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MARKSCHEME

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MATHEMATICS

Higher Level

Paper 2

22 pages

SECTION A

1. (a) (i) $n = 27$ ***(A1)***

METHOD 1

$S_{27} = \frac{14+196}{2} \times 27$ ***(M1)***

$= 2835$ ***A1***

METHOD 2

$S_{27} = \frac{27}{2}(2\times 14 + 26\times 7)$ ***(M1)***

$= 2835$ ***A1***

METHOD 3

$S_{27} = \sum_{n=1}^{27} 7 + 7n$ ***(M1)***

$= 2835$ ***A1***

(ii) $\sum_{n=1}^{27}(7+7n)$ or equivalent ***A1***

Note: Accept $\sum_{n=2}^{28} 7n$

[4 marks]

(b) $\frac{n}{2}\left(2000 - 6(n-1)\right) < 0$ ***(M1)***

$n > 334.333$

$n = 335$ ***A1***

Note: Accept working with equalities.

[2 marks]

Total [6 marks]

2. (a) **METHOD 1**

$\mu = \frac{1}{2} \times (17.1 + 21.3)$ ***(M1)***

$\mu = 19.2$ (kg) ***A1***

finding z value for the upper quartile $= 0.674489\text{K}$

$0.674489\text{K} = \dfrac{21.3 - 19.2}{\sigma}$ or $-0.674489\text{K} = \dfrac{17.1 - 19.2}{\sigma}$ ***M1***

$\sigma = 3.11$ (kg) ***A1***

METHOD 2

finding z value for the upper quartile $= 0.674489\text{K}$

from symmetry the z value for a lower quartile is -0.674489K ***M1***

forming two simultaneous equations:

$-0.674489\text{K} = \dfrac{17.1 - \mu}{\sigma}$

$0.674489\text{K} = \dfrac{21.3 - \mu}{\sigma}$ ***M1***

solving gives:

$\mu = 19.2$ (kg) ***A1***

$\sigma = 3.11$ (kg) ***A1***

[4 marks]

(b) using $100 \times \mathrm{P}(X > 22) = 100 \times 0.184241\text{K}$

$= 18$ ***A1***

Note: Accept 18.4

[1 mark]

Total [5 marks]

3. (a) $x_{\mathrm{A}} = 2.87$ ***A1***

$x_{\mathrm{B}} = 6.78$ ***A1***

[2 marks]

(b) $\int_{2.87172\mathrm{K}}^{6.77681\mathrm{K}} 1 - 2\sin x - x^2 \mathrm{e}^{-x}\, \mathrm{d}x$ ***(M1)(A1)***

$= 6.76$ ***A1***

Note: Award ***(M1)*** for definite integral and ***(A1)*** for a correct definite integral.

[3 marks]

Total [5 marks]

4. (a) **METHOD 1**

$2\arcsin\left(\frac{1.5}{4}\right)$ ***M1***

$\alpha = 0.769^{c}\ (44.0^{\circ})$ ***A1***

METHOD 2

using the cosine rule:

$3^2 = 4^2 + 4^2 - 2(4)(4)\cos\alpha$ ***M1***

$\alpha = 0.769^{c}\ (44.0^{\circ})$ ***A1***

[2 marks]

(b) one segment

$\mathrm{A}_1 = \frac{1}{2} \times 4^2 \times 0.76879 - \frac{1}{2} \times 4^2 \times \sin(0.76879)$ ***M1A1***

$= 0.58819\mathrm{K}$ ***(A1)***

$2\mathrm{A}_1 = 1.18\ (\mathrm{cm}^2)$ ***A1***

Note: Award ***M1*** only if both sector and triangle are considered.

[4 marks]

Total [6 marks]

5. expanding $(x-1)^3 = x^3 - 3x^2 + 3x - 1$ ***A1***

expanding $\left(\frac{1}{x}+2x\right)^6$ gives

$64x^6 + 192x^4 + 240x^2 + \frac{60}{x^2} + \frac{12}{x^4} + \frac{1}{x^6} + 160$ ***(M1)A1A1***

Note:	Award ***(M1)*** for an attempt at expanding using binomial. Award ***A1*** for $\frac{60}{x^2}$. Award ***A1*** for $\frac{12}{x^4}$.

