M14/5/MATHL/HP1/ENG/TZ2/XX/M

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

MARKSCHEME

May 2014

MATHEMATICS

Higher Level

Paper 1

24 pages

SECTION A

1. (a) $\mathrm{P}(A \cap B) = \mathrm{P}(A \mid B) \times P(B)$

$\mathrm{P}(A \cap B) = \frac{2}{11} \times \frac{11}{20}$ ***(M1)***

$= \frac{1}{10}$ ***A1***

[2 marks]

(b) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + P(B) - \mathrm{P}(A \cap B)$

$\mathrm{P}(A \cup B) = \frac{2}{5} + \frac{11}{20} - \frac{1}{10}$ ***(M1)***

$= \frac{17}{20}$ ***A1***

[2 marks]

(c) No – events *A* and *B* are not independent ***A1***

EITHER

$\mathrm{P}(A \mid B) \neq \mathrm{P}(A)$ ***R1***

$\left(\frac{2}{11} \neq \frac{2}{5}\right)$

OR

$\mathrm{P}(A) \times P(B) \neq \mathrm{P}(A \cap B)$

$\frac{2}{5} \times \frac{11}{20} = \frac{11}{50} \neq \frac{1}{10}$ ***R1***

Note: The numbers are required to gain ***R1*** in the '**OR**' method only.

Note: Do not award ***A1R0*** in either method.

[2 marks]

Total [6 marks]

2. METHOD 1

$2^{3(x-1)} = (2\times3)^{3x}$ ***M1***

Note: Award ***M1*** for writing in terms of 2 and 3.

$2^{3x}\times2^{-3} = 2^{3x}\times3^{3x}$

$2^{-3} = 3^{3x}$ ***A1***

$\ln\left(2^{-3}\right) = \ln\left(3^{3x}\right)$ ***(M1)***

$-3\ln2 = 3x\ln3$ ***A1***

$x = -\dfrac{\ln2}{\ln3}$ ***A1***

METHOD 2

$\ln8^{x-1} = \ln6^{3x}$ ***(M1)***

$(x-1)\ln2^3 = 3x\ln(2\times3)$ ***M1A1***

$3x\ln2 - 3\ln2 = 3x\ln2 + 3x\ln3$ ***A1***

$x = -\dfrac{\ln2}{\ln3}$ ***A1***

METHOD 3

$\ln8^{x-1} = \ln6^{3x}$ ***(M1)***

$(x-1)\ln8 = 3x\ln6$ ***A1***

$x = \dfrac{\ln8}{\ln8-3\ln6}$ ***A1***

$x = \dfrac{3\ln2}{\ln\left(\dfrac{2^3}{6^3}\right)}$ ***M1***

$x = -\dfrac{\ln2}{\ln3}$ ***A1***

Total [5 marks]

3. (a) **EITHER**

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 3 & -1 & 14 & 6 \\ 1 & 2 & 0 & -5 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$ ***M1***

row of zeroes implies infinite solutions, (or equivalent). ***R1***

Note: Award ***M1*** for any attempt at row reduction.

OR

$$\begin{vmatrix} 1 & 1 & 2 \\ 3 & -1 & 14 \\ 1 & 2 & 0 \end{vmatrix} = 0$$ ***M1***

$$\begin{vmatrix} 1 & 1 & 2 \\ 3 & -1 & 14 \\ 1 & 2 & 0 \end{vmatrix} = 0 \text{ with one valid point}$$ ***R1***

OR

$x + y + 2z = -2$

$3x - y + 14z = 6$

$x + 2y = -5 \quad \Rightarrow x = -5 - 2y$

substitute $x = -5 - 2y$ into the first two equations:

$-5 - 2y + y + 2z = -2$

$3(-5 - 2y) - y + 14z = 6$ ***M1***

$-y + 2z = 3$

$-7y + 14z = 21$

the latter two equations are equivalent (by multiplying by 7) therefore an infinite number of solutions. ***R1***

OR

for example, $7 \times R_1 - R_2$ gives $4x + 8y = -20$ ***M1***

this equation is a multiple of the third equation, therefore an infinite number of solutions. ***R1***

continued...

