begin{aligned} &\$1040 \times 1.04 \\ &= \$1000 \times 1.04 \times 1.04 \\ &= \$1000 \times (1.04)^2 \\ &= \$1081.60 \end{aligned}$$

After three years it is worth

$$\begin{aligned} &\$1081.60 \times 1.04 \\ &= \$1000 \times (1.04)^2 \times 1.04 \\ &= \$1000 \times (1.04)^3 \\ &\approx \$1124.86 \end{aligned}$$

Observe that:

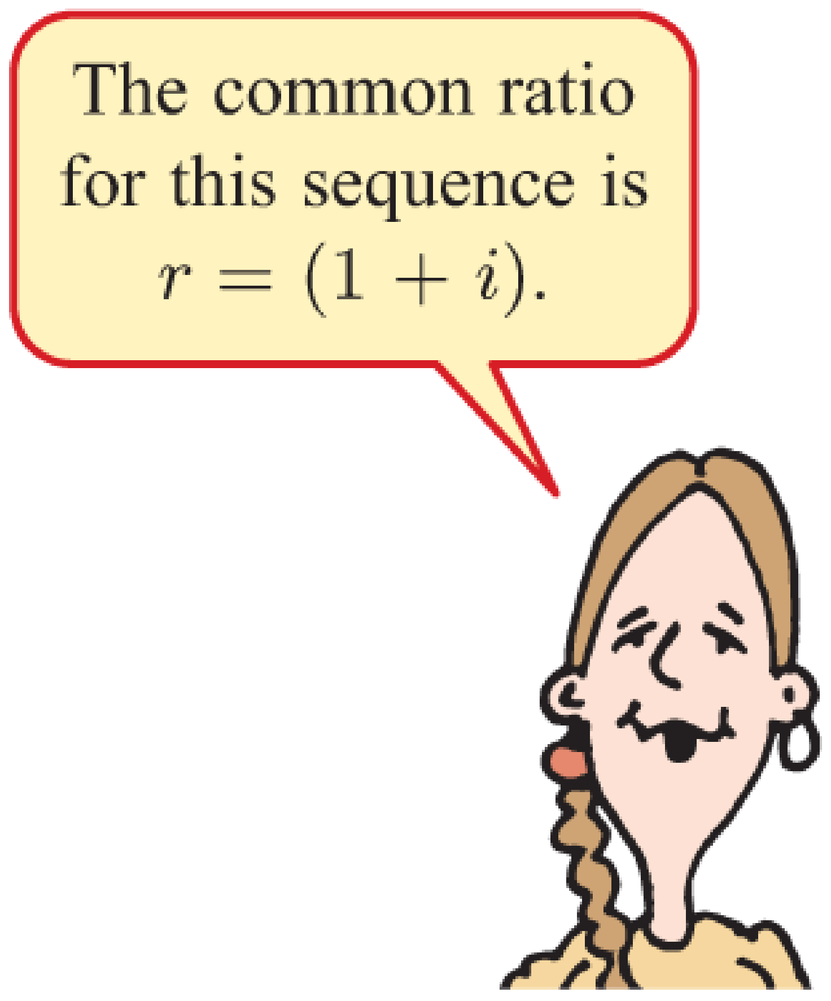
$$\begin{aligned} u_0 &= \$1000 && = \text{initial investment} \\ u_1 &= u_0 \times 1.04 && = \text{amount after } 1 \text{ year} \\ u_2 &= u_0 \times (1.04)^2 && = \text{amount after } 2 \text{ years} \\ u_3 &= u_0 \times (1.04)^3 && = \text{amount after } 3 \text{ years} \\ &\vdots \\ u_n &= u_0 \times (1.04)^n && = \text{amount after } n \text{ years} \end{aligned}$$

So the amount in the account after each year forms a geometric sequence!

The value of a compound interest investment after n time periods is

$$u_n = u_0(1+i)^n$$

where u_0 is the initial investment
and i is the interest rate per compounding period.



Example 12 **Self Tutor**

\$5000 is invested for 4 years at 7% p.a. interest compounded annually.

a What will it amount to at the end of this period?

b How much interest has been earned?

a The interest is calculated annually, so $n = 4$ time periods.

$$\begin{aligned} u_4 &= u_0 \times (1+i)^4 \\ &= 5000 \times (1.07)^4 \qquad \{7\% = 0.07\} \\ &\approx 6553.98 \end{aligned}$$

The investment will amount to \$6553.98 .

b The interest earned $= \$6553.98 - \$5000$
$= \$1553.98$

EXERCISE 5E.1

1 Lucy invested £7000 at 6% p.a. interest compounded annually. Find the value of this investment after 5 years.

2 €2000 is invested for 4 years at 2.8% p.a. interest compounded annually.

- **a** What will it amount to at the end of this period?
- **b** How much interest has been earned?

3 How much compound interest is earned by investing \$8000 at 2.9% p.a. over a 3 year period?

Example 13 **Self Tutor**

£5000 is invested for 4 years at 3% p.a. interest compounded quarterly.
Find the value of the investment at the end of this period.

There are $n = 4 \times 4 = 16$ time periods.

Each time period the investment increases by $i = \frac{3\%}{4} = 0.75\%$.

$$\therefore \text{ the amount after 4 years is } \begin{aligned}[t] u_{16} &= u_0 \times (1+i)^{16} \\ &= 5000 \times (1.0075)^{16} \qquad \{0.75\% = 0.0075\} \\ &\approx 5634.96 \end{aligned}$$

The investment will amount to £5634.96 .

4 \$20 000 is invested at 4.8% p.a. interest compounded quarterly. Find the value of this investment after:

a 1 year **b** 3 years.

5 **a** What will an investment of €30 000 at 5.6% p.a. interest compounded annually amount to after 4 years?

b How much of this is interest?

6 How much interest is earned by investing \$80 000 at 4.4% p.a. for a 3 year period with interest compounded quarterly?

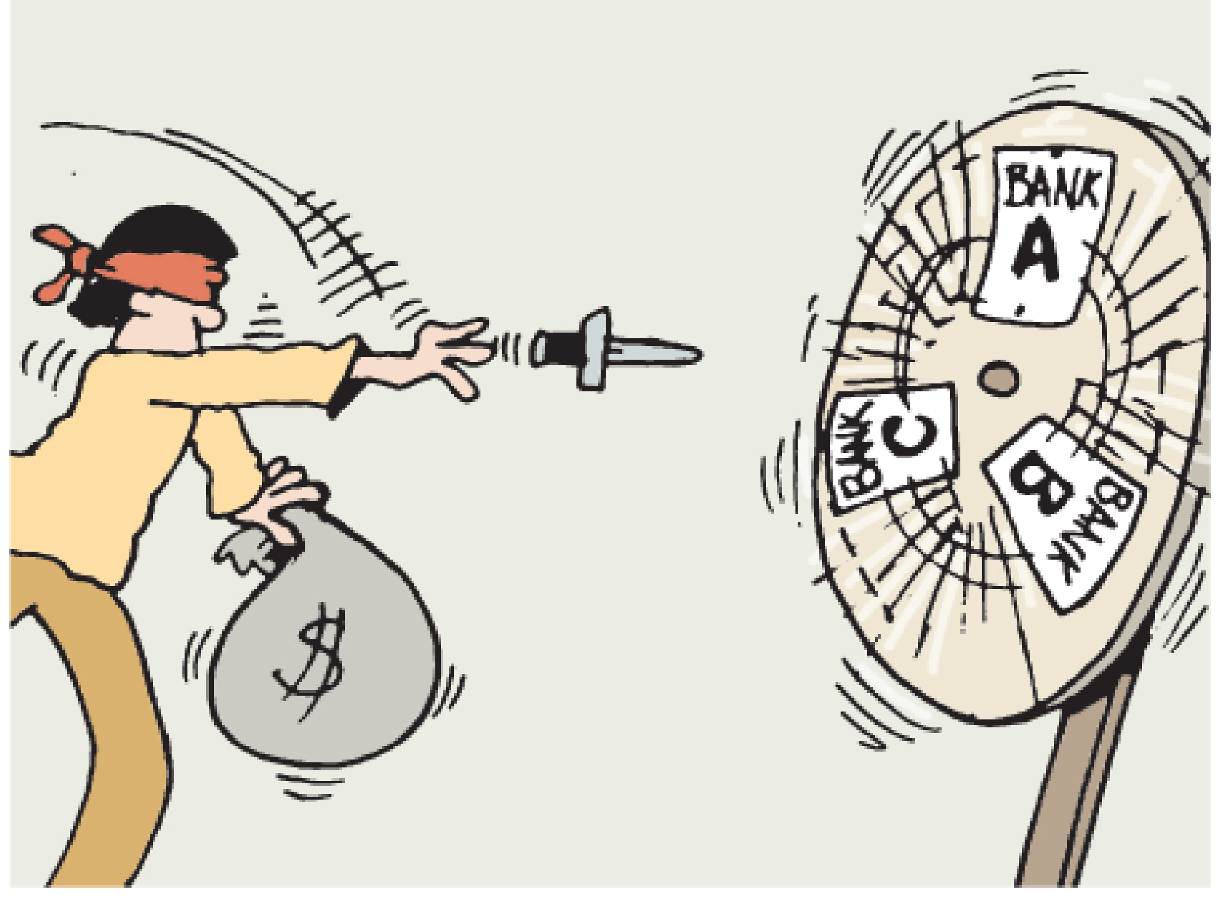
7 Jai recently inherited \$92 000. He decides to invest it for 10 years before he spends any of it. The three banks in his town offer the following terms:

Bank A: $5\frac{1}{2}\%$ p.a. compounded yearly.

Bank B: $5\frac{1}{4}\%$ p.a. compounded quarterly.

Bank C: 5% p.a. compounded monthly.

Which bank offers Jai the greatest interest on his inheritance?



Example 14 **Self Tutor**

How much does Ivana need to invest now, to get a maturing value of \$10 000 in 4 years' time, given interest at 8% p.a. compounded twice annually? Give your answer to the nearest dollar.

The initial investment u_0 is unknown.

There are $n = 4 \times 2 = 8$ time periods.

Each time period the investment increases by $i = \dfrac{8\%}{2} = 4\%$.

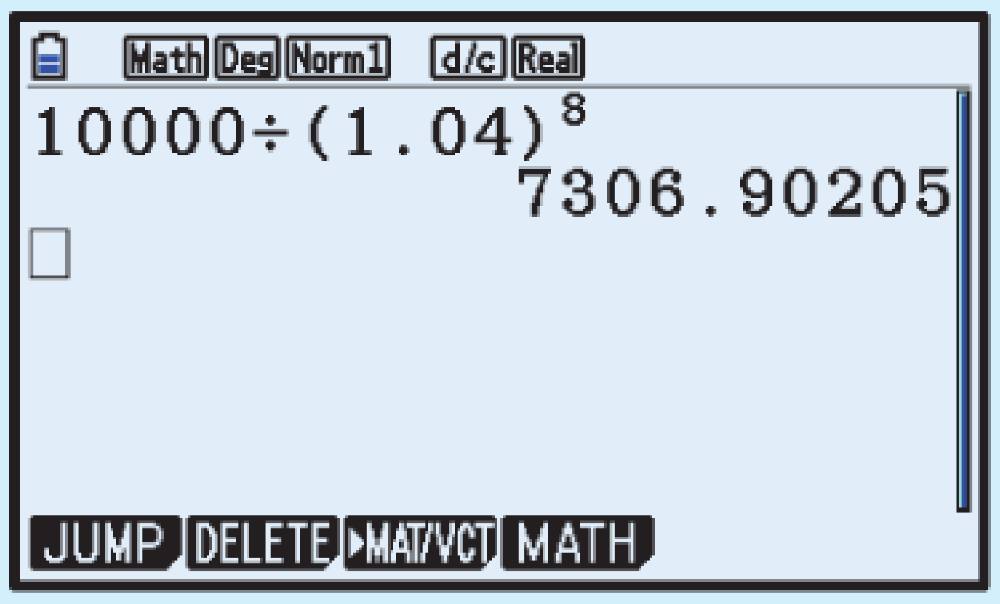
$$\text{Now} \quad u_8 = u_0 \times (1+i)^8$$

$$\therefore \quad 10\,000 = u_0 \times (1.04)^8 \qquad \{4\% = 0.04\}$$

$$\therefore \quad u_0 = \frac{10\,000}{(1.04)^8}$$

$$\therefore \quad u_0 \approx 7306.90$$

Ivana needs to invest \$7307 now.



```
10000÷(1.04)^8
                7306.90205
```

8 How much does Habib need to invest now, to get a maturing value of £20 000 in 4 years' time, given the money can be invested at a fixed rate of 7.5% p.a. compounded annually? Round your answer up to the next pound.

9 What initial investment is required to produce a maturing amount of \$15 000 in 60 months' time given a guaranteed fixed interest rate of 5.5% p.a. compounded annually? Round your answer up to the next dollar.

10 How much should I invest now to yield \$25 000 in 3 years' time, if the money can be invested at a fixed rate of 4.2% p.a. compounded quarterly?

11 What initial investment will yield ¥4 000 000 in 8 years' time if your money can be invested at 3.6% p.a. compounded monthly?

INFLATION

Inflation is the general increase in the price of goods and services over time. Inflation reduces the **purchasing power** of your money, because a fixed amount of money will buy less goods and services as prices rise over time.

Inflation needs to be taken into account when setting investment goals. For example, suppose you see a painting that you would like to buy, and it costs \$5000 today. If it takes you 3 years to accumulate the \$5000, the price of the painting is likely to have increased in that time. So, you will need to accumulate more than \$5000 to purchase the painting.

To find how much you need to accumulate, you must **index** the value of the painting for inflation.

Example 15 **Self Tutor**

Georgia would like to purchase a painting that is currently worth \$5000. She makes monthly deposits into an investment account, so that she can purchase the painting in 3 years' time.

If inflation averages 2.5% per year, calculate the value of the painting indexed for inflation for 3 years.

To index the value of the painting for inflation, we increase it by 2.5% each year for 3 years.

$$\therefore \text{ indexed value} = \$5000 \times (1.025)^3$$
$$= \$5384.45$$

EXERCISE 5E.2

1 If inflation averages 3% per year, calculate the value of:

- **a** \$8000 indexed for inflation over 2 years
- **b** \$14 000 indexed for inflation over 5 years
- **c** \$22 500 indexed for inflation over 7 years.

2 Hoang currently requires \$1000 per week to maintain his lifestyle. Assuming inflation averages 2% per year, how much will Hoang require per week for him to maintain his current lifestyle in:

- **a** 10 years' time
- **b** 20 years' time
- **c** 30 years' time?

3 A holiday package is valued at \$15 000 today.
If inflation averages 2% per year, calculate the value of the holiday package indexed for inflation over 4 years.

THE REAL VALUE OF AN INVESTMENT

To understand how well an investment will perform, we can consider its final value in terms of today's purchasing power. We call this the **real value** of the investment.

We have seen that to index a value for inflation, we *multiply* its value by the inflation multiplier each year. So, to consider the final value of an investment in today's dollars, we *divide* its value by the inflation multiplier each year.

Example 16 **Self Tutor**

Gemma invested \$4000 in an account for 5 years at 4.8% p.a. interest compounded half-yearly. Inflation over the period averaged 3% per year.

a Calculate the value of the investment after 5 years.

b Find the real value of the investment by indexing it for inflation.

a There are $n = 5 \times 2 = 10$ time periods.

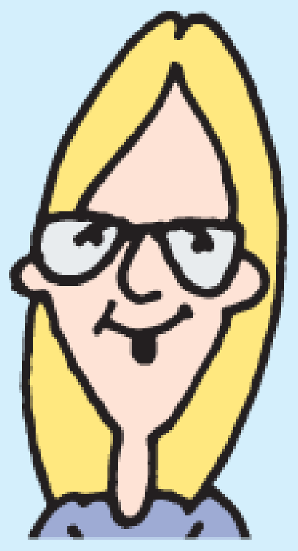
Each period, the investment increases by $i = \dfrac{4.8\%}{2} = 2.4\%$.

$$\therefore \text{ the amount after 5 years is } \quad u_{10} = u_0 \times (1+i)^{10}$$
$$= 4000 \times (1.024)^{10}$$
$$\approx 5070.60$$

The investment will amount to \$5070.60 .

b real value $\times (1.03)^5 = \$5070.60$

$$\therefore \text{ real value} = \frac{\$5070.60}{(1.03)^5}$$
$$= \$4373.94$$



EXERCISE 5E.3

1 Ernie invested \$5000 in an account for 3 years at 3.6% p.a. interest compounded quarterly. Inflation over the period averaged 2% per year.

- **a** Calculate the value of the investment after 3 years.
- **b** Find the real value of the investment by indexing it for inflation.

