Investigation into the path of the quickest descent.

Introduction:

I like downhill car-racing and like all races, in order to win, the racer has to reach the finish in the shortest amount of time. In order to minimize the time it takes to get from one point to another, it is essential that we have the right car type and the path[1] with the quickest descent. The latter will benefit all the racers and not just a select few, since all the racers will race on the same track. In addition, changing the shape of the track makes the race more interesting and thrilling since all the racing action happens much faster. In this exploration, I will be comparing the brachistochrone curve to the catenary curve in order to find out which of the two curves are the fastest. I anticipate that the catenary curve would highly resemble the brachistochrone, which means that the two would provide very similar results.

A: Introduction gives a vague aim to the exploration.

Brachistochrone vs Catenary:

Brachistochrone is an interesting mathematical design, both, historically and aesthetically. I became personally interested in the curve when I read that Johann Bernoulli had challenged the mathematicians worldwide to solve the "brachistochrone problem". He posed the problem in *Acta Eruditorum* in 1696. The best part about the problem was that Newton was deliberately tempted to solve the problem by Leibniz and Johann.

A: Full name rather than first name.

Brachistochrone is the path of quickest descent between two points. This shape is widely used in the design of downhill ski tracks. Even, skate parks are made using the brachistochrone shape[2]. The catenary in contrast is by definition a curve that an idealized chain or cable assumes under its own weight when supported only at the ends[3]. The Bernoulli brothers contributed to finding the mathematical description of both of these curves[4]. The brachistochrone was derived by the Bernoulli brothers only, whereas the expression for a catenary curve was also derived by Leibniz and Christiaan Huygens independent of each other[5].

B: Although a definition of the catenary is given, that of the brachistochrone is not.

---

[1] Track on which the cars will race.

[2] Stevehaake. "Surfing the Brachistochrone." *Engineering Sport*. N.P., 2011. Web. 06 Dec. 2016.

[3] "Catenary." *Wikipedia*. Wikimedia Foundation, n.d. Web. 06 Dec. 2016.

[4] "Brachistochrone Problem." *Brachistochrone Problem -- from Wolfram MathWorld*. N.P., n.d. Web. 06 Dec. 2016.

[5] "Catenary." *Wikipedia*. Wikimedia Foundation, n.d. Web. 06 Dec. 2016.

1

[6]

B: The graph may represent the brachistochrone but axes not labelled and function not given.

Figure 1.

The above diagram represents the brachistochrone. It is also known as the inverted cycloid curve.

B: And what is a cycloid?

Figure 2.

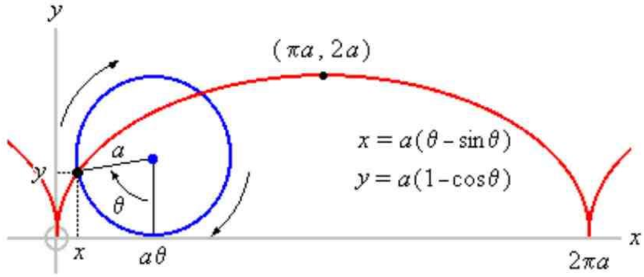
The shape shown above is a catenary curve. The metal chain has adopted this shape under its own weight.

6. Desmos graphing calculator image.

2

Equation of the Brachistochrone:

The brachistochrone is created by restricting the domain of a cycloid. "A cycloid[7] is the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slippage". We will derive the equation as a function of theta θ where θ is a real parameter corresponding to the angle through which the rolling circle has rotated. The following image explains best[8].



E: There is too much unexplained information on the diagram. For instance, in the parametric equations, the parameters are not defined.

Consider the path below.

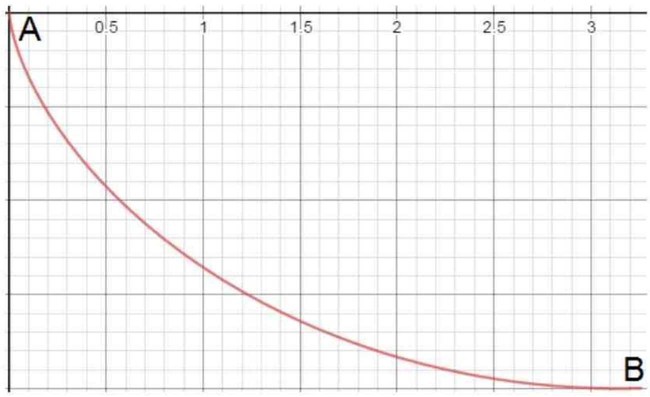


Figure 3.

We know that velocity (v) is given by $v = \frac{ds}{dt} \Rightarrow dt = \frac{ds}{v}$ where "ds" is the infinitesimal part of the path "AB". Can also be called infinitesimal part of the arc-length.

From this we have that $\int dt = \int \frac{ds}{v}$

Calculating the definite integral of time from t = 0 to t = t and integrating the right hand side from

s = A to s = B we have, $\int_0^t dt = \int_A^B \frac{ds}{v} \Rightarrow t = \int_A^B \frac{ds}{v}$ where s = position of the object.

E: It is not clear what this is with respect to.

The principle of conservation of mechanical energy states that the sum of potential energy and kinetic energy of an object is constant provided the only forces acting on the object are conservative forces.[9] This means that the change in potential energy must equal change in

[7] "Cycloid." Wikipedia. Wikimedia Foundation, n.d. Web. 18 Jan. 2017.

