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MARKSCHEME

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MATHEMATICS

Higher Level

Paper 1

18 pages

SECTION A

1. $P(2)=24+2a+b=2$, $P(-1)=-3-a+b=5$ ***M1A1A1***
$(2a+b=-22, -a+b=8)$

> **Note:** Award ***M1*** for substitution of 2 or -1 and equating to remainder, ***A1*** for each correct equation.

attempt to solve simultaneously ***M1***
$a=-10$, $b=-2$ ***A1***

[5 marks]

2. using the sum divided by 4 is 13 ***M1***
two of the numbers are 15 ***A1***
(as median is 14) we need a 13 ***A1***
fourth number is 9 ***A1***
numbers are 9, 13, 15, 15 ***N4***

[4 marks]

3. $\frac{\log 3}{\log 2}\times\frac{\log 4}{\log 3}\times\ldots\times\frac{\log 32}{\log 31}$ ***M1A1***
$=\frac{\log 32}{\log 2}$ ***A1***
$=\frac{5\log 2}{\log 2}$ ***(M1)***
$=5$ ***A1***
hence $a=5$

[5 marks]

> **Note:** Accept the above if done in a specific base *eg* $\log_2 x$.

4. $r_1+r_2+r_3=\frac{-48}{5}$ ***(M1)(A1)***
$r_1 r_2 r_3=\frac{a-2}{5}$ ***(M1)(A1)***
$\frac{-48}{5}+\frac{a-2}{5}=0$ ***M1***
$a=50$ ***A1***

> **Note:** Award ***M1A0M1A0M1A1*** if answer of 50 is found using $\frac{48}{5}$ and $\frac{2-a}{5}$.

[6 marks]

5. (a) $\cos x = 2\cos^2 \frac{1}{2}x - 1$

$\cos\frac{1}{2}x = \pm\sqrt{\frac{1+\cos x}{2}}$ ***M1***

positive as $0 \le x \le \pi$ ***R1***

$\cos\frac{1}{2}x = \sqrt{\frac{1+\cos x}{2}}$ ***AG***

[2 marks]

(b) $\cos 2\theta = 1 - 2\sin^2\theta$ ***(M1)***

$\sin\frac{1}{2}x = \sqrt{\frac{1-\cos x}{2}}$ ***A1***

[2 marks]

(c) $\sqrt{2}\int_0^{\frac{\pi}{2}} \cos\frac{1}{2}x + \sin\frac{1}{2}x \,\mathrm{d}x$ ***A1***

$= \sqrt{2}\left[2\sin\frac{1}{2}x - 2\cos\frac{1}{2}x\right]_0^{\frac{\pi}{2}}$ ***A1***

$= \sqrt{2}(0) - \sqrt{2}(0-2)$ ***(A1)***

$= 2\sqrt{2}$ ***A1***

[4 marks]

Total [8 marks]

6. (a) $x = 1$ ***A1***

[1 mark]

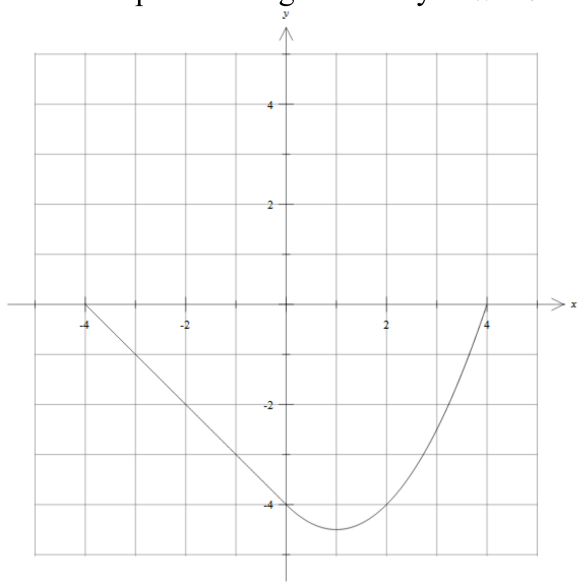
(b) ***A1*** for point $(-4, 0)$

A1 for $(0, -4)$

A1 for min at $x = 1$ in approximately the correct place

A1 for $(4, 0)$

A1 for shape including continuity at $x = 0$



[5 marks]