Question 3 continued

(b) let $y = t$ ***M1***

then $x = -5 - 2t$ ***A1***

$z = \dfrac{t+3}{2}$ ***A1***

OR

let $x = t$ ***M1***

then $y = \dfrac{-5-t}{2}$ ***A1***

$z = \dfrac{1-t}{4}$ ***A1***

OR

let $z = t$ ***M1***

then $x = 1 - 4t$ ***A1***

$y = -3 + 2t$ ***A1***

OR

attempt to find cross product of two normal vectors:

eg: $\begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = -4i + 2j + k$ ***M1A1***

$x = 1 - 4t$

$y = -3 + 2t$

$z = t$ ***A1***

(or equivalent)

Total [5 marks]

4. (a) using the formulae for the sum and product of roots:

$\alpha + \beta = -2$ ***A1***

$\alpha\beta = -\frac{1}{2}$ ***A1***

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$ ***M1***

$= (-2)^2 - 2\left(-\frac{1}{2}\right)$

$= 5$ ***A1***

[4 marks]

Note: Award ***M0*** for attempt to solve quadratic equation.

(b) $(x - \alpha^2)(x - \beta^2) = x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2$ ***M1***

$x^2 - 5x + \left(-\frac{1}{2}\right)^2 = 0$ ***A1***

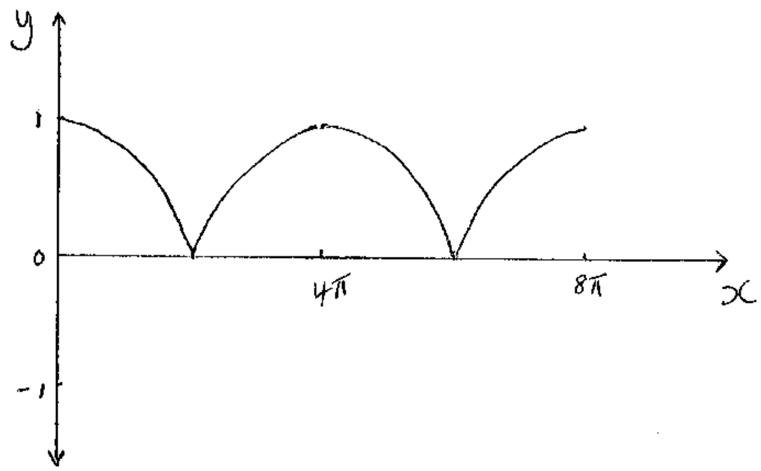
$x^2 - 5x + \frac{1}{4} = 0$

Note: Final answer must be an equation. Accept alternative correct forms.

[2 marks]

Total [6 marks]

5. (a)



A1A1

Note: Award ***A1*** for correct shape and ***A1*** for correct domain and range.

[2 marks]

continued…

Question 5 continued

(b) $\left|\cos\left(\frac{x}{4}\right)\right| = \frac{1}{2}$

$x = \frac{4\pi}{3}$ ***A1***

attempting to find any other solutions ***M1***

Note: Award ***(M1)*** if at least one of the other solutions is correct (in radians or degrees) or clear use of symmetry is seen.

$x = 8\pi - \frac{4\pi}{3} = \frac{20\pi}{3}$

$x = 4\pi - \frac{4\pi}{3} = \frac{8\pi}{3}$

$x = 4\pi + \frac{4\pi}{3} = \frac{16\pi}{3}$ ***A1***

Note: Award ***A1*** for all other three solutions correct and no extra solutions.

Note: If working in degrees, then max ***A0M1A0***.

[3 marks]

Total [5 marks]

6. (a) $\overrightarrow{PR} = \boldsymbol{a} + \boldsymbol{b}$ ***A1***

$\overrightarrow{QS} = \boldsymbol{b} - \boldsymbol{a}$ ***A1***

[2 marks]

(b) $\overrightarrow{PR} \cdot \overrightarrow{QS} = (\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{b} - \boldsymbol{a})$ ***M1***

$= |\boldsymbol{b}|^2 - |\boldsymbol{a}|^2$ ***A1***

for a rhombus $|\boldsymbol{a}| = |\boldsymbol{b}|$ ***R1***

hence $|\boldsymbol{b}|^2 - |\boldsymbol{a}|^2 = 0$ ***A1***

Note: Do not award the final ***A1*** unless ***R1*** is awarded.

hence the diagonals intersect at right angles ***AG***

[4 marks]

Total [6 marks]