$\frac{60}{x^2} \times -1 + \frac{12}{x^4} \times -3x^2$ ***(M1)***

Note: Award ***(M1)*** only if both terms are considered.

therefore coefficient x^{-2} is -96 ***A1***

Note: Accept $-96x^{-2}$

Note: Award full marks if working with the required terms only without giving the entire expansion.

[6 marks]

6. (a) (i) $0.6^3 \times 0.4^3$ ***(M1)***

Note: Award ***(M1)*** for use of the product of probabilities.

$= 0.0138$ ***A1***

(ii) binomial distribution $X : \mathrm{B}(6, 0.6)$ ***(M1)***

Note: Award ***(M1)*** for recognizing the binomial distribution.

$\mathrm{P}(X = 3) = {}^6C_3\,(0.6)^3\,(0.4)^3$
$= 0.276$ ***A1***

Note: Award ***(M1)A1*** for ${}^6C_3 \times 0.0138 = 0.276$.

[4 marks]

continued…

Question 6 continued

(b) $Y \sim \mathrm{B}(n, 0.4)$
$\mathrm{P}(Y \geq 1) > 0.995$
$1 - \mathrm{P}(Y = 0) > 0.995$
$\mathrm{P}(Y = 0) < 0.005$ ***(M1)***

Note:	Award ***(M1)*** for any of the last three lines. Accept equalities.

$0.6^n < 0.005$ ***(M1)***

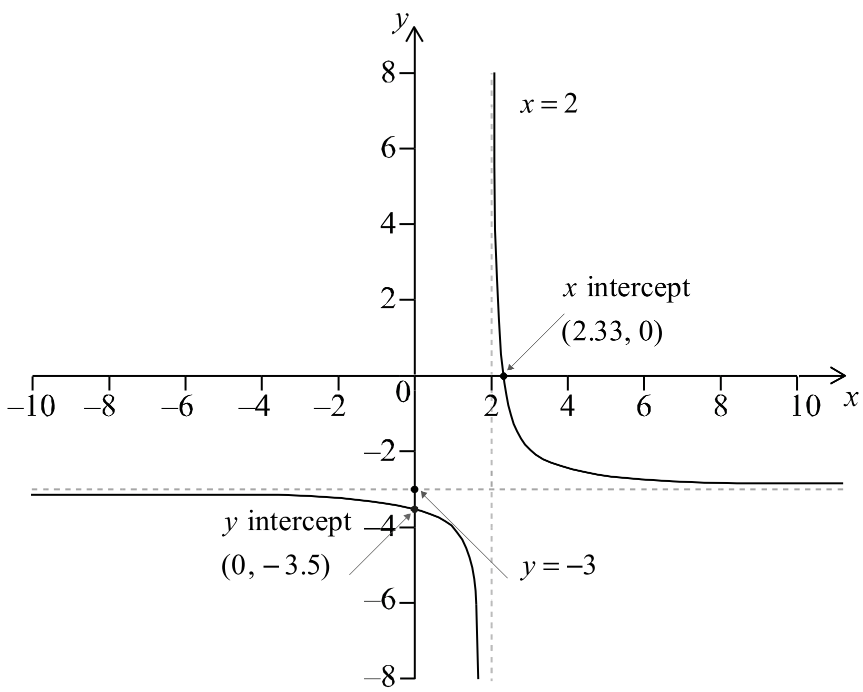
Note:	Award ***(M1)*** for attempting to solve $0.6^n < 0.005$ using any method, *eg*, logs, graphically, use of solver. Accept an equality.