2 Gino invested €20 000 in an account for 4 years at 4.2% p.a. interest compounded monthly. Inflation over the period averaged 3.4% per year.

- **a** Find the value of Gino's investment after 4 years.
- **b** Find the real value of the investment.

3 Brooke invested \$4000 in an account that pays 3% p.a. interest compounded half-yearly for 6 years.

- **a** Calculate the final value of the investment.
- **b** How much interest did Brooke earn?
- **c** Given that inflation averaged 3.2% per year over the investment period, find the real value of the investment.
- **d** Discuss the effectiveness of the investment once inflation has been considered.

4 Jerome places \$6000 in an investment account. The account pays 0.5% interest per month, and inflation is 0.1% per month.

- **a** Explain why the real interest rate is approximately 0.4% per month.
- **b** Hence find the real value of the investment after 2 years.

5 Suppose \$$u_0$ is invested in an account which pays $i\%$ interest per quarter, and that inflation is $r\%$ per quarter. Write a formula for the *real value* of the investment after y years.

DEPRECIATION

Assets such as computers, cars, and furniture lose value as time passes. This is due to wear and tear, technology becoming old, fashions changing, and other reasons.

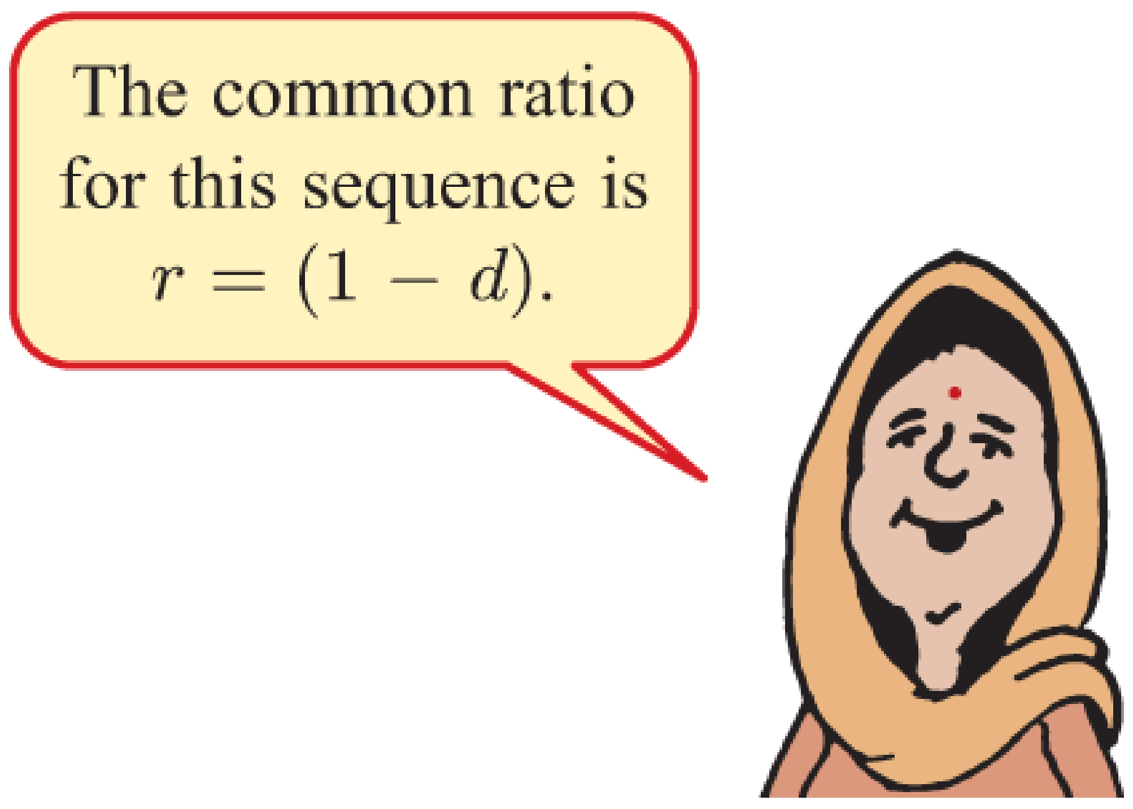
Depreciation is the loss in value of an item over time.

Mathematically, depreciation is similar to compound interest. In the case of compound interest, the investment increases by a fixed percentage each time period. For depreciation, the value *decreases* by a fixed percentage each time period.

The value of an object after n years is $u_n = u_0(1-d)^n$

where u_0 is the initial value of the object

and d is the rate of depreciation per annum.



Example 17 **Self Tutor**

An industrial dishwasher was purchased for £2400 and depreciated by 15% each year.

a Find its value after six years. **b** By how much did it depreciate?

a $u_6 = u_0 \times (1-d)^6$

$= 2400 \times (0.85)^6 \quad \{15\% = 0.15\}$

≈ 905.16

So, after 6 years the value is £905.16.

b The depreciation $=$ £2400 $-$ £905.16

$=$ £1494.84

EXERCISE 5E.4

1 A lathe is purchased by a workshop for €2500. It depreciates by 20% each year. Find the value of the lathe after 3 years.

2 A tractor was purchased for €110 000, and depreciates at 25% p.a. for 5 years.

a Find its value at the end of this period. **b** By how much has it depreciated?

3 Suppose I buy a laptop for ¥87 500 and keep it for 3 years. During this time it depreciates at an annual rate of 30%.

a Find its value after 3 years. **b** By how much has the laptop depreciated?

4 A printing press costing $250 000 was sold 4 years later for $80 000. At what yearly rate did it depreciate in value?

USING TECHNOLOGY FOR FINANCIAL MODELS

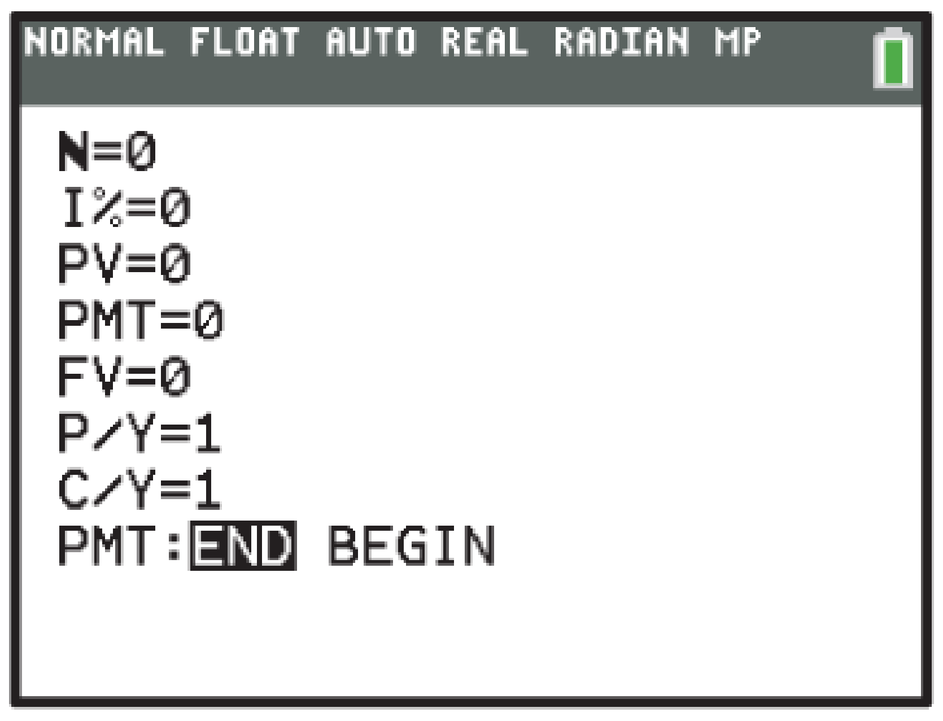
To solve more complicated problems involving financial models, we must use technology.

Most graphics calculators have an in-built **finance program** that can help with finance problems. This is called a **TVM Solver**, where **TVM** stands for **time value of money**.

The TVM Solver can be used to find any variable if all the other variables are given. For the **TI-84 Plus CE**, the abbreviations used are:

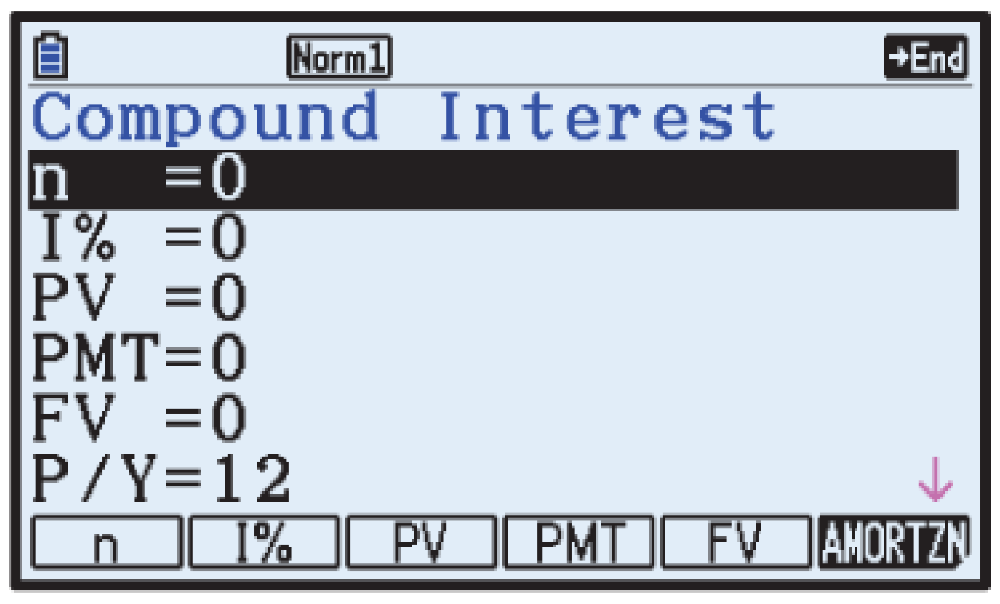
- N represents the **number of compounding periods**
- $I\%$ represents the **interest rate per year**
- PV represents the **present value** of the investment
- PMT represents the **payment each time period**
- FV represents the **future value** of the investment
- P/Y is the **number of payments per year**
- C/Y is the **number of compounding periods per year**
- PMT : END BEGIN lets you choose between payments at the end of a time period or payments at the beginning of a time period. Most interest payments are made at the end of the time periods.

TI-84 Plus CE



```
NORMAL FLOAT AUTO REAL RADIAN MP
N=0
I%=0
PV=0
PMT=0
FV=0
P/Y=1
C/Y=1
PMT:END BEGIN
```

Casio fx-CG50



```
Compound Interest
n   =0
I%  =0
PV  =0
PMT=0
FV  =0
P/Y=12
```

Click on the icon to obtain instructions for using the finance program on your calculator.

GRAPHICS CALCULATOR INSTRUCTIONS

When calculating compound interest using electronic technology, notice that:

- The initial investment is entered as a negative value, because that money is moving from you to the bank. The future value is the money you receive at the end of the investment, so FV is positive.
- N represents the number of compounding periods, not the number of years.
- I is always the percentage interest rate *per annum*.

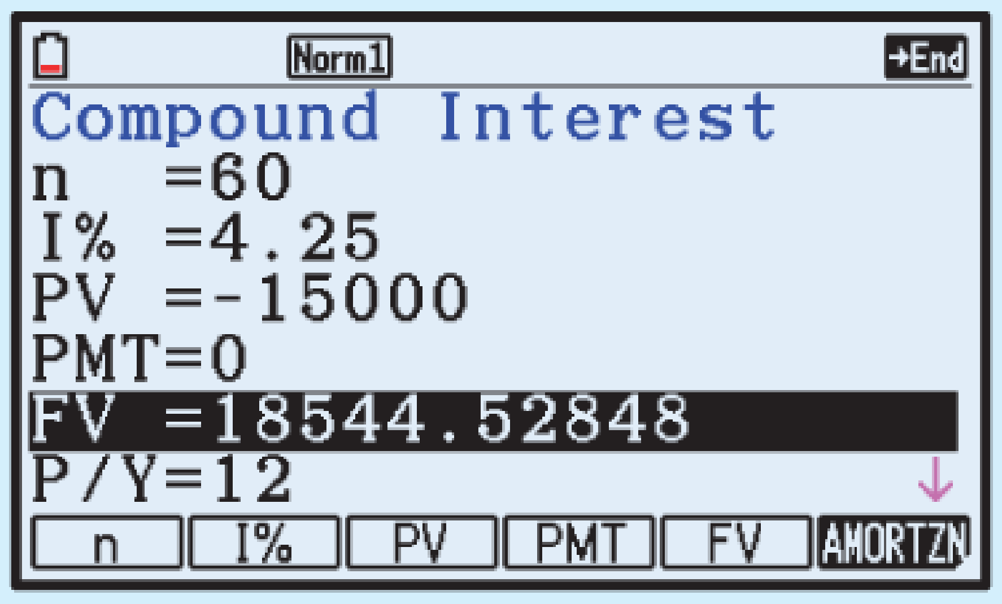
Example 18 **Self Tutor**

Sally invests $15 000 in an account that pays 4.25% p.a. compounded monthly. How much is her investment worth after 5 years?

$N = 5 \times 12 = 60$, $I\% = 4.25$, $PV = -15\,000$, $PMT = 0$, $P/Y = 12$, $C/Y = 12$

$\therefore$ $FV \approx 18\,544.53$

Sally's investment is worth $18 544.53 after 5 years.



```
Compound Interest
n   =60
I%  =4.25
PV  =-15000
PMT=0
FV  =18544.52848
P/Y=12
```

EXERCISE 5E.5

1 Enrique invests 60 000 pesos at 3.7% p.a. compounded annually. Find the value of his investment after 6 years.

2 I deposit $6000 in a bank account that pays 5% p.a. compounded monthly. How much will I have in my account after 2 years?

3 Kenneth sold his boat for $8000, and deposited the money in a bank account paying 5.6% p.a. compounded quarterly. How much will Kenneth have in his account after:

a 3 years **b** 8 years?

4 Drew invested €5000 in an account paying 4.2% p.a. interest compounded monthly.

- **a** Find the amount in the account after 7 years.
- **b** Calculate the interest earned.

5 €4000 is invested in an account that pays 1.2% interest per quarter. At this time, inflation averages 0.5% per quarter.

- **a** Find the real rate of interest per year.
- **b** Hence find the real value of the investment after 5 years.

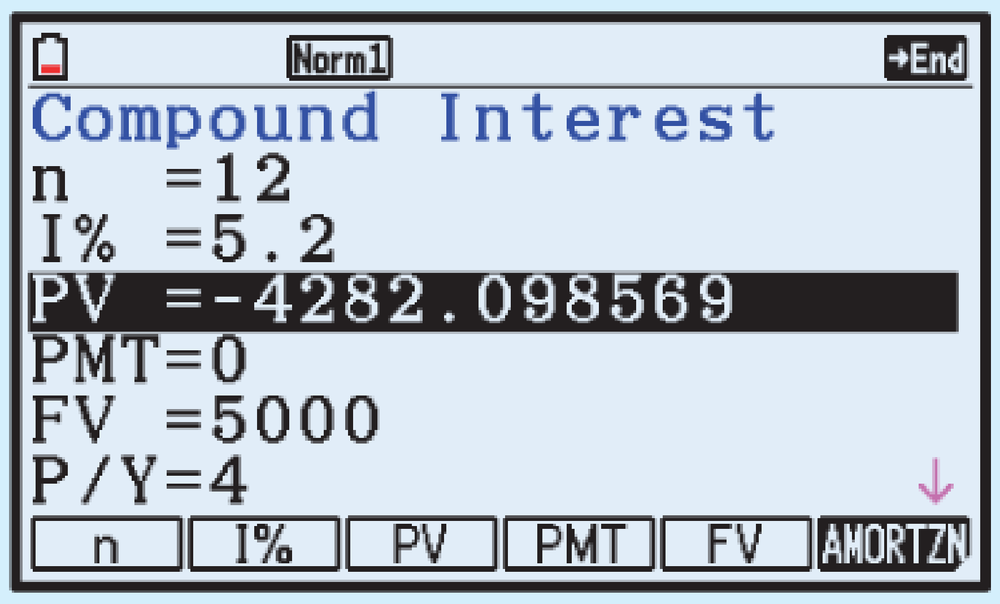
Example 19 **Self Tutor**

Halena is investing money in a term deposit paying 5.2% p.a. compounded quarterly. How much does she need to deposit now, in order to collect \$5000 at the end of 3 years?

$N = 3 \times 4 = 12, \quad I\% = 5.2, \quad PMT = 0, \quad FV = 5000,$
$P/Y = 4, \quad C/Y = 4$

$\therefore \quad PV \approx -4282.10$

Thus, \$4282.10 needs to be deposited.