[8] "How can I draw this cycloid diagram with TikZ?" Technical drawing - How can I draw this cycloid diagram with TikZ? - TeX - LaTeX Stack Exchange. N.p., n.d. Web. 16 Jan. 2017.

[9] "Conservative force." Wikipedia. Wikimedia Foundation, n.d. Web. 29 Dec. 2016.

3

kinetic energy, mathematically $mg\Delta h = \frac{1}{2}m\Delta v^2$ where m is the mass of the object and Δh is the change in height. This is only true, however, under the assumption that the surface on which that no energy is lost to friction. In other words, the surface is friction less.

For the brachistochrone Δh = y, since the height here is along the "y" axis, therefore:

$$mgy = \frac{1}{2}mv^2 \Rightarrow \frac{1}{2}v^2 = gy$$

Which leads to,

$$v = \sqrt{2gy} \qquad \text{- (1)}$$

Now, we can express the arc length "ds" in terms of "dx" and "dy" [10]:

$$(ds)^2 = (dx)^2 + (dy)^2$$

C: All bookwork, and the student, does not engage with the mathematics by showing or explaining thoroughly.

Which implies that $ds = \sqrt{(dx)^2 + (dy)^2}$ - (2)

(dy)^2 can be written as $(dy)^2 = \frac{(dy)^2}{(dx)^2} \times (dx)^2$ - (3)

Using 3 in 2:

$ds = \sqrt{(dx)^2 + \frac{(dy)^2}{(dx)^2} \times (dx)^2}$; taking $(dx)^2$ out as a common factor,

$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$, we can further rewrite $\frac{dy}{dx}$ as y':

$$ds = \sqrt{1 + y'^2}dx$$

- (4)

Using 1 and 4 in the time equation, we have:

$t_{AB} = \int_A^B \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} dx$ where t_{AB} is the time the object takes going from A to B.

C: Partial differential equation explained but not in the student's own words.

This can be further rewritten as $t_{AB} = \int_A^B \sqrt{\frac{1+y'^2}{2gy}} dx$

- (5)

The equation written above can be transformed to a partial different equation. A partial differential equation is a differential equation involving multivariable functions and their partial derivatives[11]. A partial derivative of a function is the derivative of a function of two or more variables with respect to one variable, the other(s) being treated as constant(s).[12] The Euler-Lagrange equation is a second-order differential equation whose solutions are the functions for which a given **functional**[13] is stationary[14]. It is used to find the minimum of a partial differential equation. The identity is given below:

---

[10] "Brachistochrone Problem." *Brachistochrone Problem -- from Wolfram MathWorld*. N.P., n.d. Web. 06 DEC. 2016.

[11] "Partial differential equation." Wikipedia. Wikimedia Foundation, n.d. Web. 29 Dec. 2016.

[12] As defined by "dictionary.com"

[13] Functional means the function of a function. It is also referred to as a higher order function.

[14] Partial differential equation." Wikipedia. Wikimedia Foundation, n.d. Web. 29 Dec. 2016.

4

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

"∂" is the notation used in order to denote a derivative that involves more than 2 variables. Since there is no explicit "x" variable in the function in - (5) we can say that it is a function $\boldsymbol{f}(y, y')$. Therefore $\frac{\partial f}{\partial x}$ is equal to zero and hence we can use the Beltrami identity which says[15],

$$f - y'\frac{\partial f}{\partial y'} = C$$

E: This is all transcribed bookwork and the candidate does not demonstrate understanding.

Computing $\frac{\partial f}{\partial y'}$ from - (5), we have:

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{2gy} \times \sqrt{1 + y'^2}}$$

Now, substituting values the values in the Beltrami identity, we get:

$$f - \left(y' \times \frac{\partial f}{\partial y'}\right) = \sqrt{\frac{1 + y'^2}{2gy}} - \frac{y' \times y'}{\sqrt{2gy} \times \sqrt{1 + y'^2}} = C$$

Which gives,

$$\frac{1}{\sqrt{2gy} \times \sqrt{1 + y'^2}} = C$$

Squaring both sides give,

$$C^2 = \frac{1}{2gy \times (1 + y'^2)}$$

Rearranging,

$$\left(1 + \left(\frac{dy}{dx}\right)^2\right) \times y = \frac{1}{2gC^2}$$

Let $2A = \frac{1}{2gC^2}$ where A[16] is just an arbitrary constant,

And rearrange in terms of $\frac{dy}{dx}$ to get,

$$\frac{(dy)^2}{(dx)^2} = \frac{2A - y}{y}$$

1. Let us impose the initial condition; when y = 0 $\theta = 0$ Where θ is the parameter of the parametric equations being used below. "θ" is the angle that the object makes with

[15] "Brachistochrone Problem." *Brachistochrone Problem -- from Wolfram MathWorld*. N.P., n.d. Web. 06 DEC. 2016.
[16] "Brachistochrone Problem." *Brachistochrone Problem -- from Wolfram MathWorld*. N.P., n.d. Web. 06 DEC. 2016.

the normal. At the end of its motion, the object would again make 0° angle with the normal.