$n > 10.4$
$\therefore n = 11$ ***A1***

[3 marks]

Total [7 marks]

7. (a)



A1A1A1

Note:	Award ***A1*** for correct shape, ***A1*** for $x = 2$ clearly stated and ***A1*** for $y = -3$ clearly stated.

x intercept $(2.33, 0)$ and y intercept $(0, -3.5)$ ***A1***

Note:	Accept –3.5 and 2.33 (7/3) marked on the correct axes.

[4 marks]

continued…

Question 7 continued

(b) $x = -3 + \dfrac{1}{y-2}$ ***M1***

Note: Award ***M1*** for interchanging *x* and *y* (can be done at a later stage).

$x + 3 = \dfrac{1}{y-2}$

$y - 2 = \dfrac{1}{x+3}$ ***M1***

Note: Award ***M1*** for attempting to make *y* the subject.

$f^{-1}(x) = 2 + \dfrac{1}{x+3}\left(= \dfrac{2x+7}{x+3}\right),\ x \neq -3$ ***A1A1***

Note: Award ***A1*** only if $f^{-1}(x)$ is seen. Award ***A1*** for the domain.

[4 marks]

Total [8 marks]

8. (a) $\dfrac{\mu^2 e^{-\mu}}{2!} + \dfrac{\mu^3 e^{-\mu}}{3!} = \dfrac{\mu^5 e^{-\mu}}{5!}$ ***(M1)***

$\dfrac{\mu^2}{2} + \dfrac{\mu^3}{6} - \dfrac{\mu^5}{120} = 0$

$\mu = 5.55$ ***A1***

[2 marks]

(b) $\sigma = \sqrt{5.55...} = 2.35598...$ ***(M1)***

$\mathrm{P}(3.19 \le X \le 7.9)$

$\mathrm{P}(4 \le X \le 7)$

$= 0.607$ ***A1***

[2 marks]

Total [4 marks]

9. **METHOD 1**

volume of a cone is $V = \frac{1}{3}\pi r^2 h$

given $h = r$, $V = \frac{1}{3}\pi h^3$ ***M1***

$\frac{\mathrm{d}V}{\mathrm{d}h} = \pi h^2$ ***(A1)***

when $h = 4$, $\frac{\mathrm{d}V}{\mathrm{d}t} = \pi \times 4^2 \times 0.5$ (using $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}h} \times \frac{\mathrm{d}h}{\mathrm{d}t}$) ***M1A1***

$\frac{\mathrm{d}V}{\mathrm{d}t} = 8\pi$ $(= 25.1)$ $(\text{cm}^3\,\text{min}^{-1})$ ***A1***

METHOD 2

volume of a cone is $V = \frac{1}{3}\pi r^2 h$

given $h = r$, $V = \frac{1}{3}\pi h^3$ ***M1***

$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{3}\pi \times 3h^2 \times \frac{\mathrm{d}h}{\mathrm{d}t}$ ***A1***

when $h = 4$, $\frac{\mathrm{d}V}{\mathrm{d}t} = \pi \times 4^2 \times 0.5$ ***M1A1***

$\frac{\mathrm{d}V}{\mathrm{d}t} = 8\pi$ $(= 25.1)$ $(\text{cm}^3\,\text{min}^{-1})$ ***A1***

METHOD 3

$V = \frac{1}{3}\pi r^2 h$

$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{3}\pi\left(2rh\frac{\mathrm{d}r}{\mathrm{d}t} + r^2\frac{\mathrm{d}h}{\mathrm{d}t}\right)$ ***M1A1***

Note: Award ***M1*** for attempted implicit differentiation and ***A1*** for each correct term on the RHS.

when $h = 4,\ r = 4,\ \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{3}\pi\left(2 \times 4 \times 4 \times 0.5 + 4^2 \times 0.5\right)$ ***M1A1***

$\frac{\mathrm{d}V}{\mathrm{d}t} = 8\pi$ $(= 25.1)$ $(\text{cm}^3\,\text{min}^{-1})$ ***A1***

[5 marks]

10. (a) **METHOD 1**

expanding the brackets first:

$x^4 + 2x^2y^2 + y^4 = 4xy^2$ ***M1***

$4x^3 + 4xy^2 + 4x^2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 4y^2 + 8xy\frac{dy}{dx}$ ***M1A1A1***

Note:	Award ***M1*** for an attempt at implicit differentiation. Award ***A1*** for each side correct.