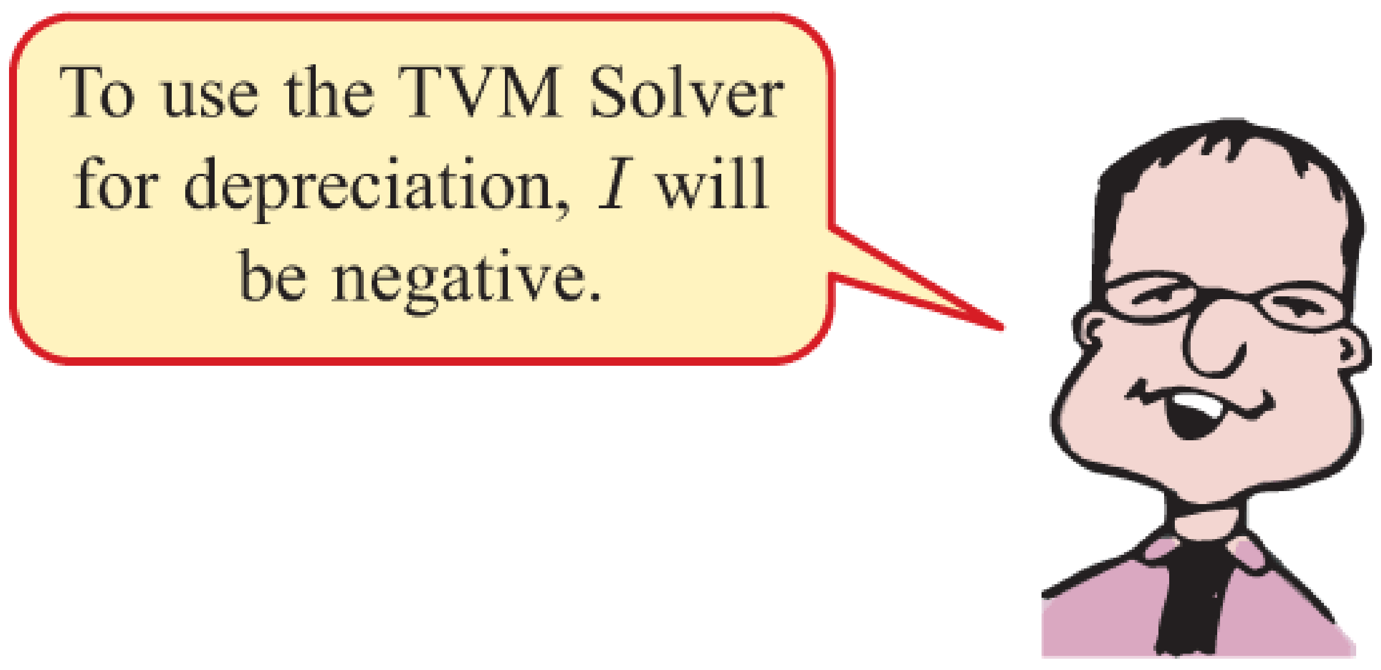


```
Compound Interest
n   =12
I%  =5.2
PV  =-4282.098569
PMT=0
FV  =5000
P/Y=4
```

6 How much would you need to invest now in order to accumulate \$2500 in 5 years' time, if the interest rate is 4.5% p.a. compounded monthly?

7 You have just won the lottery and decide to invest the money. Your accountant advises you to deposit your winnings in an account that pays 6.5% p.a. compounded annually. After four years your winnings have grown to \$102 917.31 . How much did you win in the lottery?

8 Donald bought a new stereo for \$458. If it depreciated in value by 25% p.a., find its value after 5 years.



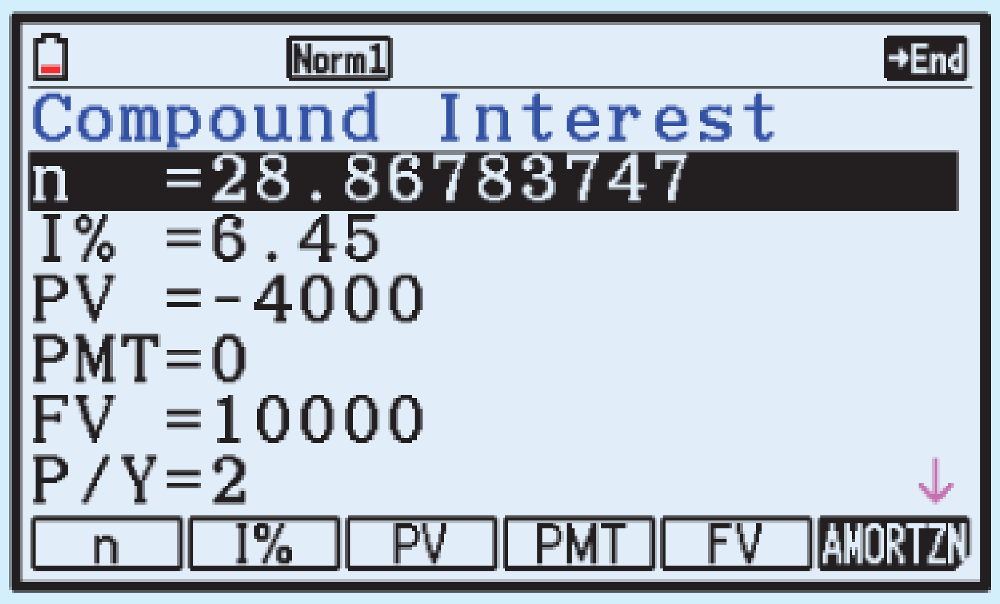
Example 20 **Self Tutor**

For how long must Magnus invest \$4000 at 6.45% p.a. compounded half-yearly, for it to amount to \$10 000?

$I\% = 6.45, \quad PV = -4000, \quad PMT = 0, \quad FV = 10\,000,$
$P/Y = 2, \quad C/Y = 2$

$\therefore \quad N \approx 28.9$

So, 29 half-years are required, which is 14.5 years.



```
Compound Interest
n   =28.86783747
I%  =6.45
PV  =-4000
PMT=0
FV  =10000
P/Y=2
```

9 A couple inherited \$40 000 and deposited it in an account paying $4\frac{1}{2}\%$ p.a. compounded quarterly. They withdrew the money as soon as they had over \$45 000. How long did they keep the money in that account?

10 A business deposits €80 000 in an account that pays $5\frac{1}{4}\%$ p.a. compounded monthly. How long will it take before they double their money?

11 Farm vehicles are known to depreciate in value by 12% each year. If Susan buys a quadrunner for \$6800, how long will it take for the value to reduce to \$1000?

Example 21

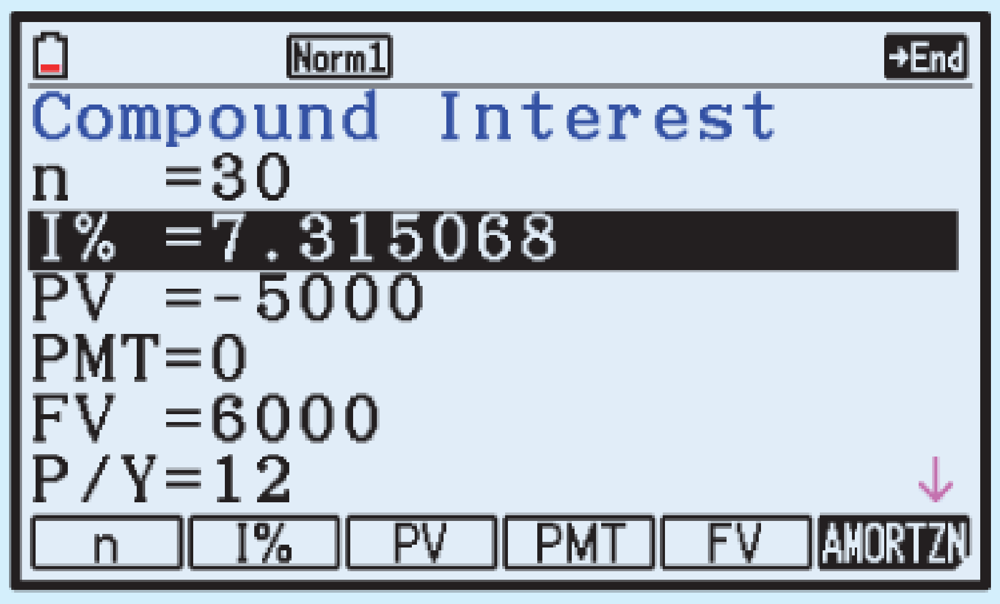
Self Tutor

Iman deposits \$5000 in an account that compounds interest monthly. 2.5 years later, the account has balance \$6000. What annual rate of interest has been paid?

$N = 2.5 \times 12 = 30, \quad PV = -5000, \quad PMT = 0, \quad FV = 6000, \quad P/Y = 12, \quad C/Y = 12$

$\therefore \quad I\% \approx 7.32$

The interest rate is 7.32% p.a.



Compound Interest
n =30
I% =7.315068
PV =-5000
PMT=0
FV =6000
P/Y=12

12 An investor has purchased rare medals for \$10 000 and hopes to sell them 3 years later for \$15 000. What must the annual percentage increase in the value of the medals be, in order for the investor's target to be reached?

13 I deposited €5000 into an account that compounds interest monthly. $3\frac{1}{2}$ years later the account has balance €6165. What annual rate of interest did the account pay?

14 A young couple invests their savings of \$9000 in an account where the interest is compounded quarterly. Three years later the account balance is \$10 493. What interest rate has been paid?

15 A new sports car devalues from £68 500 to £26 380 over 4 years. Find the annual rate of depreciation.

F SERIES

There are many situations where we are interested in finding the sum of the terms of a number sequence.

A **series** is the sum of the terms of a sequence.

For a **finite** sequence with n terms, the corresponding series is $u_1 + u_2 + u_3 + ... + u_n$.

The sum of this series is $S_n = u_1 + u_2 + u_3 + ... + u_n$ and this will always be a finite real number.

For an **infinite** sequence the corresponding series is $u_1 + u_2 + u_3 + ... + u_n + ...$

In many cases, the sum of an infinite series cannot be calculated. In some cases, however, it does **converge** to a finite number.

SIGMA NOTATION

$u_1 + u_2 + u_3 + u_4 + ... + u_n$ can be written more compactly using **sigma notation** or **summation notation**.

The symbol $\sum$ is called **sigma**. It is the equivalent of capital S in the Greek alphabet.

We write $u_1 + u_2 + u_3 + u_4 + ... + u_n$ as $\sum\limits_{k=1}^{n} u_k$.

$\sum\limits_{k=1}^{n} u_k$ reads "the sum of all numbers of the form u_k where $k = 1, 2, 3, ..., $ up to n".

Example 22 Self Tutor

Consider the sequence 1, 4, 9, 16, 25, ...

a Write down an expression for S_n.

b Find S_n for $n = 1, 2, 3, 4,$ and 5.

a $S_n = 1^2 + 2^2 + 3^2 + 4^2 + ... + n^2$ {all terms are squares}

$= \sum\limits_{k=1}^{n} k^2$

b $S_1 = 1$

$S_2 = 1 + 4 = 5$

$S_3 = 1 + 4 + 9 = 14$

$S_4 = 1 + 4 + 9 + 16 = 30$

$S_5 = 1 + 4 + 9 + 16 + 25 = 55$

Example 23 Self Tutor

Expand and evaluate:

a $\sum\limits_{k=1}^{7} (k+1)$

b $\sum\limits_{k=1}^{5} \frac{1}{2^k}$

a $\sum\limits_{k=1}^{7} (k+1) = 2 + 3 + 4 + 5 + 6 + 7 + 8$

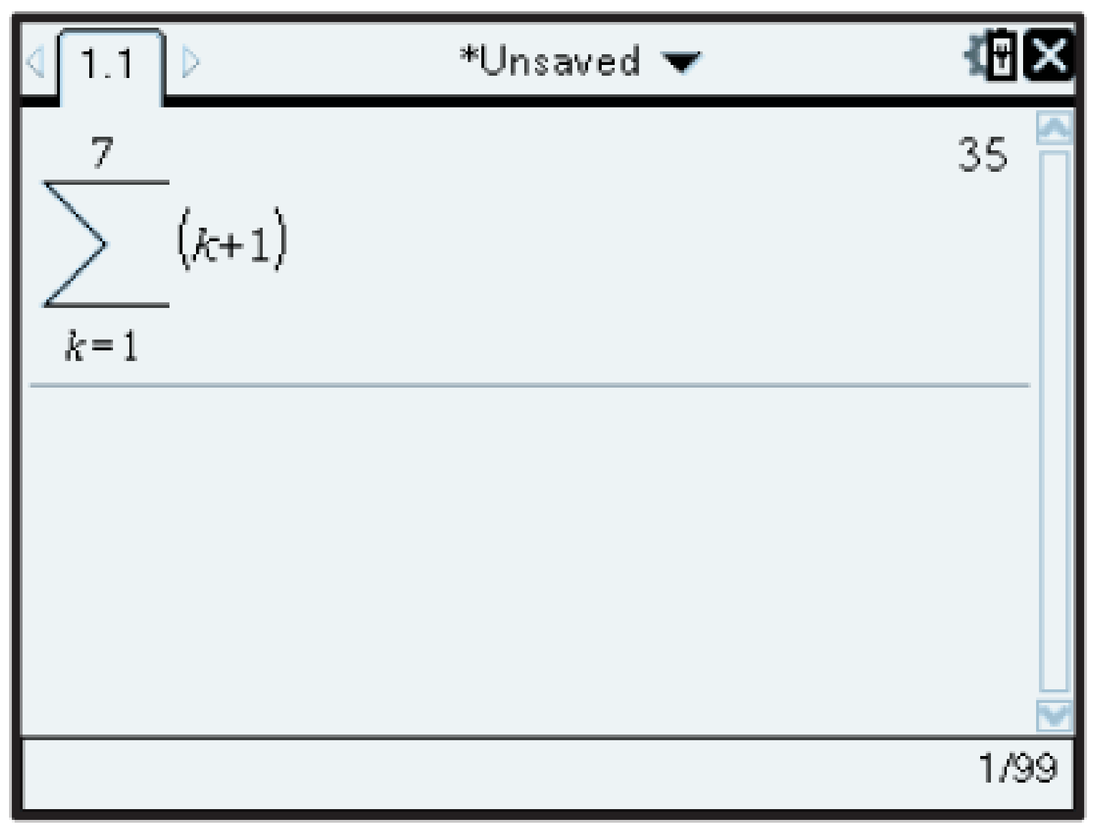
$= 35$

b $\sum\limits_{k=1}^{5} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$

$= \frac{31}{32}$

You can also use technology to evaluate the sum of a series in sigma notation. Click on the icon for instructions.

GRAPHICS CALCULATOR INSTRUCTIONS



1.1 *Unsaved

$\sum\limits_{k=1}^{7} (k+1)$ 35

1/99

PROPERTIES OF SIGMA NOTATION

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

If c is a constant, $\sum\limits_{k=1}^{n} ca_k = c\sum\limits_{k=1}^{n} a_k$ and $\sum\limits_{k=1}^{n} c = cn$.

EXERCISE 5F

1 Consider the sequence of composite numbers $4, 6, 8, 9, 10, 12, 14, 15, 16, ...$
Find:

a S_3 **b** S_5 **c** S_{12}

2 Suppose a sequence has $S_4 = 13$ and $S_5 = 20$. Find the value of u_5.

3 For each of the following sequences:

i Write down an expression for S_n. **ii** Find S_5.

a $3, 11, 19, 27, ...$ **b** $42, 37, 32, 27, ...$ **c** $12, 6, 3, 1\frac{1}{2}, ...$

d $2, 3, 4\frac{1}{2}, 6\frac{3}{4}, ...$ **e** $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$ **f** $1, 8, 27, 64, ...$

4 Expand and evaluate:

a $\sum_{k=1}^{3} 4k$ **b** $\sum_{k=1}^{6} (k+1)$ **c** $\sum_{k=1}^{4} (3k-5)$

d $\sum_{k=1}^{5} (11-2k)$ **e** $\sum_{k=1}^{7} k(k+1)$ **f** $\sum_{k=1}^{5} 10 \times 2^{k-1}$

5 For $u_n = 3n - 1$, write $u_1 + u_2 + u_3 + ... + u_{20}$ using sigma notation and evaluate the sum.

6 Show that:

a $\sum_{k=1}^{n} c = cn$ **b** $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$ **c** $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

Example 24 **Self Tutor**

a Expand $\sum_{k=1}^{n} 2k$ then write the expansion again underneath but with the terms in the reverse order.

b Add the terms vertically, and hence write an expression for the sum S_n of the first n even integers.

a $\sum_{k=1}^{n} 2k = 2 + 4 + 6 + ... + (2n-2) + 2n$

or $2n + (2n-2) + (2n-4) + ... + 4 + 2$

b
$$\begin{aligned} 2\sum_{k=1}^{n} 2k &= (2n+2) + (2n+2) + (2n+2) + ... + (2n+2) + (2n+2) \\ &= n(2n+2) \\ &= 2n(n+1) \end{aligned}$$

$$\therefore \sum_{k=1}^{n} 2k = n(n+1)$$

7 **a** Expand $\sum_{k=1}^{n} k$ then write the expansion again underneath with the terms in the reverse order.

b Add the terms vertically, and hence write an expression for the sum S_n of the first n integers.

c Hence find a and b such that $\sum_{k=1}^{n} (ak+b) = 8n^2 + 11n$ for all positive integers n.