2. Let $y = A - Acos\theta$[17], hence $dy = Asin\theta = 2A\cos\frac{\theta}{2}\sin\frac{\theta}{2}d\theta$ - (6) which gives:

$$\frac{Asin\theta}{dx} = \sqrt{\frac{2A - A + Acos\theta}{A - Acos\theta}}$$

Which gives,

$\frac{Asin\,\theta\, d\theta}{dx} = \sqrt{\frac{A+Acos\theta}{A-Acos\theta}} \equiv \sqrt{\frac{cos^2\theta/_2}{sin^2\theta/_2}}$ Using the identity $sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$ and $cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$.

Which simplifies to: $dy = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\,dx$ - (7)

D: Why is this the quickest descent path?

Using 6 and 7, we can say that

$$dx = 2Asin^2\frac{\theta}{2}\,d\theta = A(1-\cos\theta)\,d\theta$$

Finally, integrating both sides gives,
$x = A(\theta - \sin\theta) + D$ and $y = A(1-\cos\theta) + D$

From the initial condition it is clear that when $x = 0\ and\ \theta = 0, D\ had\ to\ equal\ 0.$

Therefore $x = A(\theta - \sin\theta)\ and\ y = A(1-\cos\theta)$ are the parameters for the path of the quickest descent.

This derivation has assumed that the surface is frictionless and theoretically on a frictionless surface the brachistochrone should be fastest path.

Catenary:

The derivation for this will not be included in my exploration due to the complexity in the derivation, however, the equation for it is $y = a\cosh(\frac{x}{a})$ where "a" is the parameter that determines the depth of the curve.

E: Was the derivation above simpler than the one which was shown?

INVESTIGATION:

In this investigation, I use video analysis and an online simulation[18] to compare the brachistochrone to the catenary and an incline plane. The simulation's job was to release a ball of mass 1g from the same initial height and track its motion to produce graphs of desired quantities (like velocities, displacement etc.). I am using a simulation because of the difficulty in construction of the catenary and brachistochrone. I will focus mostly on the velocity and the displacement (x-displacement and y-displacement) vectors. The x-displacement is the displacement of any object along the x-axis whereas y-displacement is the displacement

[17] "Brachistochrone Problem." *Brachistochrone Problem -- from Wolfram MathWorld*. N.P., n.d. Web. 06 Dec. 2016. https://youtu.be/atZXPk1Jfz8

[18] tobiasgurdan.de/facharbeit/Brachistochrone/Brachistochrone.html

6

along the y-axis. Displacement is a vector of measure of the distance of any object from its starting position. Mathematically, the x-displacement vector is written as $\Delta x = x_f - x_0$ where x_f is the final position and x_0 is the initial position. The y-displacement can also be called the change in height. This can also be written mathematically as $\Delta y = y_f - y_0$ where y_f is the final height and y_0 is the initial height. The y-displacement for this investigation will have a negative slope as the object will start at a greater height and stop a lower height; the x-displacement, however will have a positive since the ball will be at a more positive x-position than it started with. The steeper the displacement curve the higher the rate of change of displacement, which means that the steepest curve will cover the distance the fastest.

??

The equations of the paths I used in the simulation were:

1. Incline plane: -$\left(\frac{2}{3}\right)x$
2. Brachistochrone: {x = 1.00133$(\theta - \sin\theta)$ and y = -1.00133$(1 - \cos\theta)$}
3. Catenary: "a" $\cosh\left(\frac{x}{a}\right)$ where "a" = $\frac{3}{4}$.

E: How were the coefficients chosen and why?

B: Why use the word slope?