$\frac{dy}{dx} = \frac{-x^3 - xy^2 + y^2}{yx^2 - 2xy + y^3}$ or equivalent ***A1***

METHOD 2

$2\left(x^2 + y^2\right)\left(2x + 2y\frac{dy}{dx}\right) = 4y^2 + 8xy\frac{dy}{dx}$ ***M1A1A1***

Note:	Award ***M1*** for an attempt at implicit differentiation. Award ***A1*** for each side correct.

$\left(x^2 + y^2\right)\left(x + y\frac{dy}{dx}\right) = y^2 + 2xy\frac{dy}{dx}$

$x^3 + x^2y\frac{dy}{dx} + y^2x + y^3\frac{dy}{dx} = y^2 + 2xy\frac{dy}{dx}$ ***M1***

$\frac{dy}{dx} = \frac{-x^3 - xy^2 + y^2}{yx^2 - 2xy + y^3}$ or equivalent ***A1***

[5 marks]

(b) **METHOD 1**

at $(1, 1)$, $\frac{dy}{dx}$ is undefined ***M1A1***

$y = 1$ ***A1***

METHOD 2

gradient of normal $= -\frac{1}{\frac{dy}{dx}} = -\frac{\left(yx^2 - 2xy + y^3\right)}{\left(-x^3 - xy^2 + y^2\right)}$ ***M1***

at $(1, 1)$ gradient $= 0$ ***A1***

$y = 1$ ***A1***

[3 marks]

Total [8 marks]

SECTION B

11. (a) $a\int_0^{\frac{\pi}{2}} x\cos x\,\mathrm{d}x = 1$ ***(M1)***

integrating by parts:

$u = x \qquad v' = \cos x$

$u' = 1 \qquad v = \sin x$ ***M1***

$\int x\cos x\,\mathrm{d}x = x\sin x + \cos x$ ***A1***

$[x\sin x + \cos x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1$ ***A1***

$a = \frac{1}{\frac{\pi}{2} - 1}$ ***A1***

$= \frac{2}{\pi - 2}$ ***AG***

[5 marks]

(b) $\mathrm{P}\left(X < \frac{\pi}{4}\right) = \frac{2}{\pi - 2}\int_0^{\frac{\pi}{4}} x\cos x\,\mathrm{d}x = 0.460$ ***(M1)A1***

Note: Accept $\frac{2}{\pi-2}\left(=\frac{\pi\sqrt{2}}{8}+\frac{\sqrt{2}}{2}-1\right)$ or equivalent

[2 marks]

(c) (i) mode = 0.860 ***A1***

(x-value of a maximum on the graph over the given domain)

(ii) $\frac{2}{\pi - 2}\int_0^{m} x\cos x\,\mathrm{d}x = 0.5$ ***(M1)***

$\int_0^{m} x\cos x\,\mathrm{d}x = \frac{\pi - 2}{4}$

$m\sin m + \cos m - 1 = \frac{\pi - 2}{4}$ ***(M1)***

median $= 0.826$ ***A1***

Note: Do not accept answers containing additional solutions.

[4 marks]

continued…

Question 11 continued

(d) $\mathrm{P}\left(X < \frac{\pi}{8} \mid X < \frac{\pi}{4}\right) = \frac{\mathrm{P}\left(X < \frac{\pi}{8}\right)}{\mathrm{P}\left(X < \frac{\pi}{4}\right)}$ ***M1***

$= \frac{0.129912}{0.459826}$

$= 0.283$ ***A1***

[2 marks]

Total [13 marks]

12. (a) $C = \mathrm{AX} \times 5k + \mathrm{XB} \times k$ **(*M1*)**

Note: Award **(*M1*)** for attempting to express the cost in terms of AX, XB and *k*.