8 Write an expression for the sum of the first n positive odd integers.

9 **a** Show that $(x+1)^3 = x^3 + 3x^2 + 3x + 1$.

b Copy and complete:

$$\begin{aligned}(0+1)^3 &= 0^3 + 3(0)^2 + 3(0) + 1\\ (1+1)^3 &= 1^3 + 3(1)^2 + 3(1) + 1\\ (2+1)^3 &= 2^3 + ...\\ (3+1)^3 &= ....\\ &\vdots\\ (n+1)^3 &= ....\end{aligned}$$

c Add the terms vertically, and hence show that $(n+1)^3 = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + (n+1)$.

d Given that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, show that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

10 Given that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, write $\sum_{k=1}^{n} (k+1)(k+2)$ in simplest form.

Check your answer in the case when $n = 10$.

G ARITHMETIC SERIES

An **arithmetic series** is the sum of the terms of an arithmetic sequence.

For example: 21, 23, 25, 27, ..., 49 is a finite arithmetic sequence.

$21 + 23 + 25 + 27 + ... + 49$ is the corresponding arithmetic series.

SUM OF A FINITE ARITHMETIC SERIES

Rather than adding all the terms individually, we can use a formula to find the sum of a finite arithmetic series.

If the first term is u_1, the final term is u_n, and the common difference is d, the terms are $u_1,\ u_1+d,\ u_1+2d,\ ....,\ (u_n-2d),\ (u_n-d),\ u_n$.

$$\therefore\ S_n = u_1 + (u_1+d) + (u_1+2d) + ... + (u_n-2d) + (u_n-d) + u_n$$

But $S_n = u_n + (u_n-d) + (u_n-2d) + ... + (u_1+2d) + (u_1+d) + u_1$ {reversing them}

Adding these two equations vertically, we get:

$$2S_n = \underbrace{(u_1+u_n) + (u_1+u_n) + (u_1+u_n) + ... + (u_1+u_n) + (u_1+u_n) + (u_1+u_n)}_{n \text{ of these}}$$

$$\therefore\ 2S_n = n(u_1+u_n)$$

$$\therefore\ S_n = \frac{n}{2}(u_1+u_n) \quad \text{where} \quad u_n = u_1 + (n-1)d$$

The sum of a finite arithmetic series with first term u_1, common difference d, and last term u_n, is

$$\boldsymbol{S_n = \frac{n}{2}(u_1+u_n)} \qquad \textit{or} \qquad \boldsymbol{S_n = \frac{n}{2}(2u_1+(n-1)d)}.$$

Example 25 **Self Tutor**

Find the sum of $4 + 7 + 10 + 13 + ...$ to 50 terms.

The series is arithmetic with $u_1 = 4$, $d = 3$, and $n = 50$.

$$\text{Now } S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

$$\therefore \; S_{50} = \tfrac{50}{2}(2 \times 4 + 49 \times 3)$$

$$= 3875$$

Example 26 **Self Tutor**

Find the sum of $-6 + 1 + 8 + 15 + ... + 141$.

The series is arithmetic with $u_1 = -6$, $d = 7$, and $u_n = 141$.

First we need to find n.

$$\begin{aligned} \text{Now } u_n &= 141 \\ \therefore \; u_1 + (n-1)d &= 141 \\ \therefore \; -6 + 7(n-1) &= 141 \\ \therefore \; 7(n-1) &= 147 \\ \therefore \; n - 1 &= 21 \\ \therefore \; n &= 22 \end{aligned}$$

$$\text{Using } S_n = \frac{n}{2}(u_1 + u_n),$$

$$\begin{aligned} S_{22} &= \tfrac{22}{2}(-6 + 141) \\ &= 11 \times 135 \\ &= 1485 \end{aligned}$$

You can also use technology to evaluate series, although for some calculator models this is tedious.

GRAPHICS CALCULATOR INSTRUCTIONS

EXERCISE 5G

1 Find the sum of:

- **a** $7 + 9 + 11 + 13 + ...$ to 10 terms
- **b** $3 + 7 + 11 + 15 + ...$ to 20 terms
- **c** $\frac{1}{2} + 3 + 5\frac{1}{2} + 8 + ...$ to 50 terms
- **d** $100 + 93 + 86 + 79 + ...$ to 40 terms
- **e** $(-31) + (-28) + (-25) + (-22) + ...$ to 15 terms
- **f** $50 + 48\frac{1}{2} + 47 + 45\frac{1}{2} + ...$ to 80 terms.

2 Find the sum of:

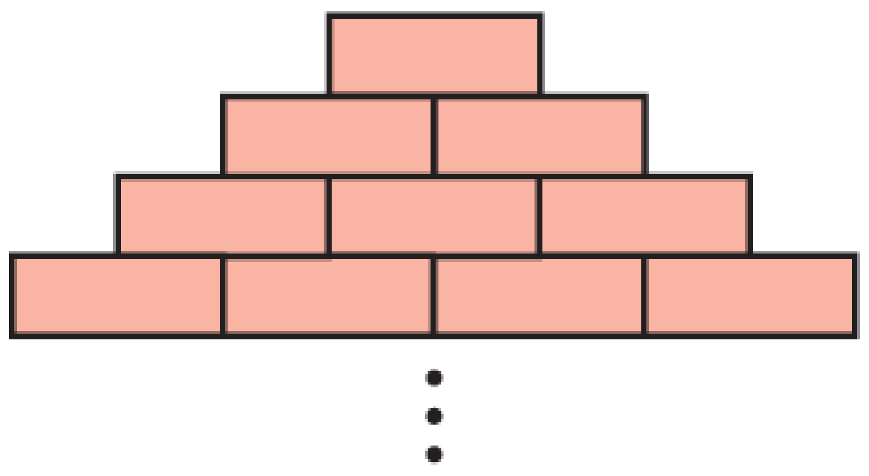
- **a** $5 + 8 + 11 + 14 + ... + 101$
- **b** $37 + 33 + 29 + 25 + ... + 9$
- **c** $50 + 49\frac{1}{2} + 49 + 48\frac{1}{2} + ... + (-20)$
- **d** $8 + 10\frac{1}{2} + 13 + 15\frac{1}{2} + ... + 83$

3 Evaluate these arithmetic series:

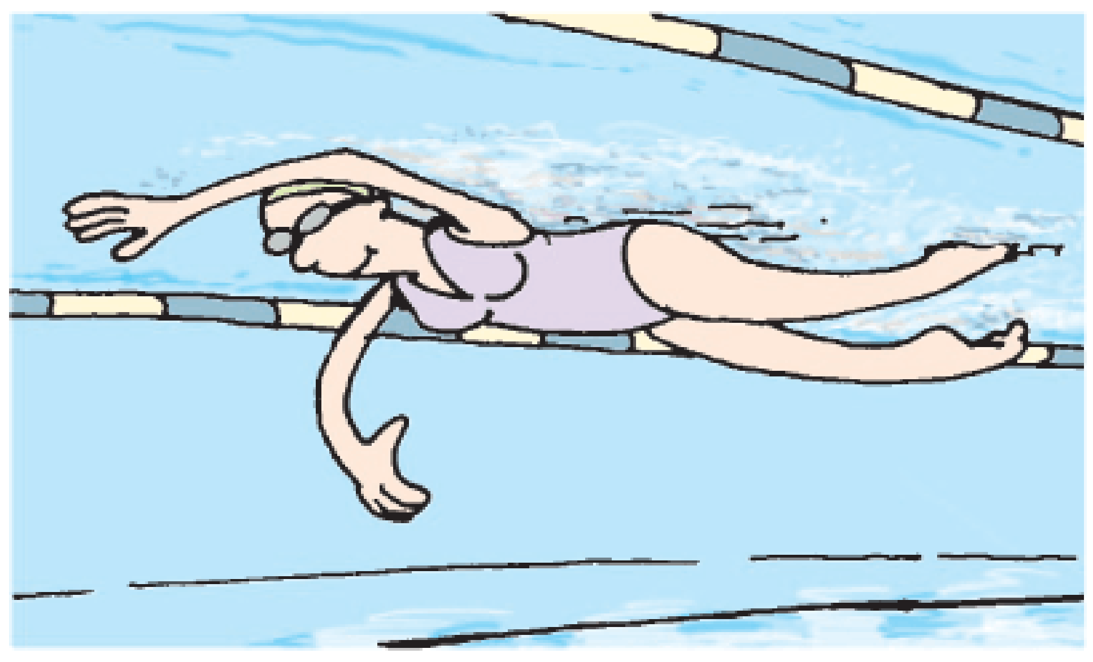
- **a** $\displaystyle\sum_{k=1}^{10} (2k + 5)$
- **b** $\displaystyle\sum_{k=1}^{15} (k - 50)$
- **c** $\displaystyle\sum_{k=1}^{20} \left(\frac{k+3}{2}\right)$

Check your answers using technology.

4 An arithmetic series has eleven terms. The first term is 6 and the last term is -27. Find the sum of the series.

5 

A bricklayer builds a triangular wall with layers of bricks as shown. If the bricklayer uses 171 bricks, how many layers did he build?



6 Vicki has 30 days to train for a swimming competition. She swims 20 laps on the first day, then each day after that she swims two more laps than the previous day. So, she swims 22 laps on the second day, 24 laps on the third day, and so on.

- **a** How many laps does Vicki swim on:
  - **i** the tenth day
  - **ii** the final day?
- **b** How many laps does Vicki swim in total?

7 A woman deposits \$100 into her son's savings account on his first birthday. She deposits \$125 on his second birthday, \$150 on his third birthday, and so on.

- **a** Calculate the amount of money she will deposit into her son's account on his 15th birthday.
- **b** Find the total amount she will have deposited over the 15 years.

8 A football stadium has 25 sections of seating. Each section has 44 rows of seats, with 22 seats in the first row, 23 in the second row, 24 in the third row, and so on. How many seats are there in:

- **a** row 44 of one section
- **b** each section
- **c** the whole stadium?

9 Find the sum of:

- **a** the first 50 multiples of 11
- **b** the multiples of 7 between 0 and 1000
- **c** the integers from 1 to 100 which are not divisible by 3
- **d** the three-digit numbers which start or end with a "4".

10 $k-1$, $2k+3$, and $27-k$ are the first three terms of an arithmetic sequence.

- **a** Find k.
- **b** Find the sum of the first 25 terms of the sequence.

11 The sixth term of an arithmetic sequence is 21, and the sum of the first seventeen terms is 0. Find the first two terms of the sequence.

12 An arithmetic series has $S_3 = 9$ and $S_6 = 90$. Find S_{10}.

Example 27 **Self Tutor**

An arithmetic sequence has first term 8 and common difference 2. The sum of the terms of the sequence is 170. Find the number of terms in the sequence.

Now $S_n = 170$, so

$$\begin{aligned}\frac{n}{2}(2u_1+(n-1)d) &= 170\\ \therefore\ \frac{n}{2}(16+2(n-1)) &= 170\\ \therefore\ 8n+n(n-1) &= 170\\ \therefore\ n^2+7n-170 &= 0\\ \therefore\ (n+17)(n-10) &= 0\\ \therefore\ n &= 10 \quad \{\text{as } n>0\}\end{aligned}$$

$\therefore$ there are 10 terms in the sequence.

13 An arithmetic sequence has first term 4 and common difference 6. The sum of the terms of the sequence is 200. Find the number of terms in the sequence.

14 An arithmetic sequence has $u_1 = 7$ and $S_2 = 17$.

a Find the common difference of the sequence. **b** Find n such that $S_n = 242$.

15 Consider the arithmetic sequence 13, 21, 29, 37, ... How many terms are needed for the sum of the sequence terms to exceed 1000?

16 Use the arithmetic sequence formula to prove that the sum of the first n integers is $\frac{n(n+1)}{2}$.

17 **a** Write a formula for the nth odd integer.

b Hence prove that the sum of the first n odd integers is n^2.

18 Three consecutive terms of an arithmetic sequence have a sum of 12 and a product of -80. Find the terms.

19 The sum of the first 15 terms of an arithmetic sequence is 480. Find the 8th term of the sequence.

20 Five consecutive terms of an arithmetic sequence have a sum of 40. The product of the first, middle, and last terms is 224. Find the terms of the sequence.

21 The sum of the first n terms of an arithmetic sequence is $\frac{n(3n+11)}{2}$.

a Find its first two terms. **b** Find the twentieth term of the sequence.

22 Find $3 - 5 + 7 - 9 + 11 - 13 + 15 - ....$ to 80 terms.

23 The sum of the first n terms of a sequence is $n^2 - \frac{9}{2}n$. Prove that the sequence is arithmetic.

24 Let $u_n = 3 + 2n$.

a For $n = 1, ..., 4$, plot the points (n, u_n) on a graph, and draw rectangles with vertices (n, u_n), $(n+1, u_n)$, $(n, 0)$, and $(n+1, 0)$.

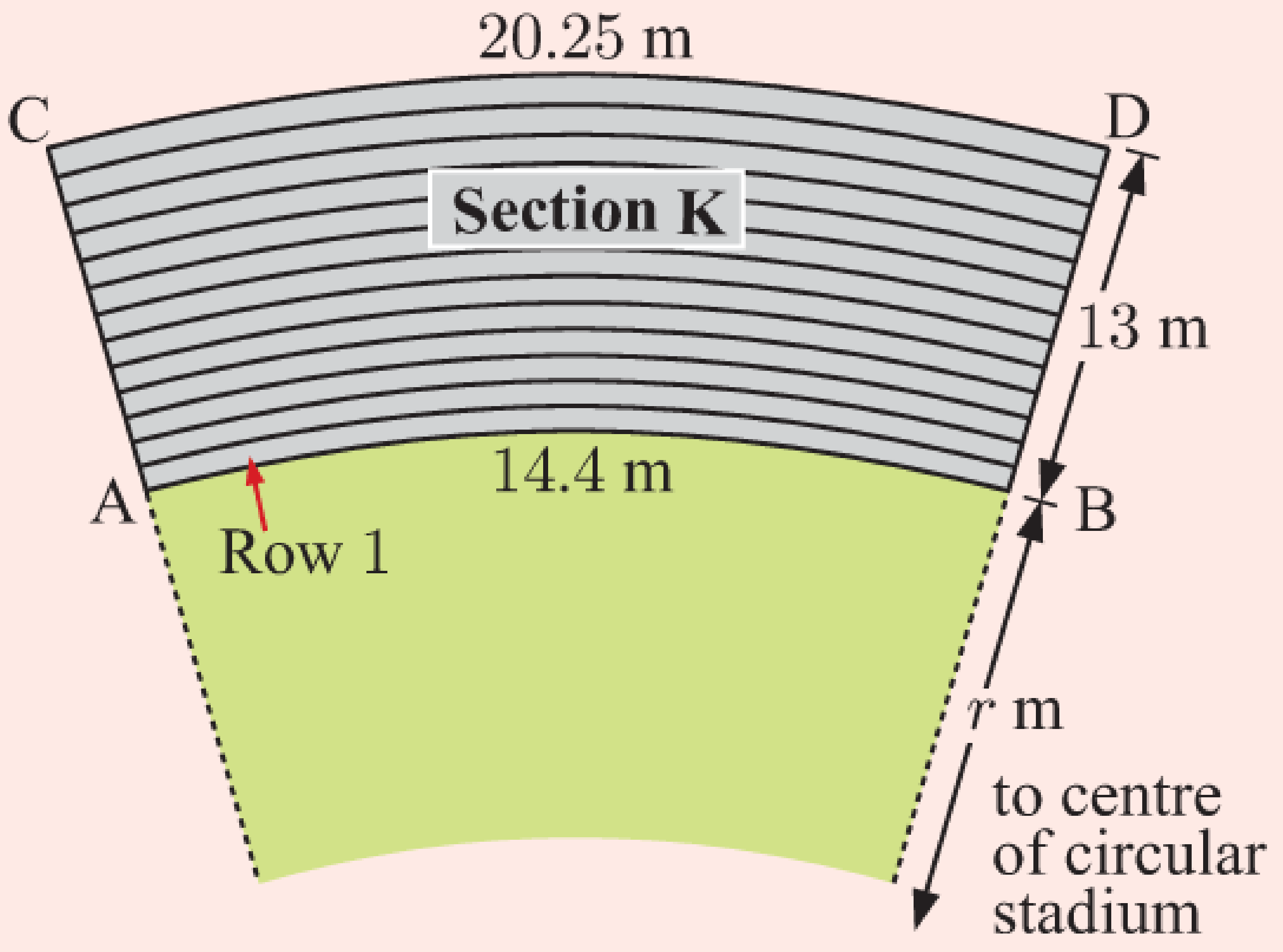
b Explain how S_n relates to the areas of the rectangles.

c Using your sketch, explain why: **i** $u_{n+1} = u_n + 2$ **ii** $S_{n+1} = S_n + u_{n+1}$.