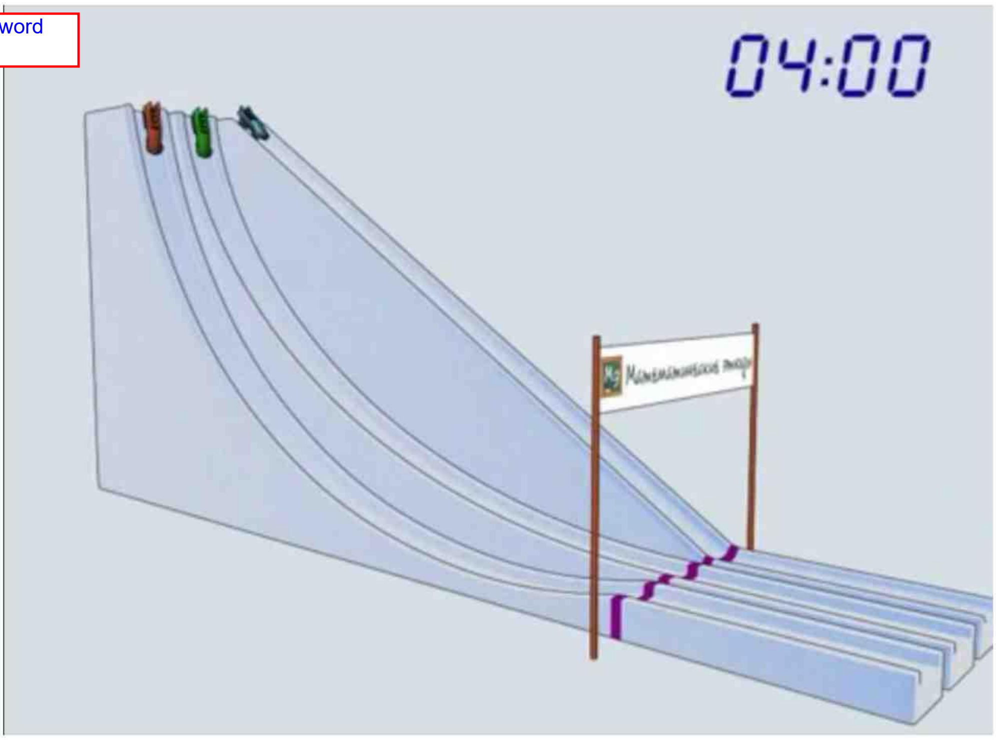


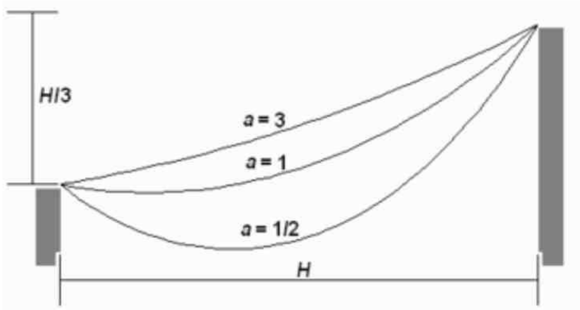
Image 1.

This is a screenshot of the simulation used. The blue car is on an incline plane, whose equation is given above, the green car is on a brachistochrone, the equation of which is also given above. Finally, the red car is on the catenary. In order to make a reasonable comparison between the catenary and other paths, a specific parametric value of "a" becomes necessary. This is important as it determines the depth of the catenary. The following Wikipedia image[19]

[19] "Catenary." Wikipedia. Wikimedia Foundation, n.d. Web. 19 Dec. 2016.

7

explains the effect of different values of “a”. In the simulation that I used, the value of “a” for the catenary was 0.75.

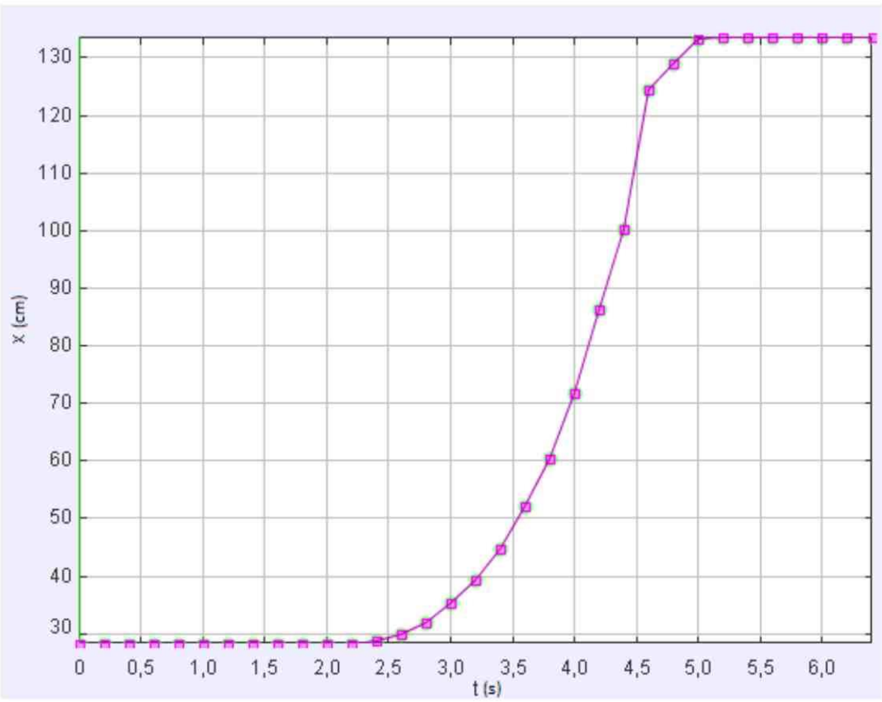


D: Missed opportunity to discuss how 0.75 was evaluated and why this was chosen.

Results from the simulation:

The incline plane / incline plane:

The x-displacement vs time graph for the incline plane:



The x-displacement (along the y-axis on the graph) was measured in centimeters and the time (along the x-axis on the graph) was measured in seconds. The stable (zero slope) beginning on the graph was before the object started rolling down the incline plane. From the graph, I can say that the object started rolling at t= $2.40\ s$. We can see that the graph has a positive slope as expected. This graph was can be considered to be an idealized result; in real life, the friction force would have an effect on the conversion of mechanical energy.

8

Higher the co-efficient of friction[20], lower the efficiency in the conversion of potential energy to kinetic energy. The ball was under free-fall* down the incline plane in the simulation. Hence, the graph obtained by values we measure manually definitely differ from the graph above.