$= 5k\sqrt{450^2 + x^2} + (1000 - x)k$ ***A1***

$= 5k\sqrt{202500 + x^2} + (1000 - x)k$ ***AG***

[2 marks]

(b) (i) $\frac{\mathrm{d}C}{\mathrm{d}x} = k\left[\frac{5 \times 2x}{2\sqrt{202500 + x^2}} - 1\right] = k\left(\frac{5x}{\sqrt{202500 + x^2}} - 1\right)$ ***M1A1***

Note: Award ***M1*** for an attempt to differentiate and ***A1*** for the correct derivative.

continued…

Question 12 continued

(ii) attempting to solve $\frac{\mathrm{d}C}{\mathrm{d}x} = 0$ ***M1***

$\frac{5x}{\sqrt{202500 + x^2}} = 1$ ***(A1)***

$x = 91.9\text{ (m)}\left(= \frac{75\sqrt{6}}{2}\text{ (m)}\right)$ ***A1***

METHOD 1

for example,

at $x = 91$ $\frac{\mathrm{d}C}{\mathrm{d}x} = -0.00895k < 0$ ***M1***

at $x = 92$ $\frac{\mathrm{d}C}{\mathrm{d}x} = 0.001506k > 0$ ***A1***

Note: Award ***M1*** for attempting to find the gradient either side of $x = 91.9$ and ***A1*** for two correct values.

thus $x = 91.9$ gives a minimum ***AG***

METHOD 2

$$\frac{\mathrm{d}^2C}{\mathrm{d}x^2} = \frac{1012500k}{\left(x^2 + 202500\right)^{\frac{3}{2}}}$$

at $x = 91.9$ $\frac{\mathrm{d}^2C}{\mathrm{d}x^2} = 0.010451k > 0$ ***(M1)A1***

Note: Award ***M1*** for attempting to find the second derivative and ***A1*** for the correct value.

Note: If $\frac{\mathrm{d}^2C}{\mathrm{d}x^2}$ is obtained and its value at $x = 91.9$ is not calculated, award ***(M1)A1*** for correct reasoning *eg*, both numerator and denominator are positive at $x = 91.9$.

thus $x = 91.9$ gives a minimum ***AG***

METHOD 3

Sketching the graph of either C versus x or $\frac{\mathrm{d}C}{\mathrm{d}x}$ versus x. ***M1***

Clearly indicating that $x = 91.9$ gives the minimum on their graph. ***A1***

[7 marks]

continued…

Question 12 continued

(c) $C_{\min} = 3205k$ ***A1***

Note:	Accept 3200*k*. Accept 3204 *k*.

[1 mark]

(d) $\arctan\left(\dfrac{450}{91.855865\text{K}}\right) = 78.463\text{K}^{\circ}$ ***M1***

$180 - 78.463\text{K} = 101.537\text{K}$

$= 102^{\circ}$ ***A1***

[2 marks]

(e) (i) when $\theta = 120^{\circ}$, $x = 260\,(\text{m})\ \left(\dfrac{450}{\sqrt{3}}(\text{m})\right)$ ***A1***

(ii) $\dfrac{133.728\text{K}}{3204.5407685\text{K}} \times 100\%$ ***M1***

$= 4.17\,(\%)$ ***A1***

[3 marks]

Total [15 marks]

13. (a) let P(n) be the proposition $z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$, $n \in ¢^+$
let $n = 1 \Rightarrow$
LHS $= r(\cos\theta + \mathrm{i}\sin\theta)$
RHS $= r(\cos\theta + \mathrm{i}\sin\theta)$, $\therefore$ P(1) is true ***R1***
assume true for $n = k \Rightarrow r^k(\cos\theta + \mathrm{i}\sin\theta)^k = r^k\left(\cos(k\theta) + \mathrm{i}\sin(k\theta)\right)$ ***M1***

Note: Only award the ***M1*** if truth is assumed.