25 An arithmetic sequence has common difference d. The series sums S_2, S_5, and S_7 themselves form an arithmetic sequence. Find, in terms of d, the common difference for this sequence.

ACTIVITY 2 STADIUM SEATING

A circular stadium consists of sections as illustrated, with aisles in between. The diagram shows the 13 tiers of concrete steps for the final section, Section K. Seats are placed along every concrete step, with each seat 0.45 m wide. The arc AB at the front of the first row is 14.4 m long, while the arc CD at the back of the back row is 20.25 m long.



1 How wide is each concrete step?

2 What is the length of the arc of the back of Row 1, Row 2, Row 3, and so on?

3 How many seats are there in Row 1, Row 2, Row 3, ..., Row 13?

4 How many sections are there in the stadium?

5 What is the total seating capacity of the stadium?

6 What is the radius r of the "playing surface"?

THEORY OF KNOWLEDGE

The sequence of odd numbers 1, 3, 5, 7, ... is defined by $u_n = 2n - 1$, $n = 1, 2, 3, 4, ...$

By studying sums of the first few terms of the sequence, we might suspect that the sum of the first n odd numbers is n^2.

$$\begin{aligned} S_1 &= & 1 &= 1 = 1^2 \\ S_2 &= & 1+3 &= 4 = 2^2 \\ S_3 &= & 1+3+5 &= 9 = 3^2 \\ S_4 &= & 1+3+5+7 &= 16 = 4^2 \\ S_5 &= & 1+3+5+7+9 &= 25 = 5^2 \end{aligned}$$

But is this enough to *prove* that the statement is true for all positive integers n?

1 Can we prove that a statement is true in all cases by checking that it is true for some specific cases?

2 How do we know when we have proven a statement to be true?

In the case of the sum of the first n odd integers, you should have proven the result in the last Exercise using the known, proven formula for the sum of an arithmetic series.

However, in mathematics not all **conjectures** turn out to be true. For example, consider the sequence of numbers $u_n = n^2 + n + 41$.

We observe that:

$u_1 = 1^2 + 1 + 41 = 43$ which is prime
$u_2 = 2^2 + 2 + 41 = 47$ which is prime
$u_3 = 3^2 + 3 + 41 = 53$ which is prime
$u_4 = 4^2 + 4 + 41 = 61$ which is prime.

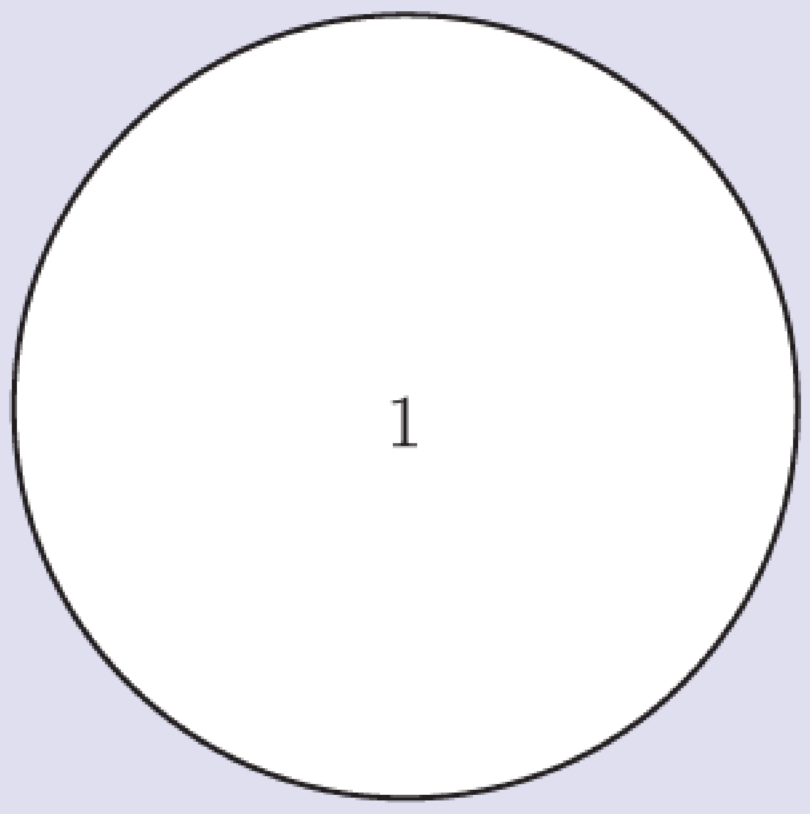
From this, we may *conjecture* that $n^2 + n + 41$ is prime for any positive integer n.

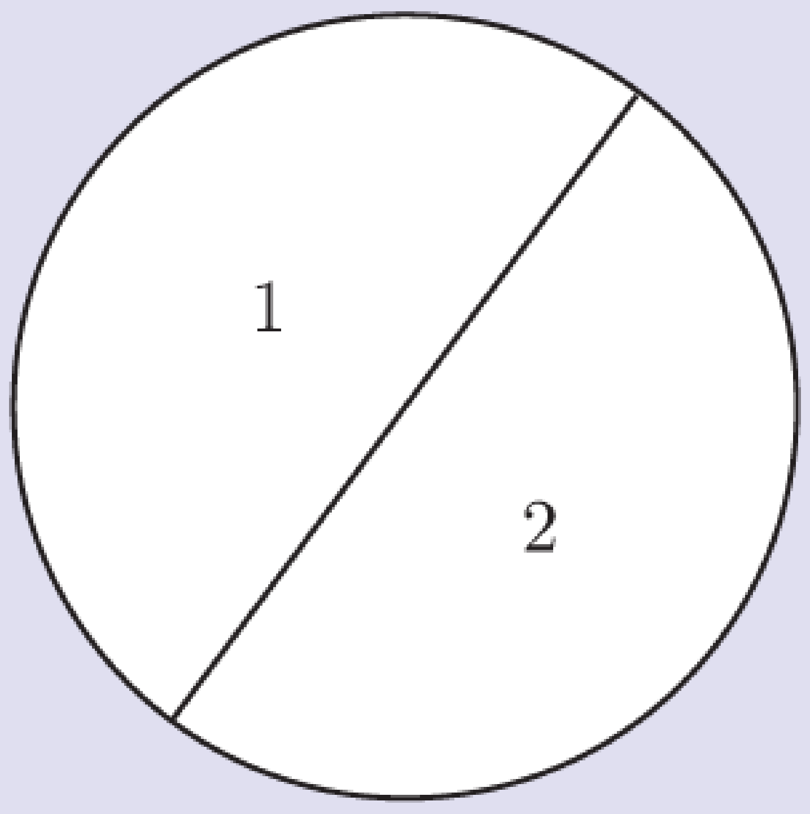
In fact, $n^2 + n + 41$ is prime for all positive integers n from 1 to 39.

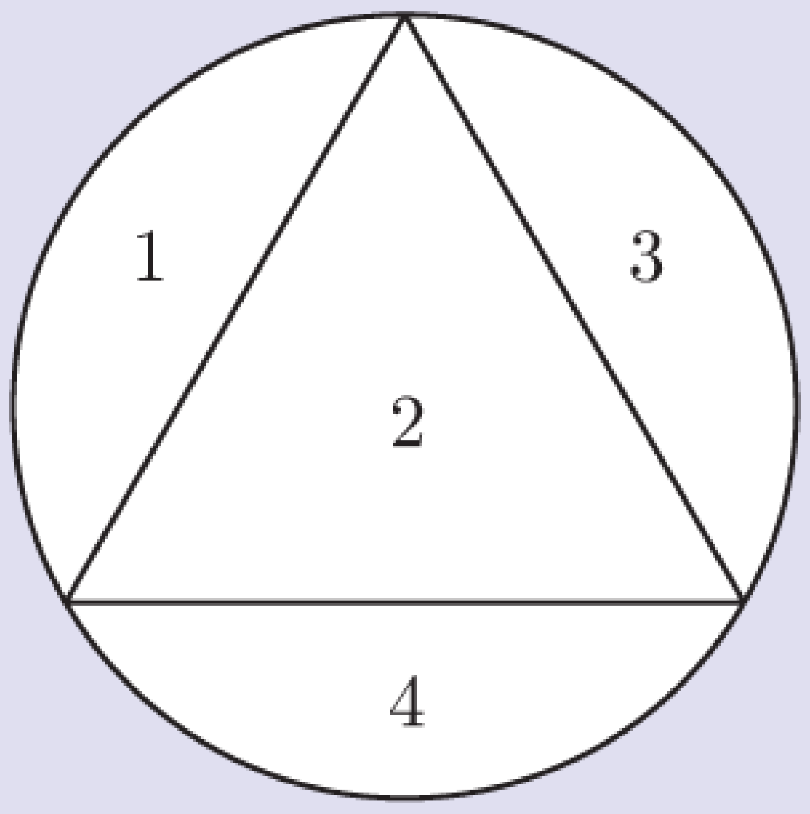
However, $u_{40} = 40^2 + 40 + 41 = 41^2$, so u_{40} is composite.

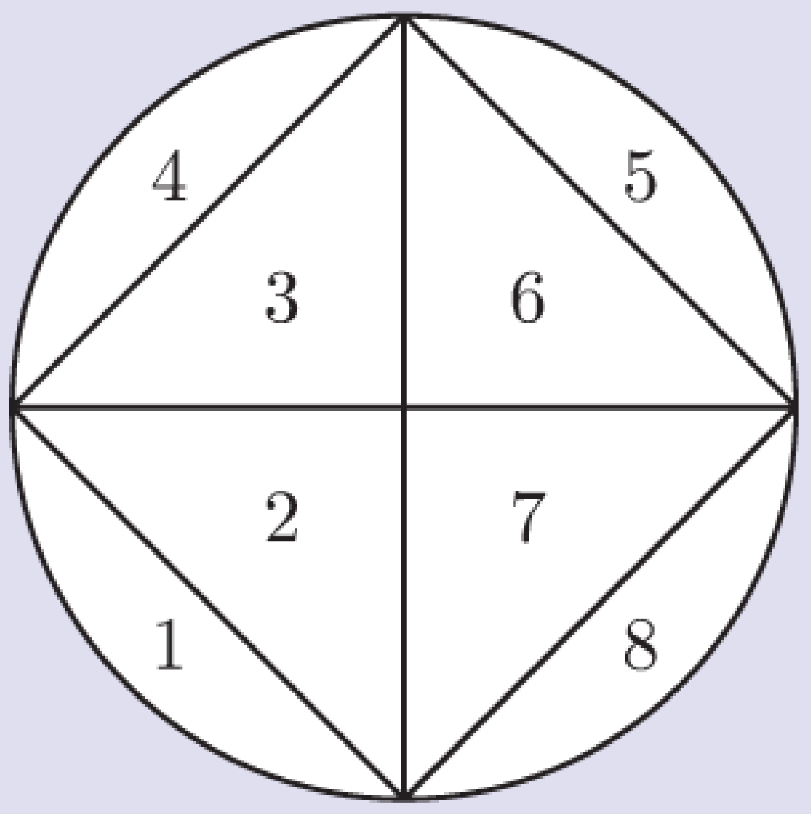
Suppose we place n points around a circle such that when we connect each point with every other point, no three lines intersect at the same point. We then count the number of regions that the circle is divided into.

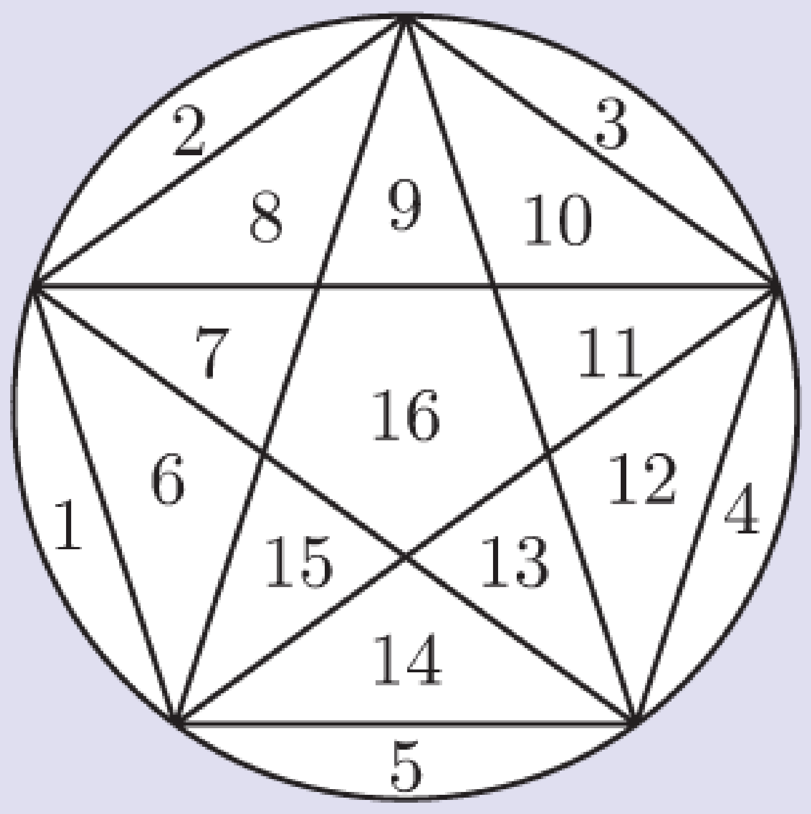
The first five cases are shown below:











From these cases we *conjecture* that for n points, the circle is divided into 2^{n-1} regions.

Draw the case $n = 6$ and see if the conjecture is true!

3 Is it reasonable for a mathematician to assume a conjecture is true until it has been formally proven?

H FINITE GEOMETRIC SERIES

A **geometric series** is the sum of the terms of a geometric sequence.

For example: 1, 2, 4, 8, 16, ..., 1024 is a finite geometric sequence.

$1 + 2 + 4 + 8 + 16 + ... + 1024$ is the corresponding finite geometric series.

If we are adding the first n terms of an infinite geometric sequence, we are then calculating a finite geometric series called the ***n*th partial sum** of the corresponding infinite series.

If we are adding all of the terms in an infinite geometric sequence, we have an **infinite geometric series**.

SUM OF A FINITE GEOMETRIC SERIES

If the first term is u_1 and the common ratio is r, then the terms are: $u_1, u_1r, u_1r^2, u_1r^3, ..., u_1r^{n-1}$.

So, $S_n = u_1 + u_1r + u_1r^2 + u_1r^3 + ... + u_1r^{n-2} + u_1r^{n-1}$

For a finite geometric series with $\boldsymbol{r \neq 1}$,

$$\boldsymbol{S_n = \frac{u_1(r^n - 1)}{r - 1}} \quad \textit{or} \quad \boldsymbol{S_n = \frac{u_1(1 - r^n)}{1 - r}}.$$

Proof:

If $S_n = u_1 + u_1r + u_1r^2 + u_1r^3 + ... + u_1r^{n-2} + u_1r^{n-1}$ $(*)$

then $rS_n = (u_1r + u_1r^2 + u_1r^3 + u_1r^4 + ... + u_1r^{n-1}) + u_1r^n$

$\therefore\ rS_n = (S_n - u_1) + u_1r^n$ {from $(*)$}

$\therefore\ rS_n - S_n = u_1r^n - u_1$

$\therefore\ S_n(r-1) = u_1(r^n - 1)$

$\therefore\ S_n = \dfrac{u_1(r^n-1)}{r-1}$ or $\dfrac{u_1(1-r^n)}{1-r}$ provided $r \neq 1$.

In the case $r = 1$ we have a sequence in which all terms are the same. The sequence is also arithmetic (with $d = 0$), and $S_n = u_1n$.

Example 28 **Self Tutor**

Find the sum of $2+6+18+54+....$ to 12 terms.