7. (a) **METHOD 1**

$$\frac{1}{2+3\text{i}}+\frac{1}{3+2\text{i}}=\frac{2-3\text{i}}{4+9}+\frac{3-2\text{i}}{9+4}$$ ***M1A1***

$$\frac{10}{w}=\frac{5-5\text{i}}{13}$$ ***A1***

$$w=\frac{130}{5-5\text{i}}$$

$$=\frac{130\times5\times(1+\text{i})}{50}$$

$$w=13+13\text{i}$$ ***A1***

[4 marks]

METHOD 2

$$\frac{1}{2+3\text{i}}+\frac{1}{3+2\text{i}}=\frac{3+2\text{i}+2+3\text{i}}{(2+3\text{i})(3+2\text{i})}$$ ***M1A1***

$$\frac{10}{w}=\frac{5+5\text{i}}{13\text{i}}$$ ***A1***

$$\frac{w}{10}=\frac{13\text{i}}{5+5\text{i}}$$

$$w=\frac{130\text{i}}{(5+5\text{i})}\times\frac{(5-5\text{i})}{(5-5\text{i})}$$

$$=\frac{650+650\text{i}}{50}$$

$$=13+13\text{i}$$ ***A1***

[4 marks]

(b) $w^*=13-13\text{i}$ ***A1***

$$z=\sqrt{338}e^{-\frac{\pi}{4}\text{i}}\left(=13\sqrt{2}\,e^{-\frac{\pi}{4}\text{i}}\right)$$ ***A1A1***

Note: Accept $\theta=\frac{7\pi}{4}$. Do not accept answers for θ given in degrees.

[3 marks]

Total [7 marks]

8. (a) $1-2(2)=-3$ and $\frac{3}{4}(2-2)^2-3=-3$ ***A1***

both answers are the same, hence f is continuous (at $x=2$) ***R1***

> **Note:** ***R1*** may be awarded for justification using a graph or referring to limits. Do not award ***A0R1***.

[2 marks]

(b) reflection in the y-axis

$$f(-x)=\begin{cases} 1+2x, & x\geq -2 \\ \frac{3}{4}(x+2)^2-3, & x<-2 \end{cases}$$ ***(M1)***

> **Note:** Award ***M1*** for evidence of reflecting a graph in y-axis.

translation $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$g(x)=\begin{cases} 2x-3, & x\geq 0 \\ \frac{3}{4}x^2-3, & x<0 \end{cases}$$ ***(M1)A1A1***

> **Note:** Award ***(M1)*** for attempting to substitute $(x-2)$ for x, or translating a graph along positive x-axis.
> Award ***A1*** for the correct domains (this mark can be awarded independent of the ***M1***).
> Award ***A1*** for the correct expressions.

[4 marks]

Total [6 marks]

9. (a) $\sin x$, $\sin 2x$ and $4\sin x\cos^2 x$

$r = \frac{2\sin x\cos x}{\sin x} = 2\cos x$ ***A1***

Note: Accept $\frac{\sin 2x}{\sin x}$.

[1 mark]

(b) **EITHER**

$|r| < 1 \Rightarrow |2\cos x| < 1$ ***M1***

OR

$-1 < r < 1 \Rightarrow -1 < 2\cos x < 1$ ***M1***

THEN

$0 < \cos x < \frac{1}{2}$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$-\frac{\pi}{2} < x < -\frac{\pi}{3}$ or $\frac{\pi}{3} < x < \frac{\pi}{2}$ ***A1A1***

[3 marks]

(c) $S_\infty = \frac{\sin x}{1 - 2\cos x}$ ***M1***

$$S_\infty = \frac{\sin\left(\arccos\left(\frac{1}{4}\right)\right)}{1 - 2\cos\left(\arccos\left(\frac{1}{4}\right)\right)}$$

$= \frac{\frac{\sqrt{15}}{4}}{\frac{1}{2}}$ ***A1A1***

Note: Award ***A1*** for correct numerator and ***A1*** for correct denominator.