For comparison purposes, time taken to cover a horizontal distance of 130 cm

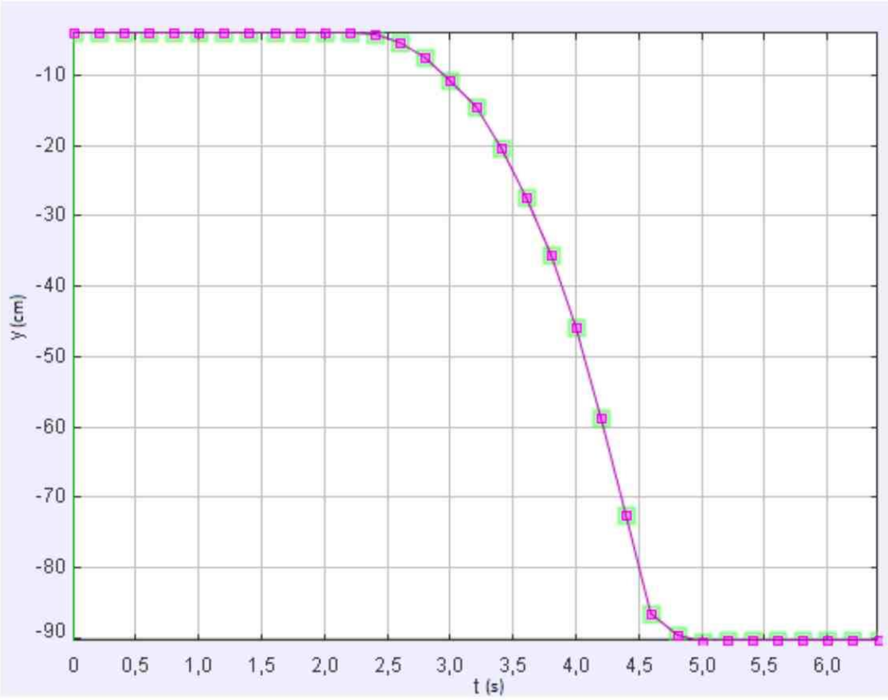
$\Delta t \approx 3.80\ s$ - (8)

D: Superficial reflection. Why was this comparison made? How is it useful?

[20] Co-efficient of friction is a value that shows the relation between the force of friction between two objects and the normal reaction between the objects.

* Free fall is used to define motion where the only force acting is that of gravity.

The y-displacement vs time graph for the incline plane:

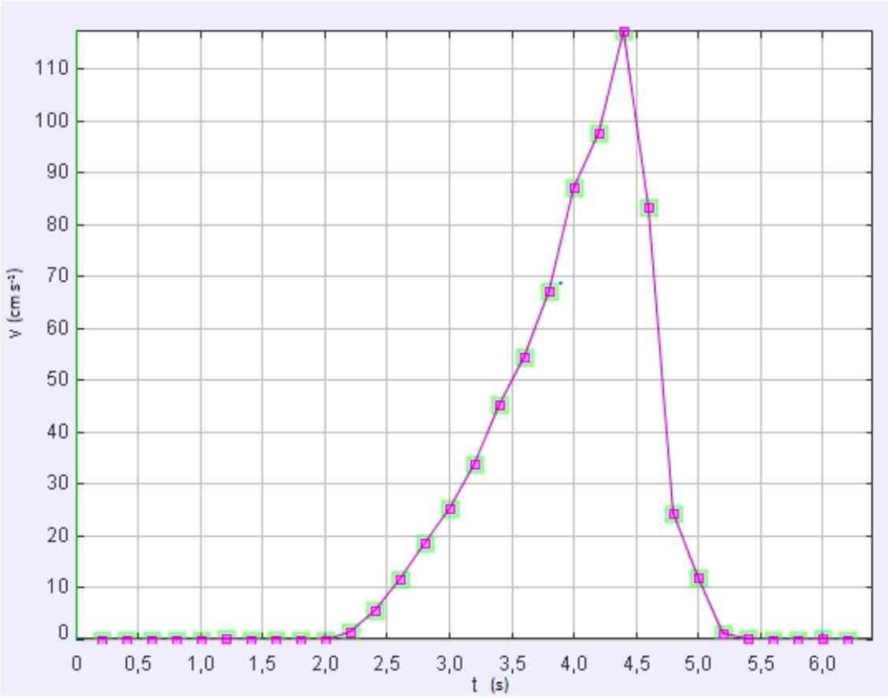


The graph as expected has a negative slope. Again the "zero slope section" of the graph is before the object started rolling. The vertical height again was measured in centimeters and time in milliseconds. The time taken for the object to roll down a height of about 90 cm $\Delta t \approx 3.80\ s$ This value was calculated by finding the time at which the object stopped moving and subtracting from it the time at which the object started moving.

D: Why are the two graphs necessary? How can they be combined?

10

The velocity vs time graph for the incline plane:



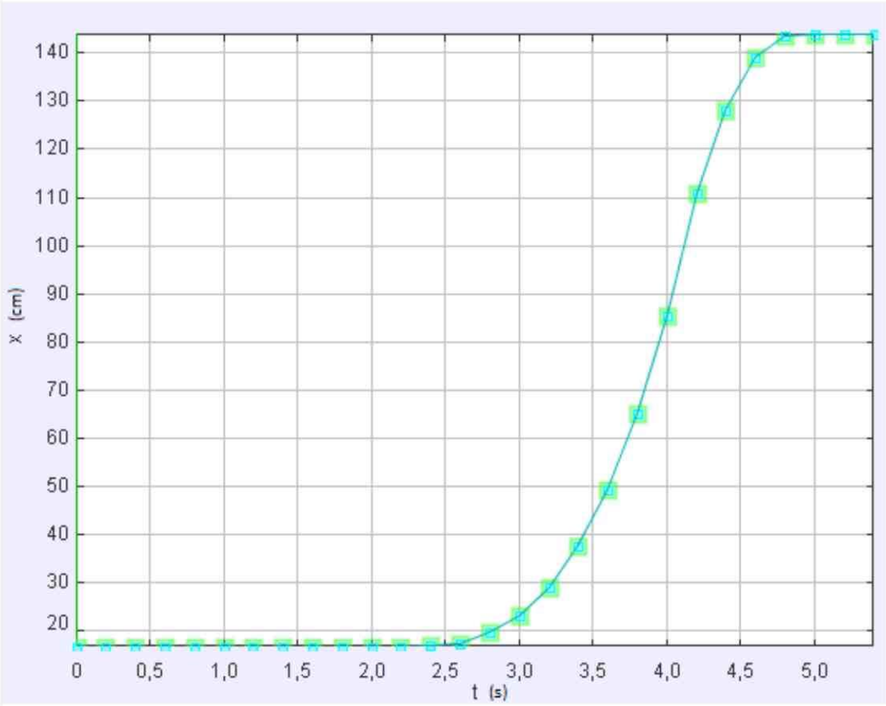
The graph above shows a positive increase in velocity from t= $2\ s$. The velocity was measured in cm s^{-1}. Therefore, the maximum velocity is about 117 cm s^{-1}. For the object to reach the maximum velocity from the time it started rolling, it took the object about $2.4s$.

C: How are the graphs being produced? How was velocity measured? Were the graphs computer generated through the simulation?

11

Brachistochrone:

The x-displacement vs time graph for the brachistochrone:

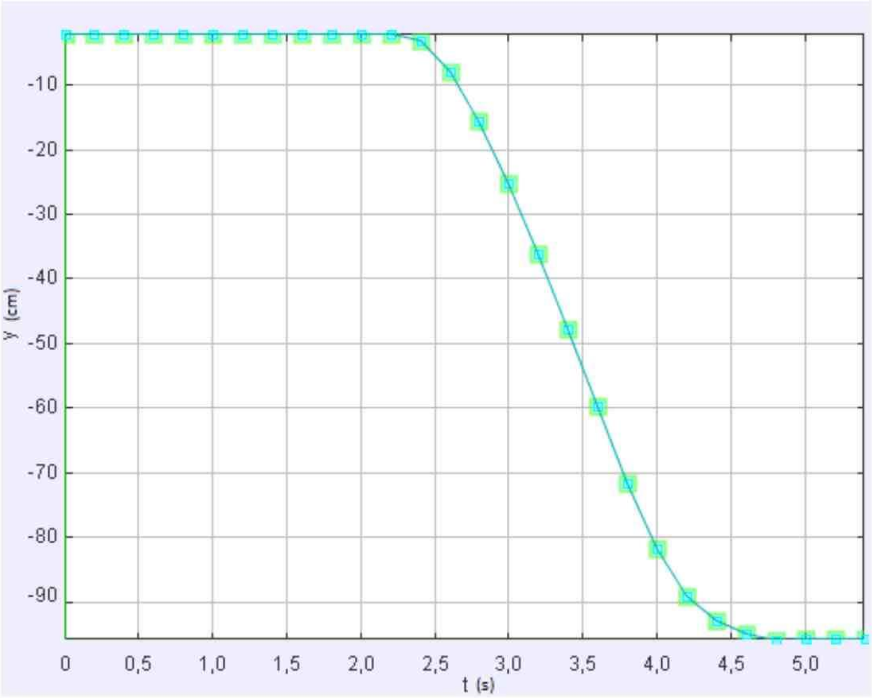


The displacement was measured in centimeters and the time was measured in seconds. We can already see that the curve for the x-displacement for the brachistochrone looks steeper than the curve for the incline plane. The time taken to cover a distance of approximately 144 cm $\Delta t = 1.20\ s$ - (9)
We can see that the brachistochrone is much faster than the incline plane.

A: Repetitive and no analysis is being done apart from the last comment which is not justified.

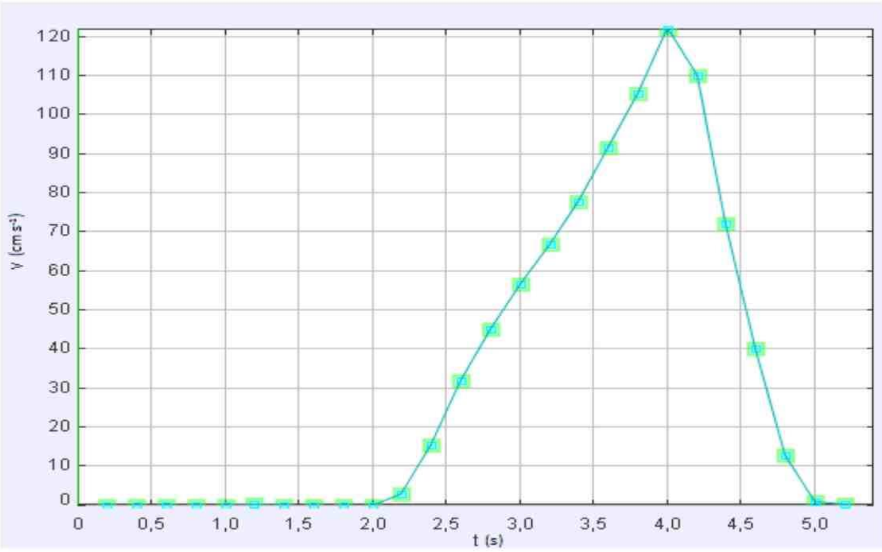
12

The y-displacement vs time graph for the brachistochrone:



The displacement and time have been measured using the same units. The graph looks steeper than the incline plane. The maximum height is the same since all the paths had the same initial height. Time taken for the ball to roll down the incline plane, is the time at which the ball stopped rolling minus the time at which ball started rolling $\Delta t \approx 1.20s$

The velocity vs time graph for the brachistochrone:

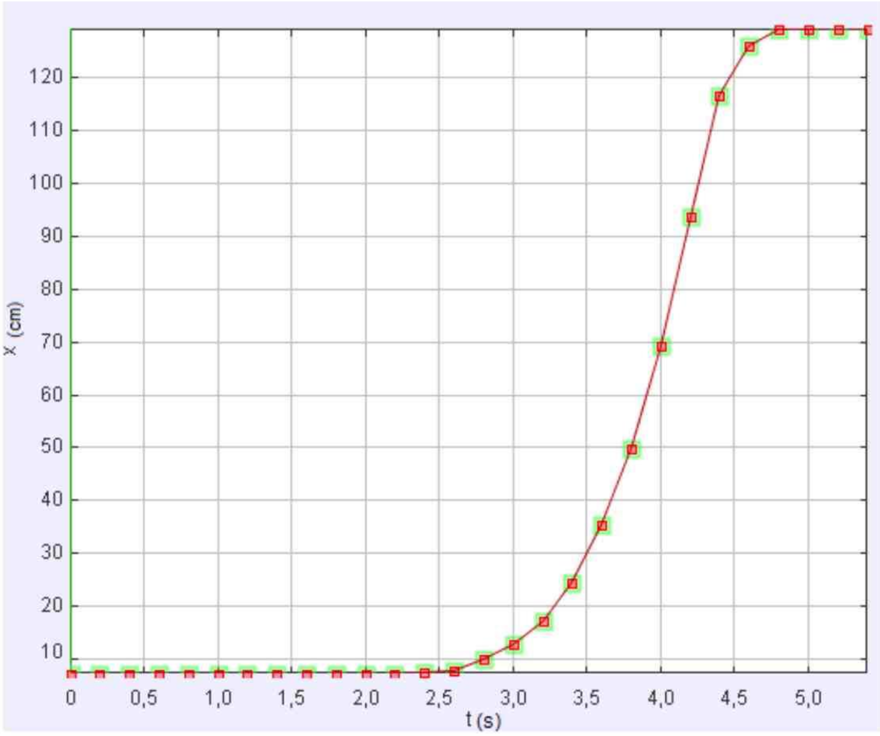


13

We can see that the velocity for the brachistochrone is higher than the velocity for the incline plane. The simulation verifies that the brachistochrone is the fastest curve, at least until now

Catenary:

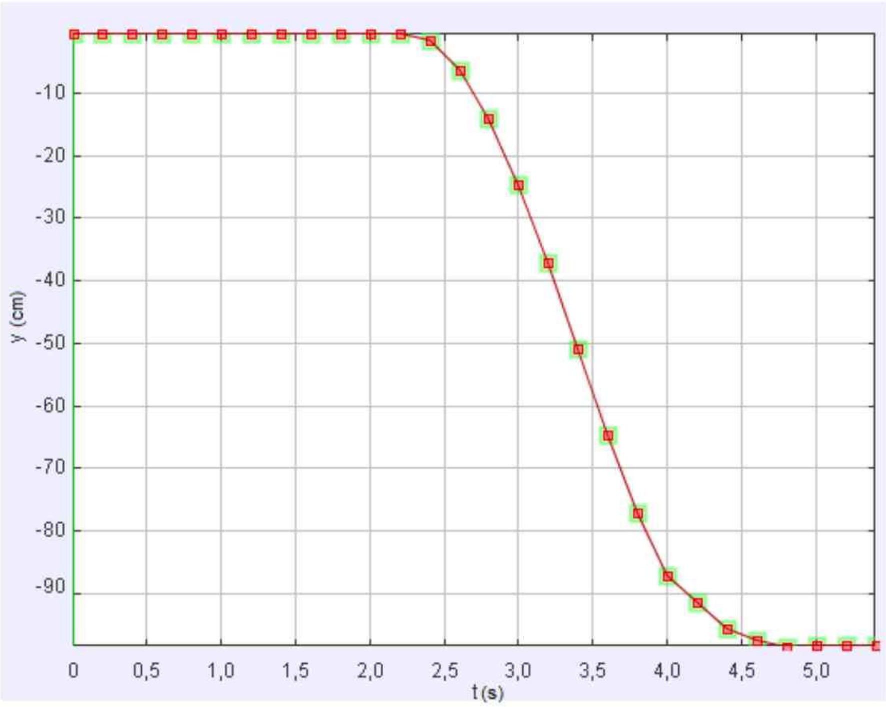
The x-displacement vs time graph for the catenary:



The graph above used the same units of measurement. The time taken for the object to cover the distance $\Delta t \approx 1.2s.$ – (10)
It can be seen that the curve is almost as steep as the brachistochrone and is definitely steeper than the incline plane. The catenary took the same time to transport the ball along the intended path.