The series is geometric with $u_1 = 2$, $r = 3$, and $n = 12$.

$$S_n = \frac{u_1(r^n - 1)}{r-1}$$

$$\therefore\ S_{12} = \frac{2(3^{12}-1)}{3-1} = 531\,440$$

Example 29 **Self Tutor**

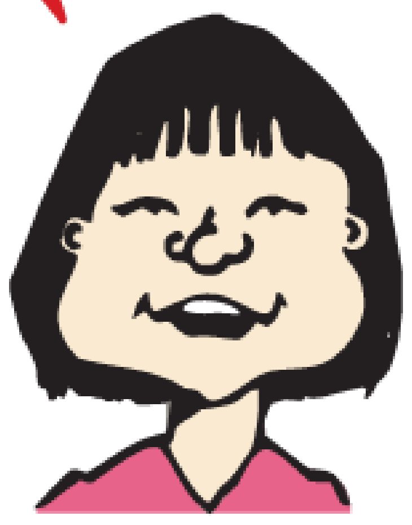
Find a formula for S_n, the sum of the first n terms of the series $9 - 3 + 1 - \frac{1}{3} + ...$

The series is geometric with $u_1 = 9$ and $r = -\frac{1}{3}$.

$$S_n = \frac{u_1(1-r^n)}{1-r} = \frac{9(1-(-\frac{1}{3})^n)}{\frac{4}{3}}$$

$$\therefore\ S_n = \tfrac{27}{4}(1-(-\tfrac{1}{3})^n)$$

This answer cannot be simplified as we do not know if n is odd or even.



EXERCISE 5H

1 Find the sum of the following series:

- **a** $2+6+18+54+....$ to 8 terms
- **b** $5+10+20+40+....$ to 10 terms
- **c** $12+6+3+1.5+....$ to 10 terms
- **d** $\sqrt{7}+7+7\sqrt{7}+49+....$ to 12 terms
- **e** $6-3+1\frac{1}{2}-\frac{3}{4}+....$ to 15 terms
- **f** $1-\frac{1}{\sqrt{2}}+\frac{1}{2}-\frac{1}{2\sqrt{2}}+....$ to 20 terms.

2 Find a formula for S_n, the sum of the first n terms of the series:

- **a** $\sqrt{3}+3+3\sqrt{3}+9+....$
- **b** $12+6+3+1\frac{1}{2}+....$
- **c** $0.9+0.09+0.009+0.0009+....$
- **d** $20-10+5-2\frac{1}{2}+....$

3 Evaluate these geometric series:

- **a** $\displaystyle\sum_{k=1}^{10} 3\times 2^{k-1}$
- **b** $\displaystyle\sum_{k=1}^{12} \left(\tfrac{1}{2}\right)^{k-2}$
- **c** $\displaystyle\sum_{k=1}^{25} 6\times(-2)^k$

4 At the end of each year, a salesperson is paid a bonus of \$2000 which is always deposited into the same account. It earns a fixed rate of interest of 6% p.a. with interest being paid annually. The total amount in the account at the end of each year will be:

$$A_1 = 2000$$
$$A_2 = A_1 \times 1.06 + 2000$$
$$A_3 = A_2 \times 1.06 + 2000 \quad \text{and so on.}$$

- **a** Show that $A_3 = 2000 + 2000\times 1.06 + 2000\times(1.06)^2$.
- **b** Show that $A_4 = 2000[1+1.06+(1.06)^2+(1.06)^3]$.
- **c** Find the total bank balance after 10 years, assuming there are no fees or withdrawals.

5 Answer the **Opening Problem** on page **90**.

6 Paula has started renting an apartment. She paid \$5000 rent in the first year, and the rent increased by 5% each year.

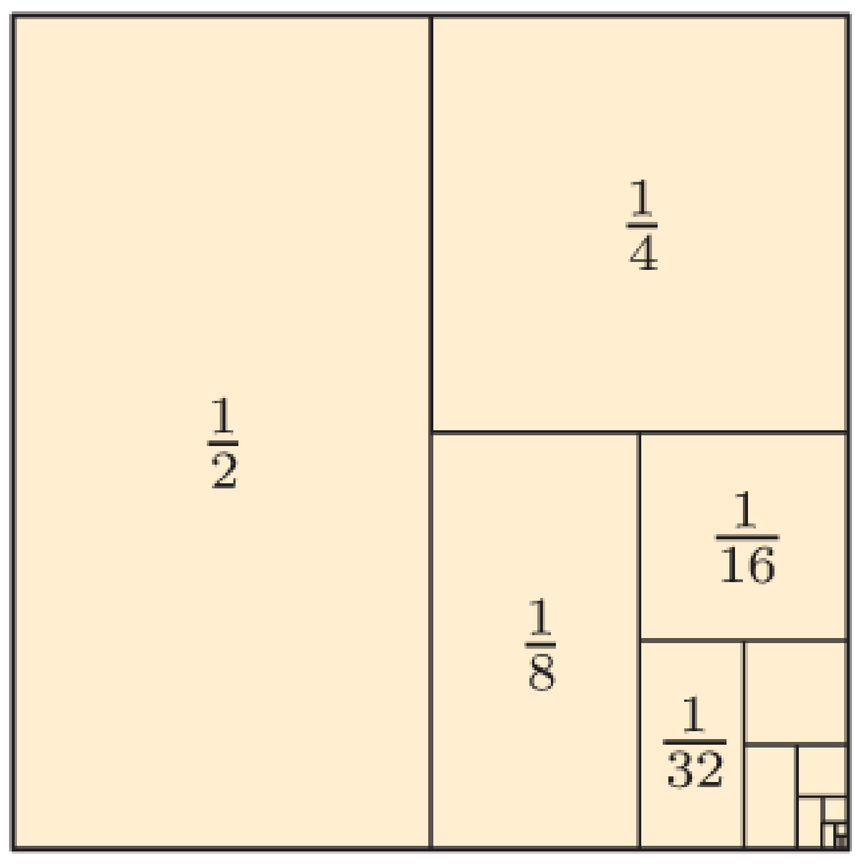
  a Find, to the nearest \$10, the rent paid by Paula in the 4th year.

  b Write an expression for the total rent paid by Paula during the first n years.

  c How much rent did Paula pay during the first 7 years? Give your answer to the nearest \$10.

7 Jim initially deposits £6000 in an account which earns 5% p.a. interest paid annually. At the end of each year, Jim invests another £1000 in the account. Find the value of the account after 8 years.

8 Consider $S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... + \frac{1}{2^n}$.



  a Find S_1, S_2, S_3, S_4, and S_5 in fractional form.

  b Hence guess the formula for S_n.

  c Find S_n using $S_n = \dfrac{u_1(1-r^n)}{1-r}$.

  d Comment on S_n as n gets very large.

  e Explain the relationship between the given diagram and **d**.

9 A geometric series has second term 6. The sum of its first three terms is -14. Find its fourth term.

10 An arithmetic and a geometric sequence both have first term 1, and their second terms are equal. The 14th term of the arithmetic sequence is three times the third term of the geometric sequence. Find the twentieth term of each sequence.

11 Suppose $u_1, u_2, ..., u_n$ is a geometric sequence with common ratio r. Show that

$$(u_1+u_2)^2 + (u_2+u_3)^2 + (u_3+u_4)^2 + ... + (u_{n-1}+u_n)^2 = \frac{2{u_1}^2(r^{2n-1}-1)}{r-1} - ({u_1}^2 + {u_n}^2).$$

Example 30

Self Tutor

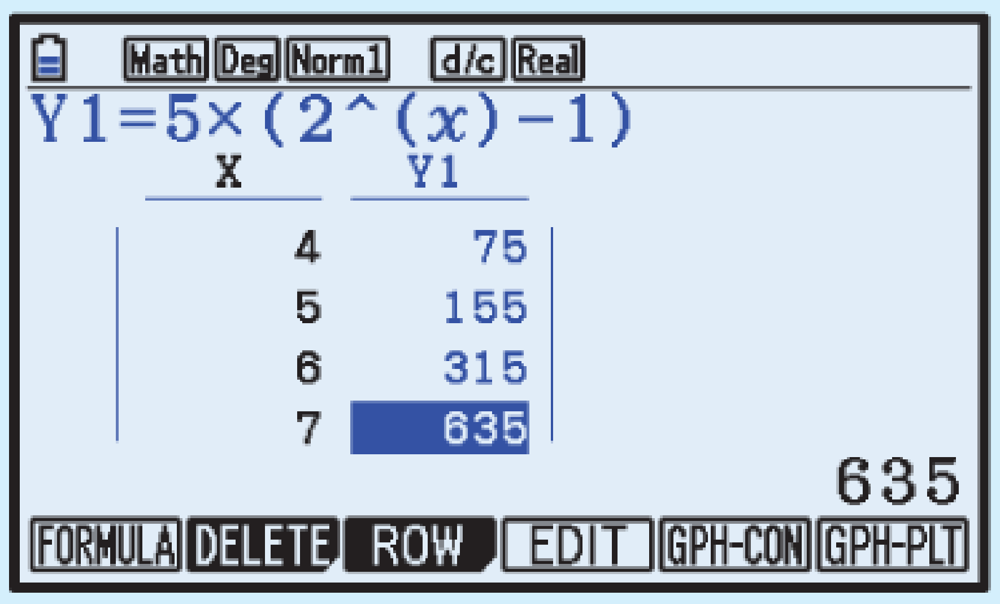
A geometric sequence has first term 5 and common ratio 2. The sum of the first n terms of the sequence is 635. Find n.

The sequence is geometric with $u_1 = 5$ and $r = 2$.

$$\therefore \quad S_n = \frac{u_1(r^n-1)}{r-1} = \frac{5(2^n-1)}{2-1} = 5(2^n-1)$$

To find n such that $S_n = 635$, we use a table of values with $Y_1 = 5 \times (2\hat{\ } X - 1)$:

Casio fx-CG50

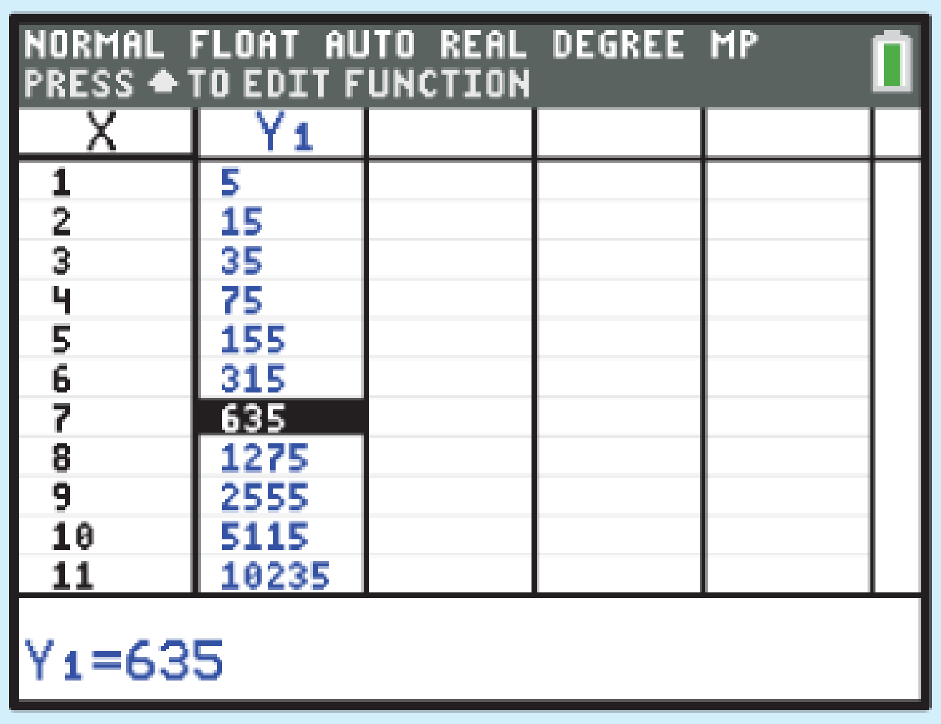


Y1=5×(2^(x)−1)

X	Y1
4	75
5	155
6	315
7	635

635

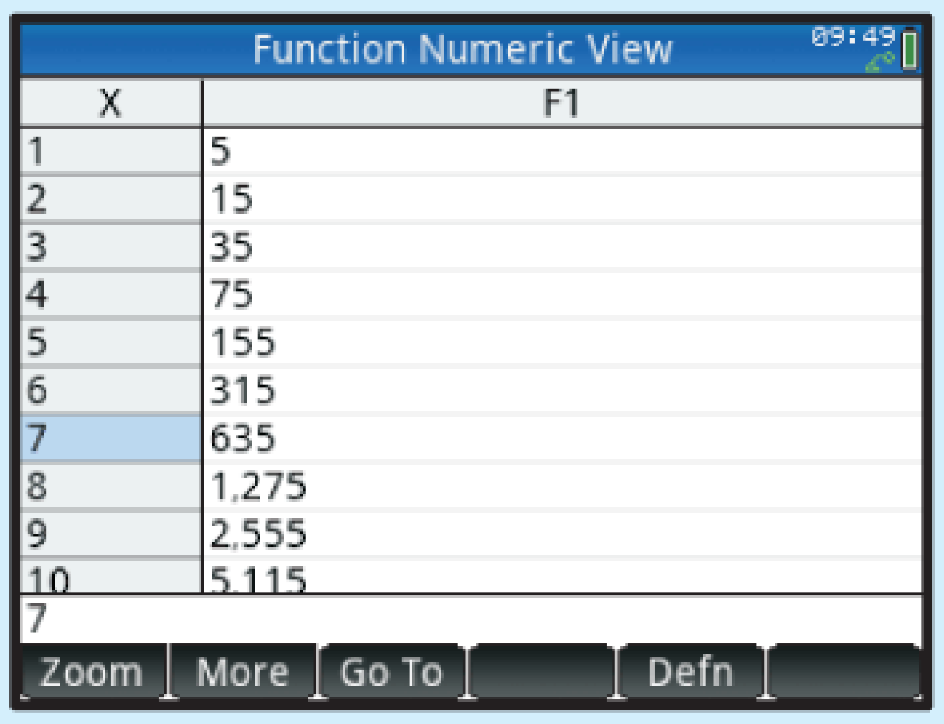
TI-84 Plus CE



X	Y₁
1	5
2	15
3	35
4	75
5	155
6	315
7	635
8	1275
9	2555
10	5115
11	10235

Y₁=635

HP Prime



X	F1
1	5
2	15
3	35
4	75
5	155
6	315
7	635
8	1,275
9	2,555
10	5,115

7

$S_7 = 635$, so $n = 7$.

12 A geometric sequence has first term 6 and common ratio 1.5. The sum of the first n terms of the sequence is 79.125. Find n.

13 Find n given that $\sum_{k=1}^{n} 2 \times 3^{k-1} = 177\,146$.

14 Felicity is offered a new job, and is given two salary options to choose from:

Option A: $40 000 in the first year, and 5% extra each subsequent year.