$= \frac{\sqrt{15}}{2}$ ***AG***

[3 marks]

Total [7marks]

10. $x = a\sec\theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$ ***(A1)***

new limits:

$x = a\sqrt{2} \Rightarrow \theta = \frac{\pi}{4}$ and $x = 2a \Rightarrow \theta = \frac{\pi}{3}$ ***(A1)***

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{a\sec\theta\tan\theta}{a^3\sec^3\theta\sqrt{a^2\sec^2\theta - a^2}}\,d\theta$ ***M1***

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos^2\theta}{a^3}\,d\theta$ ***A1***

using $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$ ***M1***

$\frac{1}{2a^3}\left[\frac{1}{2}\sin 2\theta + \theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ or equivalent ***A1***

$= \frac{1}{4a^3}\left(\frac{\sqrt{3}}{2} + \frac{2\pi}{3} - 1 - \frac{\pi}{2}\right)$ or equivalent ***A1***

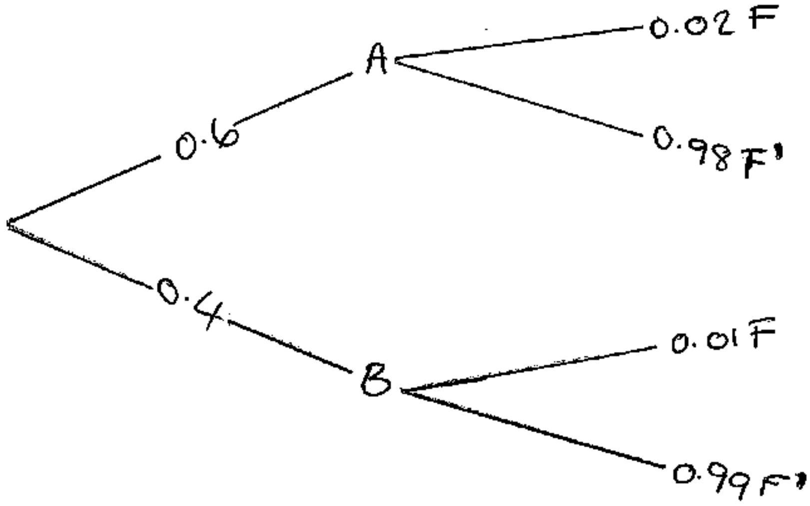
$= \frac{1}{24a^3}\left(3\sqrt{3} + \pi - 6\right)$ ***AG***

[7 marks]

Total [7 marks]

SECTION B

11. (a) (i)



A1A1

> **Note:** Award ***A1*** for a correctly labelled tree diagram and ***A1*** for correct probabilities.

(ii) $\mathrm{P}(F) = 0.6 \times 0.02 + 0.4 \times 0.01$ ***(M1)***

$= 0.016$ ***A1***

(iii) $\mathrm{P}(A \mid F) = \dfrac{\mathrm{P}(A \cap F)}{\mathrm{P}(F)}$

$= \dfrac{0.6 \times 0.02}{0.016} \left(= \dfrac{0.012}{0.016} \right)$ ***M1***

$= 0.75$ ***A1***

[6 marks]

continued…

Question 11 continued

(b) (i) **METHOD 1**

$\mathrm{P}(X=2)=\dfrac{{}^{3}C_{2}\times{}^{4}C_{1}}{{}^{7}C_{3}}$ ***(M1)***

$=\dfrac{12}{35}$ ***A1***

METHOD 2

$\dfrac{3}{7}\times\dfrac{2}{6}\times\dfrac{4}{5}\times 3$ ***(M1)***

$=\dfrac{12}{35}$ ***A1***

(ii)

x	0	1	2	3
$\mathrm{P}(X=x)$	$\dfrac{4}{35}$	$\dfrac{18}{35}$	$\dfrac{12}{35}$	$\dfrac{1}{35}$

A2

Note: Award ***A1*** if $\dfrac{4}{35}$, $\dfrac{18}{35}$ or $\dfrac{1}{35}$ is obtained.

(iii) $\mathrm{E}(X)=\sum x\,\mathrm{P}(X=x)$

$\mathrm{E}(X)=0\times\dfrac{4}{35}+1\times\dfrac{18}{35}+2\times\dfrac{12}{35}+3\times\dfrac{1}{35}$ ***M1***

$=\dfrac{45}{35}=\left(\dfrac{9}{7}\right)$ ***A1***

[6 marks]

Total [12 marks]

12. (a) direction vector $\vec{AB} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$ or $\vec{BA} = \begin{pmatrix} -1 \\ -3 \\ 5 \end{pmatrix}$ ***A1***

$\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$ or $\boldsymbol{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$ or equivalent ***A1***

> **Note:** Do not award final ***A1*** unless '$r = \ldots$' (or equivalent) seen.
> Allow FT on direction vector for final ***A1***.