now show $n = k$ true implies $n = k + 1$ also true
$r^{k+1}(\cos\theta + \mathrm{i}\sin\theta)^{k+1} = r^{k+1}(\cos\theta + \mathrm{i}\sin\theta)^k(\cos\theta + \mathrm{i}\sin\theta)$ ***M1***
$= r^{k+1}\left(\cos(k\theta) + \mathrm{i}\sin(k\theta)\right)(\cos\theta + \mathrm{i}\sin\theta)$
$= r^{k+1}\left(\cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta + \mathrm{i}\left(\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta\right)\right)$ ***A1***
$= r^{k+1}\left(\cos(k\theta + \theta) + \mathrm{i}\sin(k\theta + \theta)\right)$ ***A1***
$= r^{k+1}\left(\cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta\right) \Rightarrow n = k + 1$ is true ***A1***
P(k) true implies P($k+1$) true and P(1) is true, therefore by mathematical induction statement is true for $n \geq 1$ ***R1***

Note: Only award the final ***R1*** if the first 4 marks have been awarded.

[7 marks]

(b) (i) $u = 2\mathrm{cis}\left(\frac{\pi}{3}\right)$ ***A1***

$v = \sqrt{2}\mathrm{cis}\left(-\frac{\pi}{4}\right)$ ***A1***

Notes: Accept 3 sf answers only. Accept equivalent forms. Accept $2e^{\frac{\pi}{3}i}$ and $\sqrt{2}\,e^{-\frac{\pi}{4}i}$.

(ii) $u^3 = 2^3\mathrm{cis}(\pi) = -8$
$v^4 = 4\mathrm{cis}(-\pi) = -4$ ***(M1)***
$u^3v^4 = 32$ ***A1***

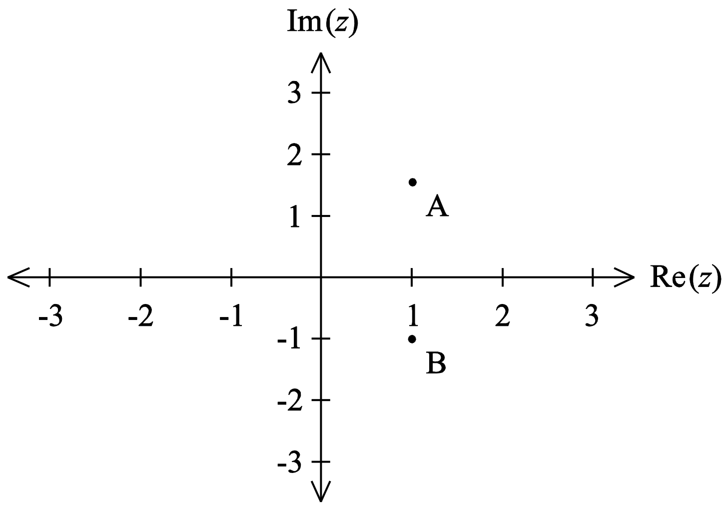
Notes: Award ***(M1)*** for an attempt to find u^3 and v^4. Accept equivalent forms.

[4 marks]

continued...

Question 13 continued

(c)



A1

Note: Award **A1** if A or $1+\sqrt{3}i$ and B or $1-i$ are in their correct quadrants, are aligned vertically and it is clear that $|u|>|v|$.

[1 mark]

(d) $\text{Area}=\frac{1}{2}\times 2\times\sqrt{2}\times\sin\left(\frac{5\pi}{12}\right)$ ***M1A1***

$=1.37\left(=\frac{\sqrt{2}}{4}\left(\sqrt{6}+\sqrt{2}\right)\right)$ ***A1***

Notes: Award ***M1A0A0*** for using $\frac{7\pi}{12}$.

[3 marks]

(e) $(z-1+\text{i})(z-1-\text{i})=z^2-2z+2$ ***M1A1***

Note: Award ***M1*** for recognition that a complex conjugate is also a root.