Total [6 marks]

7. **METHOD 1**

$\text{AD}^2 = 2^2 + 3^2 - 2\times2\times3\times\cos 60^\circ$ ***M1***

$\left(\text{or } \text{AD}^2 = 1^2 + 3^2 - 2\times1\times3\times\cos 60^\circ\right)$

> **Note:** ***M1*** for use of cosine rule with 60° angle.

$\text{AD}^2 = 7$ ***A1***

$\cos \text{D}\hat{\text{A}}\text{C} = \dfrac{9+7-4}{2\times3\times\sqrt{7}}$ ***M1A1***

> **Note:** ***M1*** for use of cosine rule involving $\text{D}\hat{\text{A}}\text{C}$.

$= \dfrac{2}{\sqrt{7}}$ ***A1***

METHOD 2

let point E be the foot of the perpendicular from D to AC

$\text{EC} = 1$ (by similar triangles, or triangle properties) ***M1A1***

(or $\text{AE} = 2$)

$\text{DE} = \sqrt{3}$ and $\text{AD} = \sqrt{7}$ (by Pythagoras) ***(M1)A1***

$\cos \text{D}\hat{\text{A}}\text{C} = \dfrac{2}{\sqrt{7}}$ ***A1***

[5 marks]

> **Note:** If first ***M1*** not awarded but remainder of the question is correct award ***M0A0M1A1A1***.

8. $\dfrac{\mathrm{d}v}{\mathrm{d}s} = 2s^{-3}$ ***M1A1***

> **Note:** Award ***M1*** for $2s^{-3}$ and ***A1*** for the whole expression.

$a = v\dfrac{\mathrm{d}v}{\mathrm{d}s}$ ***(M1)***

$a = -\dfrac{1}{s^2}\times\dfrac{2}{s^3}\left(= -\dfrac{2}{s^5}\right)$ ***(A1)***

when $s = \dfrac{1}{2}$, $a = -\dfrac{2}{(0.5)^5}$ $(= -64)$ (m s^{-2}) ***M1A1***

> **Note:** ***M1*** is for the substitution of 0.5 into their equation for acceleration.
> Award ***M1A0*** if $s = 50$ is substituted into the correct equation.

[6 marks]

9. (a) **METHOD 1**

$$\frac{2x}{1+x^4}+\frac{2y}{1+y^4}\frac{dy}{dx}=0$$ ***M1A1A1***

Note: Award ***M1*** for implicit differentiation, ***A1*** for LHS and ***A1*** for RHS.

$$\frac{dy}{dx}=-\frac{x\left(1+y^4\right)}{y\left(1+x^4\right)}$$ ***A1***

METHOD 2

$$y^2=\tan\left(\frac{\pi}{4}-\arctan x^2\right)$$

$$=\frac{\tan\frac{\pi}{4}-\tan\left(\arctan x^2\right)}{1+\left(\tan\frac{\pi}{4}\right)\left(\tan\left(\arctan x^2\right)\right)}$$ ***(M1)***

$$=\frac{1-x^2}{1+x^2}$$ ***A1***

$$2y\frac{dy}{dx}=\frac{-2x\left(1+x^2\right)-2x\left(1-x^2\right)}{\left(1+x^2\right)^2}$$ ***M1***

$$2y\frac{dy}{dx}=\frac{-4x}{\left(1+x^2\right)^2}$$

$$\frac{dy}{dx}=-\frac{2x}{y\left(1+x^2\right)^2}$$ ***A1***

$$\left(=\frac{2x\sqrt{1+x^2}}{\sqrt{1-x^2}\left(1+x^2\right)^2}\right)$$

[4 marks]

continued …

Question 9 continued

(b) $y^2 = \tan\left(\frac{\pi}{4} - \arctan\frac{1}{2}\right)$ ***(M1)***

$= \dfrac{\tan\frac{\pi}{4} - \tan\left(\arctan\frac{1}{2}\right)}{1 + \left(\tan\frac{\pi}{4}\right)\left(\tan\left(\arctan\frac{1}{2}\right)\right)}$ ***(M1)***

Note: The two ***M1***s may be awarded for working in part (a).