[2 marks]

(b) both lines expressed in parametric form:

L_1:
$x = 1 + t$
$y = 3t$
$z = 4 - 5t$

L_2:
$x = 1 + 3s$
$y = -2 + s$ ***M1A1***
$z = -2s + 1$

> **Notes:** Award ***M1*** for an attempt to convert L_2 from Cartesian to parametric form.
> Award ***A1*** for correct parametric equations for L_1 and L_2.
> Allow ***M1A1*** at this stage if same parameter is used in both lines.

attempt to solve simultaneously for x and y: ***M1***
$1 + t = 1 + 3s$
$3t = -2 + s$
$t = -\frac{3}{4}$, $s = -\frac{1}{4}$ ***A1***

substituting both values back into z values respectively gives $z = \frac{31}{4}$

and $z = \frac{3}{2}$ so a contradiction ***R1***

therefore L_1 and L_2 are skew lines ***AG***

[5 marks]

continued…

Question 12 continued

(c) finding the cross product:

$\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ ***(M1)***

$= -\boldsymbol{i} - 13\boldsymbol{j} - 8\boldsymbol{k}$ ***A1***

Note: Accept $\boldsymbol{i} + 13\boldsymbol{j} + 8\boldsymbol{k}$

$-1(0) - 13(1) - 8(-2) = 3$ ***(M1)***

$\Rightarrow -x - 13y - 8z = 3$ or equivalent ***A1***

[4 marks]

(d) (i) $(\cos\theta =)\dfrac{\begin{pmatrix} k \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}{\sqrt{k^2+1+1} \times \sqrt{1+1}}$ ***M1***

Note: Award ***M1*** for an attempt to use angle between two vectors formula.

$\dfrac{\sqrt{3}}{2} = \dfrac{k+1}{\sqrt{2(k^2+2)}}$ ***A1***

obtaining the quadratic equation

$4(k+1)^2 = 6(k^2+2)$ ***M1***

$k^2 - 4k + 4 = 0$

$(k-2)^2 = 0$

$k = 2$ ***A1***

Note: Award ***M1A0M1A0*** if $\cos 60^{\circ}$ is used $(k = 0$ or $k = -4)$.

continued…

Question 12 continued

(ii) $\boldsymbol{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

substituting into the equation of the plane Π_2:

$3 + 2\lambda + \lambda = 12$ ***M1***

$\lambda = 3$ ***A1***

point P has the coordinates:

$(9, 3, -2)$ ***A1***

> **Notes:** Accept $9\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$ and $\begin{pmatrix} 9 \\ 3 \\ -2 \end{pmatrix}$.
>
> Do not allow FT if two values found for *k*.

[7 marks]

Total [18 marks]

13. (a) $f'(x) = \dfrac{(x^2+1) - 2x(x+1)}{(x^2+1)^2} \left(= \dfrac{-x^2 - 2x + 1}{(x^2+1)^2} \right)$ ***M1A1***

[2 marks]

(b) $\dfrac{-x^2 - 2x + 1}{(x^2+1)^2} = 0$

$x = -1 \pm \sqrt{2}$ ***A1***

[1 mark]

continued…

Question 13 continued

(c) $f''(x)=\dfrac{(-2x-2)(x^2+1)^2-2(2x)(x^2+1)(-x^2-2x+1)}{\left(x^2+1\right)^4}$ ***A1A1***

Note: Award ***A1*** for $(-2x-2)\left(x^2+1\right)^2$ or equivalent.

Note: Award ***A1*** for $-2(2x)\left(x^2+1\right)\left(-x^2-2x+1\right)$ or equivalent.

$=\dfrac{(-2x-2)\left(x^2+1\right)-4x\left(-x^2-2x+1\right)}{\left(x^2+1\right)^3}$

$=\dfrac{2x^3+6x^2-6x-2}{\left(x^2+1\right)^3}$ ***A1***

$\left(=\dfrac{2\left(x^3+3x^2-3x-1\right)}{\left(x^2+1\right)^3}\right)$

[3 marks]

(d) recognition that $(x-1)$ is a factor ***(R1)***

$(x-1)\left(x^2+bx+c\right)=\left(x^3+3x^2-3x-1\right)$ ***M1***

$\Rightarrow x^2+4x+1=0$ ***A1***

$x=-2\pm\sqrt{3}$ ***A1***

Note: Allow long division / synthetic division.