Option B: $60 000 in the first year, and $1000 more each subsequent year.

a If Felicity believed that she would work for 3 years in this new job, explain why *Option B* would be best for her.

b Write down an expression for the amount of money earned in the nth year if she selects:

i *Option A* **ii** *Option B*.

c Find the minimum length of time Felicity would need to work before the amount of money earned per year from *Option A* exceeds that of *Option B*.

d Felicity decides that the best way to compare the two options is to consider the *total* income accumulated after the first n years in each case. If T_A and T_B represent the total income earned over n years for *Options A* and *B* respectively, show that:

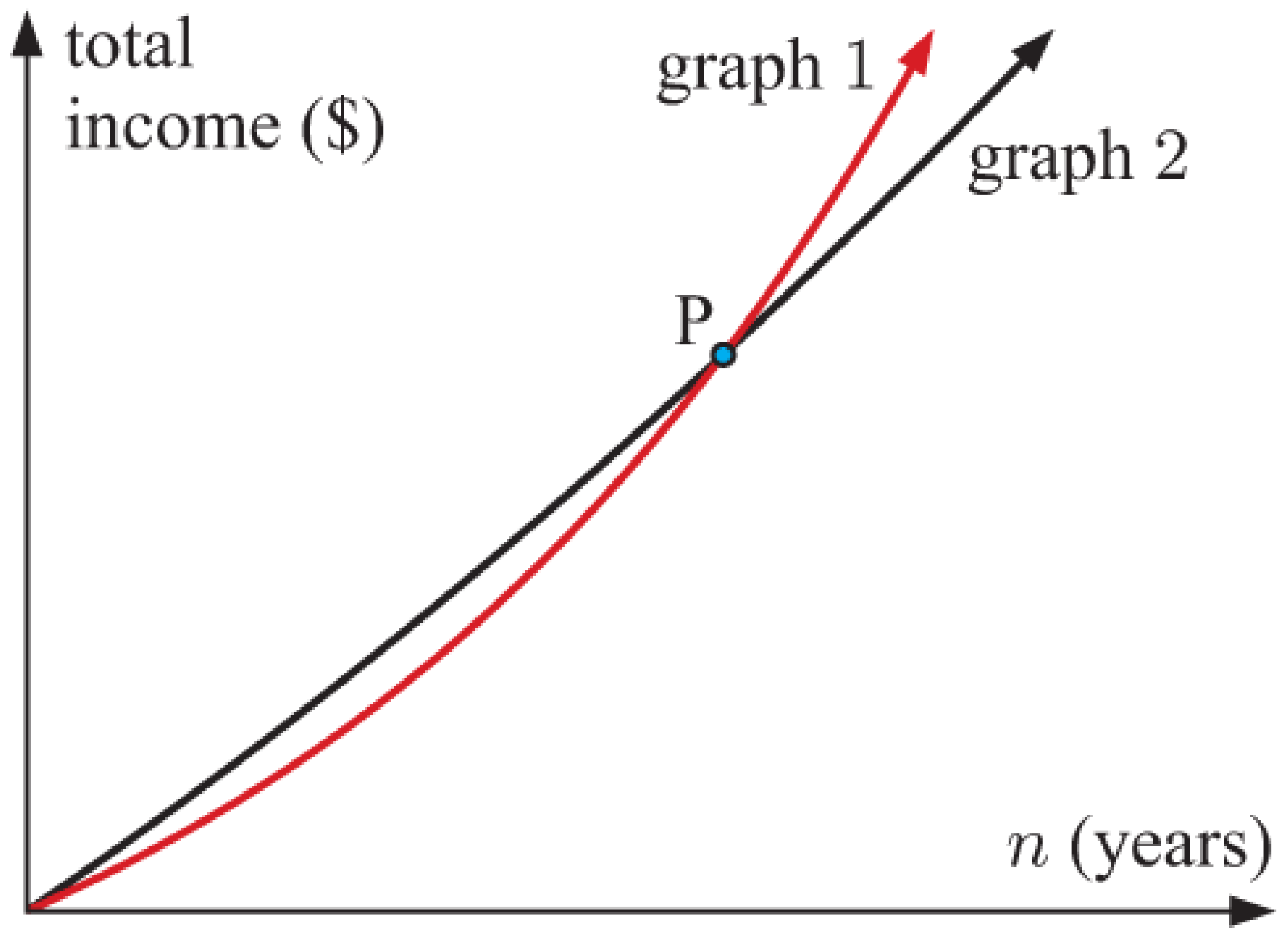
i $T_A = 800\,000(1.05^n - 1)$ dollars **ii** $T_B = 500n^2 + 59\,500n$ dollars

e The graph alongside shows T_A and T_B graphed against n.

i Which graph represents T_A and which graph represents T_B?

ii Use technology to find the coordinates of the point P, where T_A and T_B intersect.

iii Hence write down a time interval, in whole years, for which *Option B* provides the greater total income.



15 $8000 is borrowed over a 2-year period at a rate of 12% p.a. compounded quarterly. Quarterly repayments are made and the interest is adjusted each quarter, which means that at the end of each quarter, interest is charged on the previous balance and then the balance is reduced by the amount repaid.

There are $2 \times 4 = 8$ repayments and the interest per quarter is $\frac{12\%}{4} = 3\%$.

At the end of the first quarter, the amount owed is given by $A_1 = \$8000 \times 1.03 - R$, where R is the amount of each repayment.

At the end of the second quarter, the amount owed is given by:

$$\begin{aligned} A_2 &= A_1 \times 1.03 - R \\ &= (\$8000 \times 1.03 - R) \times 1.03 - R \\ &= \$8000 \times (1.03)^2 - 1.03R - R \end{aligned}$$

a Write an expression for the amount owed at the end of the third quarter, A_3.

b Write an expression for the amount owed at the end of the eighth quarter, A_8.

c Given that $A_8 = 0$ for the loan to be fully repaid, deduce the value of R.

d Now suppose the amount borrowed was $\$P$, r is the interest rate per repayment interval (as a decimal), and there are m repayments. Show that each repayment is

$$R = \frac{P(1+r)^m \times r}{(1+r)^m - 1} \text{ dollars.}$$

I INFINITE GEOMETRIC SERIES

To examine the sum of all the terms of an infinite geometric sequence, we need to consider $S_n = \frac{u_1(1-r^n)}{1-r}$ when n gets very large.

If $|r| > 1$, the series is said to be **divergent** and the sum becomes infinitely large.

For example, when $r = 2$, $1+2+4+8+16+....$ is infinitely large.

If $|r| < 1$, or in other words $-1 < r < 1$, then as n becomes very large, r^n approaches 0.

This means that S_n will get closer and closer to $\frac{u_1}{1-r}$.

$|r|$ is the *size* of r. If $|r| > 1$ then $r < -1$ or $r > 1$.

If $|r| < 1$, an infinite geometric series of the form $u_1 + u_1 r + u_1 r^2 + ... = \sum_{k=1}^{\infty} u_1 r^{k-1}$ will **converge** to the **limiting sum** $\boldsymbol{S = \frac{u_1}{1-r}}$.

Proof: If the first term is u_1 and the common ratio is r, the terms are $u_1, u_1 r, u_1 r^2, u_1 r^3, ...$

Suppose the sum of the corresponding infinite series is

$S = u_1 + u_1 r + u_1 r^2 + u_1 r^3 + ...$ $(*)$

$$\therefore \quad rS = u_1 r + u_1 r^2 + u_1 r^3 + u_1 r^4 + ...$$

$$\therefore \quad rS = S - u_1 \qquad \{\text{comparing with } (*)\}$$

$$\therefore \quad S(r-1) = -u_1$$

$$\therefore \quad S = \frac{u_1}{1-r} \qquad \{\text{provided } r \neq 1\}$$

This result can be used to find the value of recurring decimals.

Example 31 **Self Tutor**

Write $0.\overline{7}$ as a rational number.

$0.\overline{7} = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10\,000} + ...$ is an infinite geometric series with $u_1 = \frac{7}{10}$ and $r = \frac{1}{10}$.

$$\therefore \quad S = \frac{u_1}{1-r} = \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{7}{9}$$

$\therefore \quad 0.\overline{7} = \frac{7}{9}$

EXERCISE 5I

1 **a** Explain why $0.\overline{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + ...$ is an infinite geometric series.

b Hence show that $0.\overline{3} = \frac{1}{3}$.

2 Write as a rational number: **a** $0.\overline{4}$ **b** $0.\overline{16}$ **c** $0.\overline{312}$

3 Use $S = \dfrac{u_1}{1-r}$ to check your answer to **Exercise 5H** question **8 d**.

4 Find the sum of each of the following infinite geometric series:

a $18 + 12 + 8 + \frac{16}{3} + ...$ **b** $18.9 - 6.3 + 2.1 - 0.7 + ...$

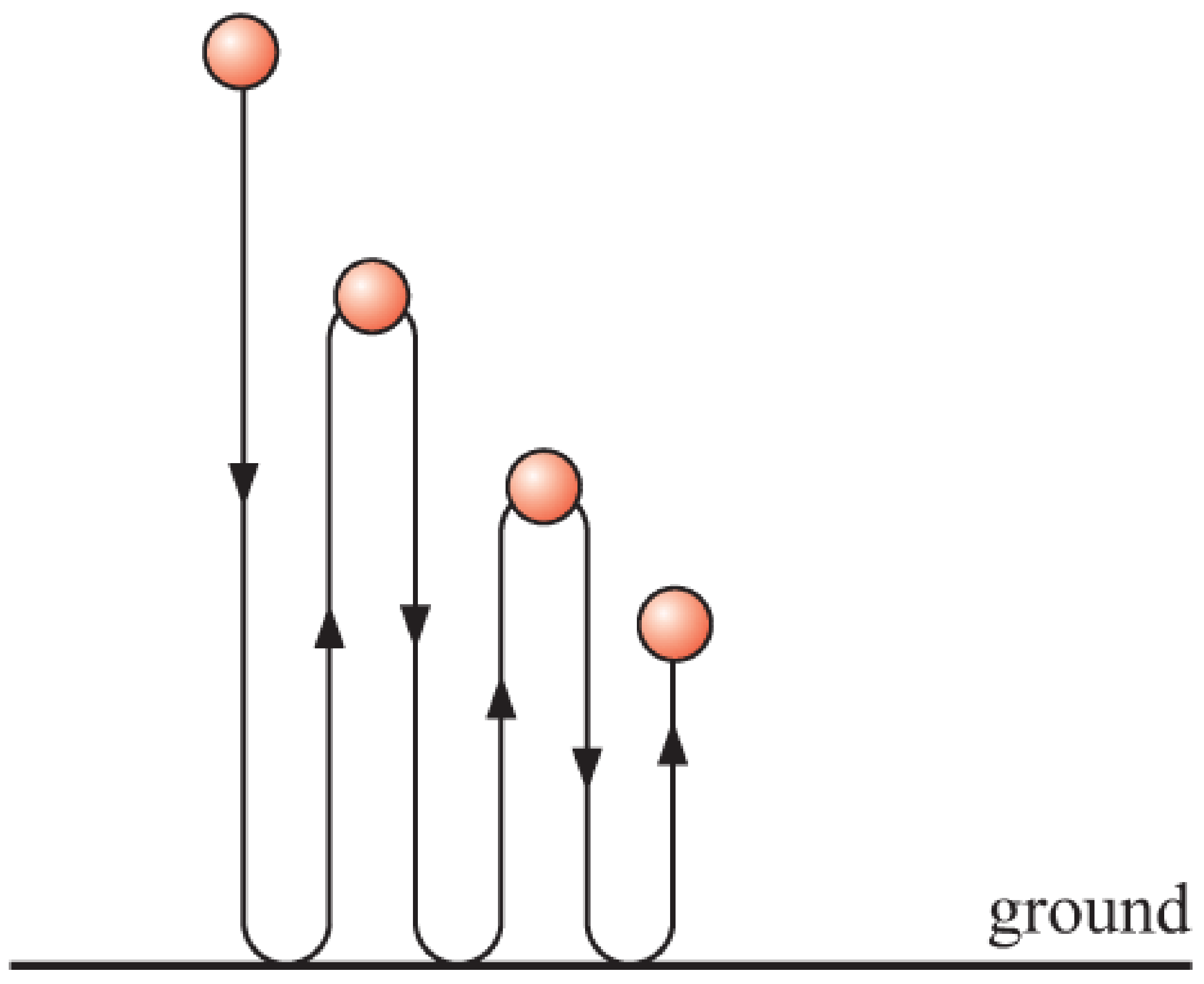
5 Find: **a** $\displaystyle\sum_{k=1}^{\infty} \frac{3}{4^k}$ **b** $\displaystyle\sum_{k=0}^{\infty} 6\left(-\tfrac{2}{5}\right)^k$

6 The sum of the first three terms of a convergent infinite geometric series is 19. The sum of the series is 27. Find the first term and the common ratio.

7 The second term of a convergent infinite geometric series is $\frac{8}{5}$. The sum of the series is 10. Show that there are two possible series, and find the first term and the common ratio in each case.

8 An infinite geometric series has $S = \frac{64}{3}$ and $S_3 = 21$. Find S_5.

9



When dropped, a ball takes 1 second to hit the ground. It then takes 90% of this time to rebound to its new height, and this continues until the ball comes to rest.

a Show that the total time of motion is given by $1 + 2(0.9) + 2(0.9)^2 + 2(0.9)^3 + ...$

b Find S_n for the series in **a**.

c How long does it take for the ball to come to rest?

10 When a ball is dropped, it rebounds 75% of its height after each bounce. If the ball travels a total distance of 490 cm, from what height was the ball dropped?

11 **a** Explain why $0.\overline{9} = 1$ exactly. **b** Show that if $u_n = \dfrac{9}{10^n}$, then $S_n = 1 - \dfrac{1}{10^n}$.

c On a graph, plot the points (n, u_n) and (n, S_n) for $n = 1, 2, ..., 10$. Connect each set of points with a smooth curve.

12 Find x if $\displaystyle\sum_{k=1}^{\infty}\left(\frac{3x}{2}\right)^{k-1} = 4$.

13 Suppose $u_1 + u_2 + u_3 + u_4 + ...$ is a convergent infinite geometric series, with $u_1, u_2, u_3, u_4, ... > 0$.

a Explain why $u_1 - u_2 + u_3 - u_4 + ...$ and $\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3} + \sqrt{u_4} + ...$ must also be convergent infinite geometric series.

b Given $u_1 - u_2 + u_3 - u_4 + ... = \frac{81}{10}$ and $\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3} + \sqrt{u_4} + ... = \frac{9}{2}$, find $u_1 + u_2 + u_3 + u_4 + ...$.

14 Show that $\frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \frac{5}{64} + ... = 1$.

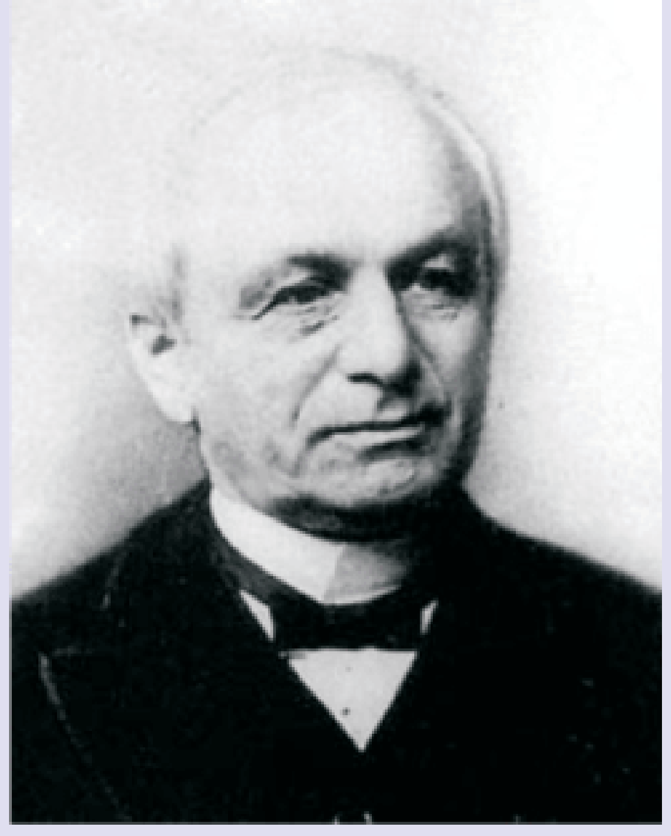
ACTIVITY 3

Click on the icon to run a card game for sequences and series.

CARD GAME

THEORY OF KNOWLEDGE

The German mathematician Leopold Kronecker (1823 - 1891) made important contributions in number theory and algebra. Several things are named after him, including formulae, symbols, and a theorem.



Leopold Kronecker

Kronecker made several well-known quotes, including:

> "*God made integers; all else is the work of man.*"
>
> "*A mathematical object does not exist unless it can be constructed from natural numbers in a finite number of steps.*"

1 What do you understand by the term *infinity*?

2 If the entire world were made of grains of sand, could you count them? Would the number of grains of sand be infinite?

3 There are clearly an infinite number of positive integers, and an infinite number of positive even integers.

- **a** Construct an argument that:
  - **i** there are *more* positive integers than positive even integers
  - **ii** there is the *same number* of positive integers as positive even integers.
- **b** Can the traditional notions of "more than", "less than", and "equal to" be extended to infinity?

Consider an infinite geometric series with first term u_1 and common ratio r.

If $|\,r\,| < 1$, the series will converge to the sum $S = \dfrac{u_1}{1-r}$.

4 Can we explain through *intuition* how a sum of non-zero terms, which goes on and on for ever and ever, could actually be a finite number?

In the case $r = -1$, the terms are $u_1, -u_1, u_1, -u_1, ...$

If we take partial sums of the series, the answer is always u_1 or 0.

5 What is the sum of the infinite series when $r = -1$? Is it infinite? Is it defined?

Substituting $r = -1$ into the formula above gives $S = \dfrac{u_1}{2}$. Could this possibly be the answer?

ACTIVITY 4 VON KOCH'S SNOWFLAKE CURVE

The Haese Mathematics logo is the 4th member of a sequence of diagrams. The **limiting curve** which the sequence approaches is called **von Koch's snowflake curve**.

Click on the icon to obtain this Activity.

VON KOCH'S SNOWFLAKE CURVE

REVIEW SET 5A

1 Consider the number sequence 5, 9, 11, 12, 15, 19. Find:

a u_2 **b** u_6 **c** S_4.