$= \dfrac{1-\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}$ ***A1***

$y = -\dfrac{1}{\sqrt{3}}$ ***A1***

substitution into $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

$= \dfrac{4\sqrt{6}}{9}$ ***A1***

Note: Accept $\dfrac{8\sqrt{3}}{9\sqrt{2}}$ *etc.*

[5 marks]

Total [9 marks]

10. $\sin^2 x + \cos^2 x + 2\sin x\cos x = \frac{4}{9}$ ***(M1)(A1)***

using $\sin^2 x + \cos^2 x = 1$ ***(M1)***

$2\sin x\cos x = -\frac{5}{9}$

using $2\sin x\cos x = \sin 2x$ ***(M1)***

$\sin 2x = -\frac{5}{9}$

$\cos 4x = 1 - 2\sin^2 2x$ ***M1***

Note:	Award this ***M1*** for decomposition of $\cos 4x$ using double angle formula anywhere in the solution.

$= 1 - 2 \times \frac{25}{81}$

$= \frac{31}{81}$ ***A1***

[6 marks]

SECTION B

11. (a) $f'(x) = \dfrac{x \times \frac{1}{x} - \ln x}{x^2}$ ***M1A1***

$= \dfrac{1 - \ln x}{x^2}$ ***AG***

[2 marks]

(b) $\dfrac{1 - \ln x}{x^2} = 0$ has solution $x = \mathrm{e}$ ***M1A1***

$y = \dfrac{1}{\mathrm{e}}$ ***A1***

hence maximum at the point $\left(\mathrm{e}, \dfrac{1}{\mathrm{e}}\right)$

[3 marks]

(c) $f''(x) = \dfrac{x^2\left(-\frac{1}{x}\right) - 2x(1 - \ln x)}{x^4}$ ***M1A1***

$= \dfrac{2\ln x - 3}{x^3}$

Note: The ***M1A1*** should be awarded if the correct working appears in part (b).

point of inflexion where $f''(x) = 0$ ***M1***

so $x = \mathrm{e}^{\frac{3}{2}}$, $y = \dfrac{3}{2}\mathrm{e}^{\frac{-3}{2}}$ ***A1A1***

C has coordinates $\left(\mathrm{e}^{\frac{3}{2}}, \dfrac{3}{2}\mathrm{e}^{\frac{-3}{2}}\right)$

[5 marks]

(d) $f(1) = 0$ ***A1***

$f'(1) = 1$ ***(A1)***

$y = x + c$ ***(M1)***

through $(1, 0)$

equation is $y = x - 1$ ***A1***

[4 marks]

continued …

Question 11 continued

(e) **METHOD 1**

area $= \int_1^{e} x - 1 - \frac{\ln x}{x}\,dx$ ***M1A1A1***

> **Note:** Award ***M1*** for integration of difference between line and curve, ***A1*** for correct limits, ***A1*** for correct expressions in either order.

$\int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2}(+c)$ ***(M1)A1***

$\int (x-1)\,dx = \frac{x^2}{2} - x(+c)$ ***A1***

$= \left[\frac{1}{2}x^2 - x - \frac{1}{2}(\ln x)^2\right]_1^{e}$

$= \left(\frac{1}{2}e^2 - e - \frac{1}{2}\right) - \left(\frac{1}{2} - 1\right)$

$= \frac{1}{2}e^2 - e$ ***A1***

METHOD 2

area $=$ area of triangle $- \int_1^{e} \frac{\ln x}{x}\,dx$ ***M1A1***

> **Note:** ***A1*** is for correct integral with limits and is dependent on the ***M1***.