[4 marks]

(e) $\int_{-1}^{0}\dfrac{x+1}{x^2+1}\mathrm{d}x$ ***M1***

$\int\dfrac{x+1}{x^2+1}\mathrm{d}x=\int\dfrac{x}{x^2+1}\mathrm{d}x+\int\dfrac{1}{x^2+1}\mathrm{d}x$ ***M1***

$=\dfrac{1}{2}\ln\left(x^2+1\right)+\arctan(x)$ ***A1A1***

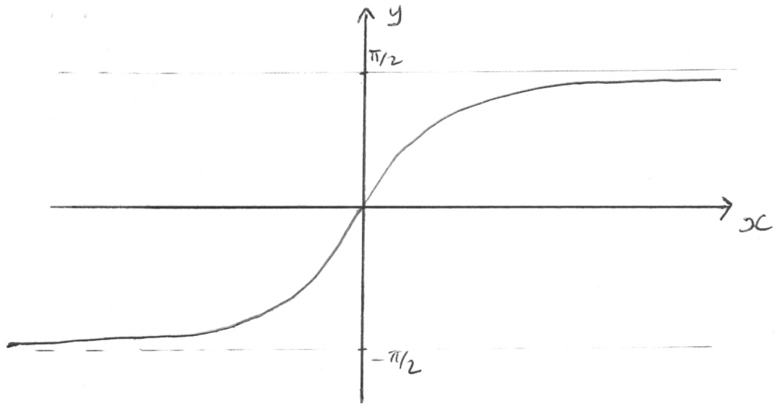
$=\left[\dfrac{1}{2}\ln\left(x^2+1\right)+\arctan(x)\right]_{-1}^{0}=\dfrac{1}{2}\ln 1+\arctan 0-\dfrac{1}{2}\ln 2-\arctan(-1)$ ***M1***

$=\dfrac{\pi}{4}-\ln\sqrt{2}$ ***A1***

[6 marks]

Total [16 marks]

14. (a)



A1A1

> **Note:** ***A1*** for correct shape, ***A1*** for asymptotic behaviour at $y = \pm\frac{\pi}{2}$.

[2 marks]

(b) $h \circ g(x) = \arctan\left(\frac{1}{x}\right)$ ***A1***

domain of $h \circ g$ is equal to the domain of g : $x \in \mathbb{R}$, $x \neq 0$ ***A1***

[2 marks]

(c) (i) $f(x) = \arctan(x) + \arctan\left(\frac{1}{x}\right)$

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \times -\frac{1}{x^2}$$ ***M1A1***

$$f'(x) = \frac{1}{1+x^2} + \frac{-\frac{1}{x^2}}{\frac{x^2+1}{x^2}}$$ ***(A1)***

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2}$$

$$= 0$$ ***A1***

continued…

Question 14 continued

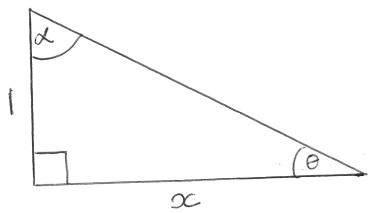
(ii) **METHOD 1**

f is a constant **R1**
when $x > 0$

$f(1) = \frac{\pi}{4} + \frac{\pi}{4}$ **M1A1**

$= \frac{\pi}{2}$ **AG**

METHOD 2



from diagram

$\theta = \arctan \frac{1}{x}$ **A1**

$\alpha = \arctan x$ **A1**

$\theta + \alpha = \frac{\pi}{2}$ **R1**

hence $f(x) = \frac{\pi}{2}$ **AG**

METHOD 3

$\tan(f(x)) = \tan\left(\arctan(x) + \arctan\left(\frac{1}{x}\right)\right)$ **M1**

$= \frac{x + \frac{1}{x}}{1 - x\left(\frac{1}{x}\right)}$ **A1**

denominator $= 0$, so $f(x) = \frac{\pi}{2}$ (for $x > 0$) **R1**

[7 marks]

continued…

Question 14 continued

(d) (i) Nigel is correct. ***A1***

METHOD 1

$\arctan(x)$ is an odd function and $\frac{1}{x}$ is an odd function

composition of two odd functions is an odd function and sum of two odd functions is an odd function ***R1***

METHOD 2

$f(-x) = \arctan(-x) + \arctan\left(-\frac{1}{x}\right) = -\arctan(x) - \arctan\left(\frac{1}{x}\right) = -f(x)$

therefore f is an odd function. ***R1***

(ii) $f(x) = -\frac{\pi}{2}$ ***A1***

[3 marks]

Total [14 marks]