2 Find k if $3k$, $k-2$, and $k+7$ are consecutive terms of an arithmetic sequence.

3 A sequence is defined by $u_n = 6\left(\frac{1}{2}\right)^{n-1}$.

a Prove that the sequence is geometric. **b** Find u_1 and r.

c Find the 16th term of the sequence to 3 significant figures.

4 Determine the general term of a geometric sequence given that its sixth term is $\frac{16}{3}$ and its tenth term is $\frac{256}{3}$.

5 Insert six numbers between 23 and 9 so that all eight numbers are in arithmetic sequence.

6 Theodore is squeezing the juice from some lemons. When he has squeezed 6 lemons, he has collected 274.3 mL of juice.

a Find the average amount of juice collected from each lemon.

b Hence write an arithmetic sequence u_n which approximates the amount of juice collected from squeezing n lemons.

c Predict the amount of juice collected from squeezing 13 lemons.

7 Find the sum of each of the following infinite geometric series:

a $18 - 12 + 8 - ....$ **b** $8 + 4\sqrt{2} + 4 + ...$

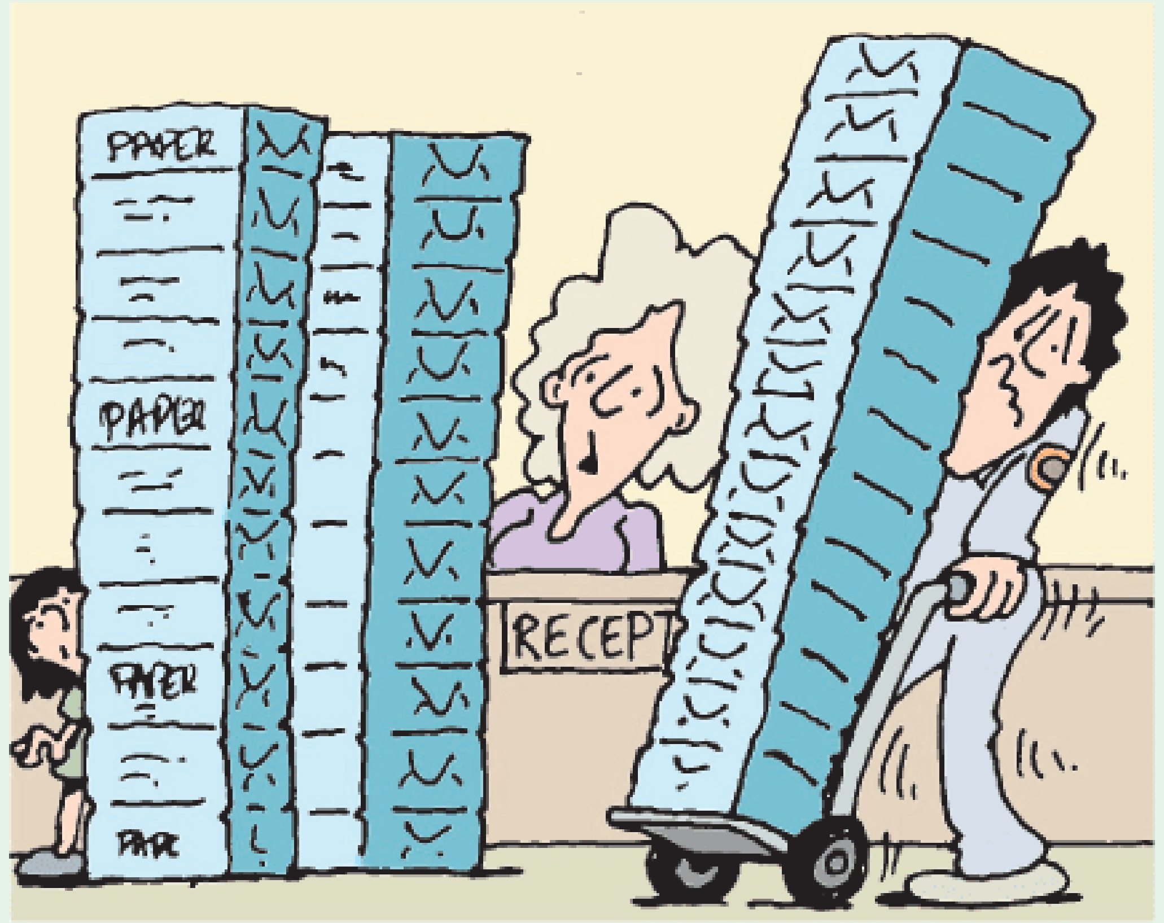
8 Find the sum of:

a $7 + 11 + 15 + 19 + ... + 99$ **b** $35 + 33\frac{1}{2} + 32 + 30\frac{1}{2} + ... + 20$

9 Each year, a school manages to use only 90% as much paper as the previous year. In the year 2010, they used 700 000 sheets of paper.

a Find how much paper the school used in the years 2011 and 2012.

b How much paper did the school use in total in the decade from 2008 to 2018?



10 Expand and hence evaluate: **a** $\sum_{k=1}^{7} k^2$ **b** $\sum_{k=1}^{4} \frac{k+3}{k+2}$

ACTIVITY 4 VON KOCH'S SNOWFLAKE CURVE

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Click on the icon to obtain this Activity.

VON KOCH'S SNOWFLAKE CURVE

REVIEW SET 5A

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c Predict the amount of juice collected from squeezing 13 lemons.

7 Find the sum of each of the following infinite geometric series:

a $18 - 12 + 8 - \ldots.$ **b** $8 + 4\sqrt{2} + 4 + \ldots.$

8 Find the sum of:

a $7 + 11 + 15 + 19 + \ldots. + 99$ **b** $35 + 33\frac{1}{2} + 32 + 30\frac{1}{2} + \ldots. + 20$

9 Each year, a school manages to use only 90% as much paper as the previous year. In the year 2010, they used 700 000 sheets of paper.

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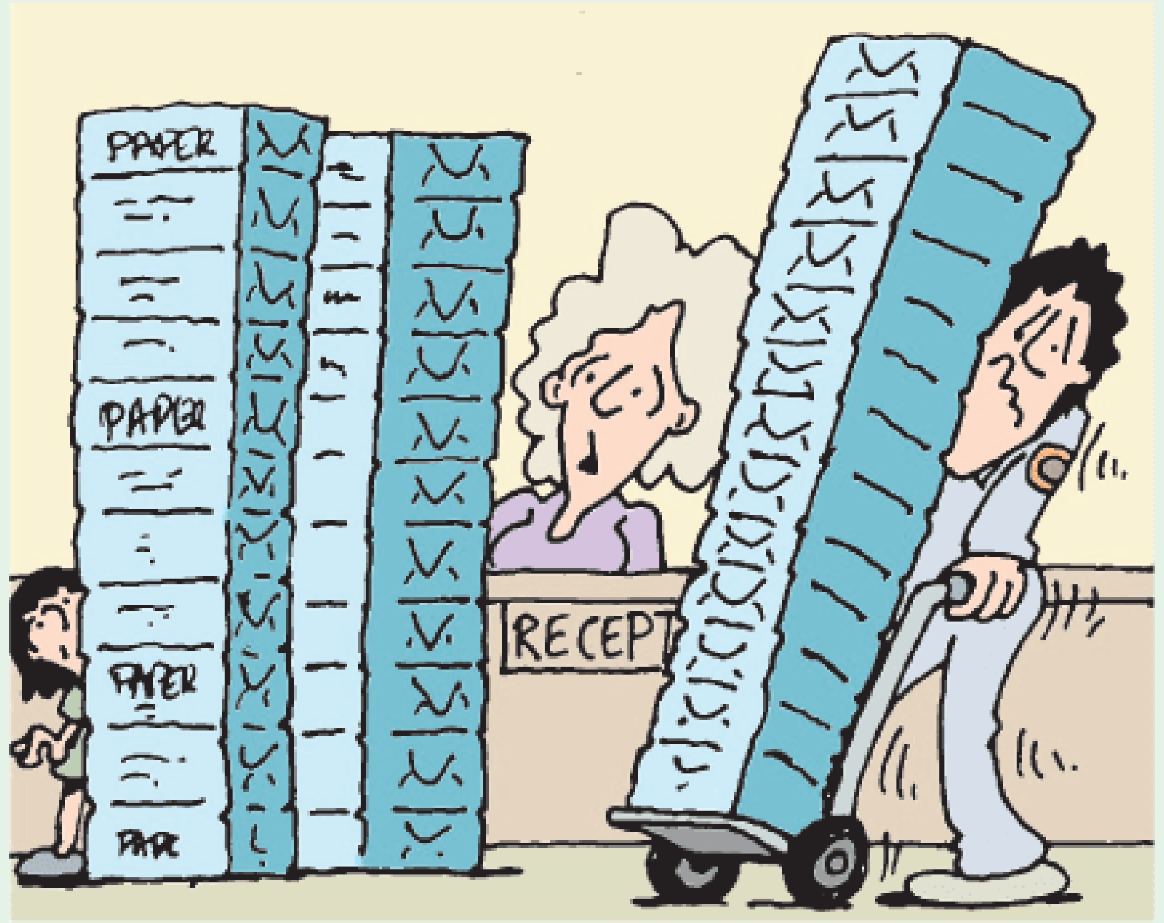
b How much paper did the school use in total in the decade from 2008 to 2018?



10 Expand and hence evaluate: **a** $\sum_{k=1}^{7} k^2$ **b** $\sum_{k=1}^{4} \frac{k+3}{k+2}$

24 A competition offers three options for the first prize, each of which pays the winner a monthly sum for 24 months.

Option 1: \$8000 per month.

Option 2: \$1000 in the first month, then each successive month pays \$600 more than the previous month.

Option 3: \$500 in the first month, then each successive month pays 20% more than the previous month.

- **a** Calculate the total prize value for *Option 1*.
- **b** For *Option 2*:
  - **i** Write down the amount won in each of the first three months.
  - **ii** Calculate the total amount won over the 24 month period.
- **c** For *Option 3*:
  - **i** Write down the amount won in each of the first three months.
  - **ii** Calculate the total amount won over the 24 month period.
- **d** Which option is worth the greatest amount of money overall?
- **e** The amount won in the first month under *Option 3* is to be altered so that the total prize over 24 months is \$250 000. Calculate the new initial amount, giving your answer to the nearest cent.

25 $2x$ and $x-2$ are the first two terms of a convergent geometric series. The sum of the series is $\frac{18}{7}$. Find x, clearly explaining why there is only one possible value.

26 a, b, and c are consecutive terms of an arithmetic sequence. Prove that the following are also consecutive terms of an arithmetic sequence:

- **a** $b+c$, $c+a$, and $a+b$
- **b** $\dfrac{1}{\sqrt{b}+\sqrt{c}}$, $\dfrac{1}{\sqrt{c}+\sqrt{a}}$, and $\dfrac{1}{\sqrt{a}+\sqrt{b}}$

REVIEW SET 5B

1 Evaluate the first five terms of the sequence:

- **a** $\{(\frac{1}{3})^n\}$
- **b** $\{12+5n\}$
- **c** $\left\{\dfrac{4}{n+2}\right\}$

2 A sequence is defined by $u_n = 68-5n$.

- **a** Prove that the sequence is arithmetic.
- **b** Find u_1 and d.
- **c** Find the 37th term of the sequence.
- **d** State the first term of the sequence which is less than -200.

3
- **a** Find the general term of the arithmetic sequence with $u_7 = 31$ and $u_{15} = -17$.
- **b** Hence find the value of u_{34}.

4 Find the sum of the first 12 terms of:

- **a** $3+9+15+21+....$
- **b** $24+12+6+3+....$

5 Stacy runs a hot dog stand at a local fair. On the first day she served 25 customers and made £60 profit. On the second day she served 43 customers and made £135 profit.

- **a** Assuming that her profit from serving n customers forms an arithmetic sequence, find a model which approximates the profit from serving n customers.

b Explain the significance of the common difference and the constant term in your model.

c On the third day, Stacy served 36 customers. Use your model to estimate her profit.

6 Find the first term of the sequence 5, 10, 20, 40, ... which exceeds 10 000.

7 A ball bounces from a height of 3 metres and returns to 80% of its previous height on each bounce. Find the total distance travelled by the ball until it stops bouncing.

8 \$7000 is invested at 6% p.a. compound interest. Find the value of the investment after 3 years if interest is compounded:

a annually **b** quarterly **c** monthly.

9 **a** Find k given that 4, k, and $k^2 - 1$ are consecutive terms of a geometric sequence.

b For each value of k, find the common ratio of the sequence.

10 Seve is training for a long distance walk. He walks for 10 km in the first week, then each week thereafter he walks 500 m further than the previous week. If he continues this pattern for a year, how far does Seve walk:

a in the last week **b** in total?

11 Find the sum of the infinite geometric series:

a $1.21 - 1.1 + 1 - ....$ **b** $\frac{14}{3} + \frac{4}{3} + \frac{8}{21} + ...$

12 Find the first term of the sequence $24, 8, \frac{8}{3}, \frac{8}{9}, ...$ which is less than 0.001 .

13 Vijay deposits 200 000 rupees in an account that compounds interest half-yearly. 6 years later, the account has balance 250 680 rupees. Use technology to calculate the annual rate of interest.

14 Frederik invests €5000 at 5.8% p.a. compounded monthly. Use technology to find how long it will take to amount to €12 000.

15 Richard sold his car for \$7500. He invested the money in an account paying 3.7% p.a. interest compounded quarterly for 8 years. Inflation averaged 3.1% per year over this period.

a Find the future value of Richard's investment.

b Find the real value of Richard's investment.

16 A photocopier bought for \$9800 will depreciate by 26% each year. Find its value after 5 years.

17 Evaluate:

a $\sum_{k=1}^{8} \left(\frac{31 - 3k}{2}\right)$ **b** $\sum_{k=1}^{15} 50(0.8)^{k-1}$ **c** $\sum_{k=7}^{\infty} 5\left(\frac{2}{5}\right)^{k-1}$

18 A geometric sequence has $u_6 = 24$ and $u_{11} = 768$.

a Determine the general term of the sequence. **b** Find the sum of the first 15 terms.

19 In 2004 there were 3000 iguanas on a Galapagos island. Since then, the population of iguanas on the island has increased by 5% each year.

a How many iguanas were on the island in 2007?

b In what year will the population first exceed 10 000?

20 **a** Under what conditions will the series $\sum_{k=1}^{\infty} 50(2x-1)^{k-1}$ converge? Explain your answer.

b Find $\sum_{k=1}^{\infty} 50(2x-1)^{k-1}$ if $x = 0.3$.

21 Suppose $u_1, u_2, u_3, ...$ is an arithmetic sequence.

a Show that $2^{u_1}, 2^{u_2}, 2^{u_3}, ...$ is a geometric sequence.

b Given that $u_3 = -4$ and $u_5 = -10$, find the sum of the infinite geometric series $2^{u_1} + 2^{u_2} + 2^{u_3} + ...$

22 Michael is saving to buy a house and needs $400 000.

a Three years ago, he invested a sum of money in an account paying 6.5% p.a. interest compounded half-yearly. This investment has just matured at $100 000. How much did Michael invest three years ago?

b Michael decides to reinvest his $100 000 lump sum into an account for a period of n years at 6.0% p.a. interest compounded annually.

Copy and complete the table below showing the value V_n of Michael's investment after n years.

n (years)	0	1	2	3	4
V_n ($)	100 000	106 000	112 360		

c Write a formula for V_n in terms of n.

d Michael also decides to start an additional saving plan, whereby he deposits $6000 into a safe at the end of each year. Write down a formula for S_n, the amount of money in Michael's safe after n years.

e The total amount of money Michael has for his house after n years is given by $T_n = V_n + S_n$. Calculate the missing values in the table below.

n (years)	0	1	2	3	4
T_n ($)	100 000	112 000	124 360		

f After how many whole years will Michael have the $400 000 needed to buy his house?

23 The sum of an infinite geometric series is 49 and the second term of the series is 10. Find the possible values for the sum of the first three terms of the series.

24 Suppose n consecutive geometric terms are inserted between 1 and 2. Write the sum of these n terms, in terms of n.

25 The 3rd, 4th, and 8th terms of an arithmetic sequence are the first three terms of a geometric sequence, $r \neq 1$.

a Find the common ratio for the geometric sequence.

b Show that the 24th term of the arithmetic sequence is the 4th term of the geometric sequence.

26 Notice that $11 - 2 = 9 = 3^2$ and $1111 - 22 = 1089 = 33^2$.

Show that $\underbrace{(111111 .... 1)}_{2n \text{ lots of } 1} - \underbrace{(22222 .... 2)}_{n \text{ lots of } 2}$ is a perfect square.