$\int \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2}(+c)$ ***(M1)A1***

area of triangle $= \frac{1}{2}(e-1)(e-1)$ ***M1A1***

$\frac{1}{2}(e-1)(e-1) - \left(\frac{1}{2}\right) = \frac{1}{2}e^2 - e$ ***A1***

[7 marks]

Total [21 marks]

12. (a) $|\vec{OA}| = |\vec{CB}| = |\vec{OC}| = |\vec{AB}| = 6$ (therefore a rhombus) ***A1A1***

Note: Award ***A1*** for two correct lengths, ***A2*** for all four.

Note: Award ***A1A0*** for $\vec{OA} = \vec{CB} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$ or $\vec{OC} = \vec{AB} = \begin{pmatrix} 0 \\ -\sqrt{24} \\ \sqrt{12} \end{pmatrix}$ if no magnitudes are shown.

$\vec{OA} \text{g} \vec{OC} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \text{g} \begin{pmatrix} 0 \\ -\sqrt{24} \\ \sqrt{12} \end{pmatrix} = 0$ (therefore a square) ***A1***

Note: Other arguments are possible with a minimum of three conditions.

[3 marks]

(b) $\text{M}\left(3, -\frac{\sqrt{24}}{2}, \frac{\sqrt{12}}{2}\right)\left(= \left(3, -\sqrt{6}, \sqrt{3}\right)\right)$ ***A1***

[1 mark]

(c) **METHOD 1**

$\vec{OA} \times \vec{OC} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -\sqrt{24} \\ \sqrt{12} \end{pmatrix} = \begin{pmatrix} 0 \\ -6\sqrt{12} \\ -6\sqrt{24} \end{pmatrix} \left(= \begin{pmatrix} 0 \\ -12\sqrt{3} \\ -12\sqrt{6} \end{pmatrix} \right)$ ***M1A1***

Note: Candidates may use other pairs of vectors.

equation of plane is $-6\sqrt{12}y - 6\sqrt{24}z = d$

any valid method showing that $d = 0$ ***M1***

$\Pi : y + \sqrt{2}z = 0$ ***AG***

METHOD 2

equation of plane is $ax + by + cz = d$

substituting O to find $d = 0$ ***(M1)***

substituting two points (A, B, C or M) ***M1***

eg

$6a = 0$, $-\sqrt{24}b + \sqrt{12}c = 0$ ***A1***

$\Pi : y + \sqrt{2}z = 0$ ***AG***

[3 marks]

continued ...

Question 12 continued

(d) $\boldsymbol{r} = \begin{pmatrix} 3 \\ -\sqrt{6} \\ \sqrt{3} \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ \sqrt{2} \end{pmatrix}$ ***A1A1A1***

Note: Award ***A1*** for $\boldsymbol{r} =$, ***A1A1*** for two correct vectors.

[3 marks]

(e) Using $y = 0$ to find λ ***M1***
Substitute their λ into their equation from part (d) ***M1***
D has coordinates $\left(3, 0, 3\sqrt{3}\right)$ ***A1***

[3 marks]

(f) λ for point E is the negative of the λ for point D ***(M1)***

Note: Other possible methods may be seen.

E has coordinates $\left(3, -2\sqrt{6}, -\sqrt{3}\right)$ ***A1A1***

Note: Award ***A1*** for each of the y and z coordinates.

[3 marks]

(g) (i) $\overrightarrow{\text{DA}}\,\text{g}\,\overrightarrow{\text{DO}} = \begin{pmatrix} 3 \\ 0 \\ -3\sqrt{3} \end{pmatrix} \text{g} \begin{pmatrix} -3 \\ 0 \\ -3\sqrt{3} \end{pmatrix} = 18$ ***M1A1***

$\cos \text{O}\hat{\text{D}}\text{A} = \dfrac{18}{\sqrt{36}\sqrt{36}} = \dfrac{1}{2}$ ***M1***

hence $\text{O}\hat{\text{D}}\text{A} = 60^{\circ}$ ***A1***

Note: Accept method showing OAD is equilateral.