$\left(z-1-\sqrt{3}\text{i}\right)\left(z-1+\sqrt{3}\text{i}\right)=z^2-2z+4$ ***A1***

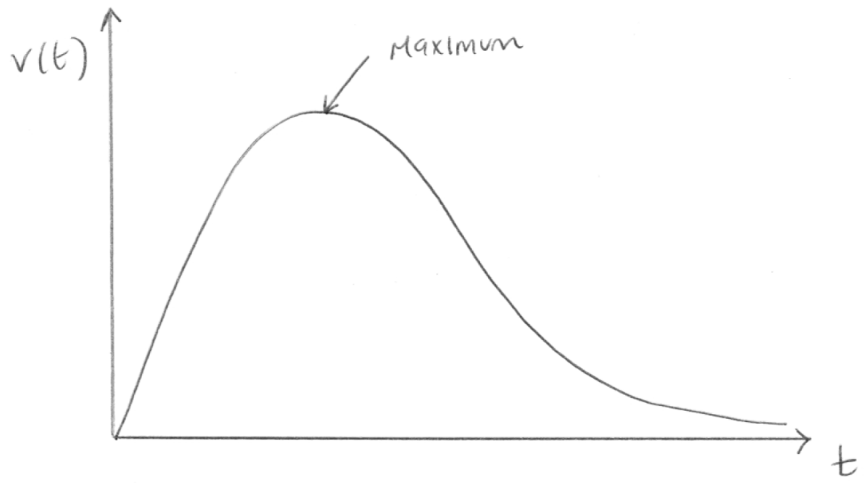
$\left(z^2-2z+2\right)\left(z^2-2z+4\right)=z^4-4z^3+10z^2-12z+8$ ***M1A1***

Note: Award ***M1*** for an attempt to expand two quadratics.

[5 marks]

Total [20 marks]

14. (a)



A1

A1 for correct shape and correct domain

(1.41, 0.0884) $\left(\sqrt{2}, \frac{\sqrt{2}}{16}\right)$ *A1*

[2 marks]

(b) **EITHER**

$u = t^2$

$\frac{\mathrm{d}u}{\mathrm{d}t} = 2t$ *A1*

OR

$t = u^{\frac{1}{2}}$

$\frac{\mathrm{d}t}{\mathrm{d}u} = \frac{1}{2}u^{-\frac{1}{2}}$ *A1*

THEN

$\int \frac{t}{12+t^4}\,\mathrm{d}t = \frac{1}{2}\int \frac{\mathrm{d}u}{12+u^2}$ *M1*

$= \frac{1}{2\sqrt{12}}\arctan\left(\frac{u}{\sqrt{12}}\right)(+c)$ *M1*

$= \frac{1}{4\sqrt{3}}\arctan\left(\frac{t^2}{2\sqrt{3}}\right)(+c)$ or equivalent *A1*

[4 marks]

continued…

Question 14 continued

(c) $\int_0^6 \frac{t}{12+t^4}\,\mathrm{d}t$ ***(M1)***

$= \left[\frac{1}{4\sqrt{3}}\arctan\left(\frac{t^2}{2\sqrt{3}}\right)\right]_0^6$ ***M1***

$= \frac{1}{4\sqrt{3}}\left(\arctan\left(\frac{36}{2\sqrt{3}}\right)\right)\left(= \frac{1}{4\sqrt{3}}\left(\arctan\left(\frac{18}{\sqrt{3}}\right)\right)\right)(\text{m})$ ***A1***

Note: Accept $\frac{\sqrt{3}}{12}\arctan\left(6\sqrt{3}\right)$ or equivalent.

[3 marks]

(d) $\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{1}{2\sqrt{s(1-s)}}$ ***(A1)***

$a = v\frac{\mathrm{d}v}{\mathrm{d}s}$

$a = \arcsin\left(\sqrt{s}\right)\times\frac{1}{2\sqrt{s(1-s)}}$ ***(M1)***

$a = \arcsin\left(\sqrt{0.1}\right)\times\frac{1}{2\sqrt{0.1\times0.9}}$

$a = 0.536\ \left(\text{ms}^{-2}\right)$ ***A1***

[3 marks]

Total [12 marks]