14

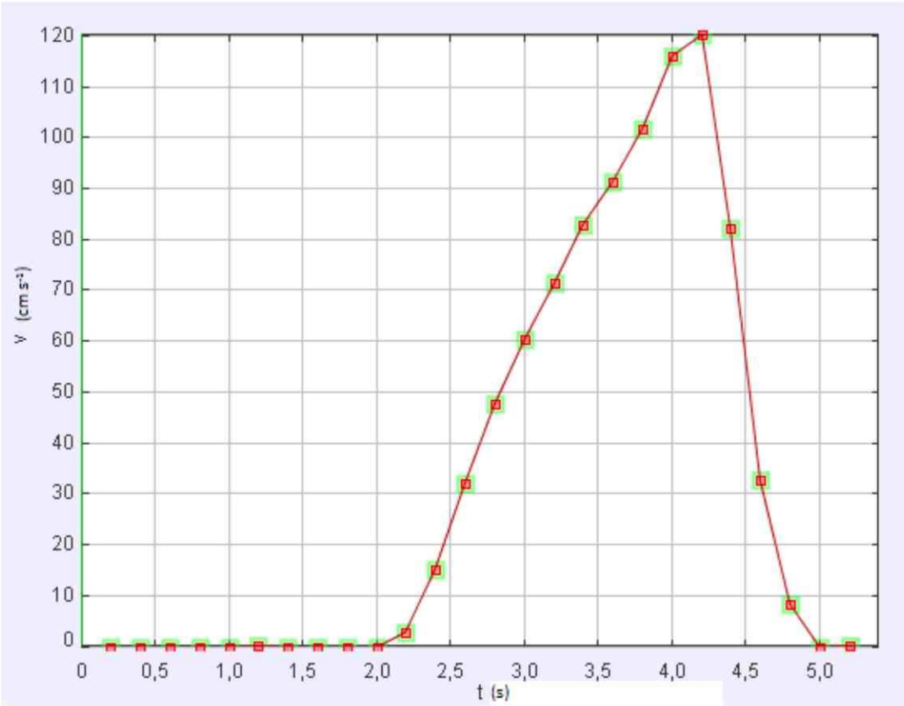
The y-displacement vs time graph for the catenary:



From the graph we can see the characteristic of the catenary being exhibited here. We know that the catenary is the shape adopted by a free hanging string which implies that the height should start increasing at some time, but, that is not exhibited here as the parameter "a" was 1 which means that the curve was rather stiff and therefore resembled a brachistochrone.

15

Finally, the velocity vs time graph for the catenary:



This is the interesting bit of the results section; the maximum velocity of the object along a catenary is about 120 cm s^{-1}. Though the time it took the object to travel down was almost the same, the velocity of the object along the brachistochrone was higher. This shows that the brachistochrone accelerated the ball faster. The simulation software worked out the graph by analyzing the trajectory 5 frames per second; if more frames per second were used, something like 1 frame per second, a better analysis would have been produced showing that the brachistochrone is a significantly faster path.

B: It would have been better to compile the graphs for the three tracks so that comparisons are easier. In this way the student could have reflected more.

Conclusion:

From the comparisons, it is clear that the brachistochrone was actually the fastest curve and that my hypothesis was correct. The highest velocity award goes to the brachistochrone. One important thing to say here is the friction was not taken into account for the simulation. Friction of the surface would lead to energy loss and hence the results obtained here might not be precise. So, the assumption in the derivation of the brachistochrone to assume a frictionless surface shows that the brachistochrone is in fact the fastest path of descent!

Bibliography:

Stevehaake. "Surfing the Brachistochrone." *Engineering Sport*. N.P., 2011. Web. 06 Dec. 2016.

"Catenary." *Wikipedia*. Wikimedia Foundation, n.d. Web. 06 Dec. 2016.

"Brachistochrone Problem." *Brachistochrone Problem -- from Wolfram MathWorld*. N.P., n.d. Web. 06 Dec. 2016.

"How can I draw this cycloid diagram with TikZ?" Technical drawing - How can I draw this cycloid diagram with TikZ? - TeX - LaTeX Stack Exchange. N.p., n.d. Web. 16 Jan. 2017.

"Conservative force." Wikipedia. Wikimedia Foundation, n.d. Web. 29 Dec. 2016.

"Partial differential equation." Wikipedia. Wikimedia Foundation, n.d. Web. 29 Dec. 2016.

Functional means the function of a function. It is also referred to as a higher order function.

Partial differential equation." Wikipedia. Wikimedia Foundation, n.d. Web. 29 Dec. 2016.

"Brachistochrone Problem." *Brachistochrone Problem -- from Wolfram MathWorld*. N.P., n.d. Web. 06 DEC. 2016.

tobiasgurdan.de/facharbeit/Brachistochrone/Brachistochrone.html, n.d. 29th Oct 2016

Co-efficient of friction is a value that shows the relation between the force of friction between two objects and the normal reaction between the objects.

The simulation that I used has been taken down from the internet and therefore I can’t the URL to cite it here.

Acknowledgements:

1. Desmos
2. Dictionary.com
3. Tracker