(ii) OABCDE is a regular octahedron (accept equivalent description) ***A2***

Note: ***A2*** for saying it is made up of 8 equilateral triangles Award ***A1*** for two pyramids, ***A1*** for equilateral triangles. (can be either stated or shown in a sketch – but there must be clear indication the triangles are equilateral)

[6 marks]

Total [22 marks]

13. (a) $r = 1 + \mathrm{i}$ ***(A1)***

$u_4 = 3(1+\mathrm{i})^3$ ***M1***

$= -6 + 6\mathrm{i}$ ***A1***

[3 marks]

(b) $S_{20} = \dfrac{3\left((1+\mathrm{i})^{20} - 1\right)}{\mathrm{i}}$ ***(M1)***

$= \dfrac{3\left((2i)^{10} - 1\right)}{\mathrm{i}}$ ***(M1)***

> **Note:** Only one of the two ***M1***s can be implied. Other algebraic methods may be seen.

$= \dfrac{3\left(-2^{10} - 1\right)}{\mathrm{i}}$ ***(A1)***

$= 3\mathrm{i}\left(2^{10} + 1\right)$ ***A1***

[4 marks]

(c) (i) **METHOD 1**

$v_n = \left(3(1+\mathrm{i})^{n-1}\right)\left(3(1+\mathrm{i})^{n-1+k}\right)$ ***M1***

$9(1+\mathrm{i})^k (1+i)^{2n-2}$ ***A1***

$= 9(1+\mathrm{i})^k ((1+i)^2)^{n-1} \left(= 9(1+\mathrm{i})^k (2\mathrm{i})^{n-1}\right)$

this is the general term of a geometrical sequence ***R1AG***

> **Notes:** Do not accept the statement that the product of terms in a geometric sequence is also geometric unless justified further.
> If the final expression for v_n is $9(1+\mathrm{i})^k (1+i)^{2n-2}$ award ***M1A1R0***.

METHOD 2

$\dfrac{v_{n+1}}{v_n} = \dfrac{u_{n+1}u_{n+k+1}}{u_n u_{n+k}}$ ***M1***

$= (1+\mathrm{i})(1+\mathrm{i})$ ***A1***

this is a constant, hence sequence is geometric ***R1AG***

> **Note:** Do not allow methods that do not consider the general term.

(ii) $9(1+\mathrm{i})^k$ ***A1***

(iii) common ratio is $(1+i)^2 (= 2i)$ (which is independent of k) ***A1***

[5 marks]

continued …

Question 13 continued

(d) (i) **METHOD 1**

$w_n = \left|3(1+i)^{n-1} - 3(1+\mathrm{i})^n\right|$ ***M1***

$= 3\left|1+i\right|^{n-1}\left|1-(1+\mathrm{i})\right|$ ***M1***

$= 3\left|1+i\right|^{n-1}$ ***A1***

$\left(= 3\left(\sqrt{2}\right)^{n-1}\right)$

this is the general term for a geometric sequence ***R1AG***

METHOD 2

$w_n = \left|u_n - (1+\mathrm{i})u_n\right|$ ***M1***

$= \left|u_n\right|\left|-\mathrm{i}\right|$

$= \left|u_n\right|$ ***A1***

$= \left|3(1+\mathrm{i})^{n-1}\right|$

$= 3\left|(1+\mathrm{i})\right|^{n-1}$ ***A1***

$\left(= 3\left(\sqrt{2}\right)^{n-1}\right)$

this is the general term for a geometric sequence ***R1AG***

Note: Do not allow methods that do not consider the general term.

(ii) distance between successive points representing u_n in the complex plane forms a geometric sequence ***R1***

Note: Various possibilities but must mention distance between successive points.

[5 marks]

Total [17 marks]