International Baccalaureate®
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MARKSCHEME

November 2013

MATHEMATICS

Higher Level

Paper 1

SECTION A

1. $f(-2)=0 \ (\Rightarrow -24+4p-2q-2=0)$ ***M1***
$f(-1)=4 \ (\Rightarrow -3+p-q-2=4)$ ***M1***

Note: In each case award the ***M*** marks if correct substitution attempted and right-hand side correct.

attempt to solve simultaneously ($2p-q=13$, $p-q=9$) ***M1***
$p=4$ ***A1***
$q=-5$ ***A1***

Total [5 marks]

2. (a) $\frac{1}{6}+\frac{1}{2}+\frac{3}{10}+a=1 \Rightarrow a=\frac{1}{30}$ ***A1***

[1 mark]

(b) $E(X)=\frac{1}{2}+2\times\frac{3}{10}+3\times\frac{1}{30}$ ***M1***
$=\frac{6}{5}$ ***A1***

Note: Do not award ***FT*** marks if a is outside $[0, 1]$.

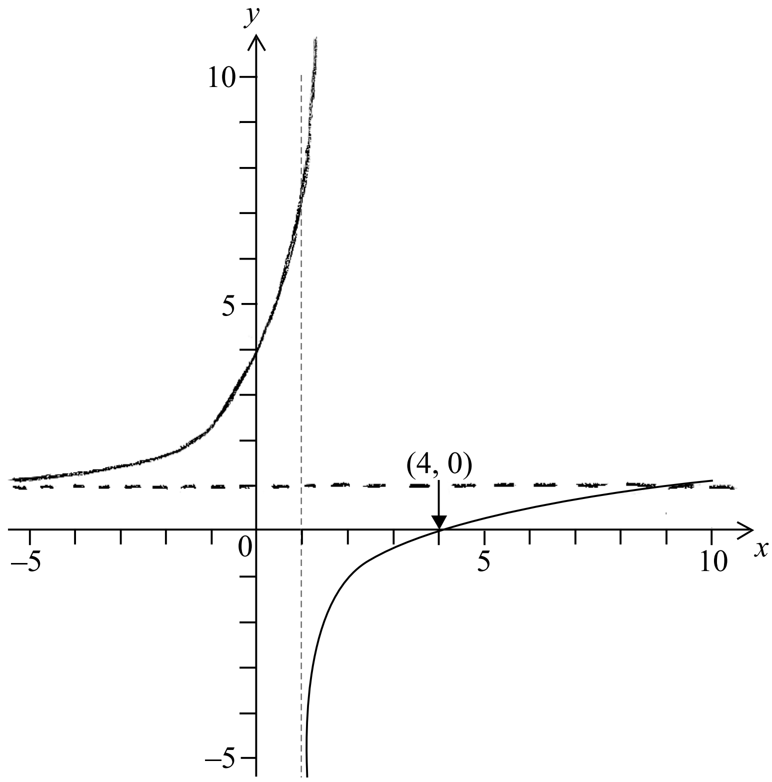
[2 marks]

(c) $\mathrm{E}(X^2)=\frac{1}{2}+2^2\times\frac{3}{10}+3^2\times\frac{1}{30}=2$ ***(A1)***
attempt to apply $\mathrm{Var}(X)=\mathrm{E}\left(X^2\right)-\left(\mathrm{E}(X)\right)^2$ ***M1***
$\left(=2-\frac{36}{25}\right)=\frac{14}{25}$ ***A1***

[3 marks]

Total [6 marks]

3. (a)



shape with y-axis intercept $(0, 4)$ ***A1***

Note: Accept curve with an asymptote at $x = 1$ suggested.

correct asymptote $y = 1$ ***A1***

[2 marks]

(b) range is $f^{-1}(x) > 1$ (or $]1, \infty[$) ***A1***

Note: Also accept $]1, 10]$ or $]1, 10[$.

Note: Do not allow follow through from incorrect asymptote in (a).

[1 mark]

(c) $(4, 0) \Rightarrow \ln(4a + b) = 0$ ***M1***

$\Rightarrow 4a + b = 1$ ***A1***

asymptote at $x = 1 \Rightarrow a + b = 0$ ***M1***

$\Rightarrow a = \frac{1}{3},\ b = -\frac{1}{3}$ ***A1***

[4 marks]

Total [7 marks]

4. (if $\boldsymbol{A}$ is singular then) det $\boldsymbol{A} = 0$ (seen anywhere) ***R1***

det $\boldsymbol{A} = b(b+1) - a(a+1)$ ***M1***

$b + b^2 - a - a^2 = 0$

EITHER

$b - a + (b+a)(b-a) = 0$ ***(M1)***

$(b-a)(1+b+a) = 0$ ***A1***

OR

$b - a = a^2 - b^2$ ***(M1)***

$b - a = (a+b)(a-b)$ or $-(a-b) = (a+b)(a-b)$ ***A1***

THEN

$a + b = -1$ ***A1***

Total [5 marks]

5. $3x^2y^2 + 2x^3y\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2 - 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 9\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1M1A1***

Note: First ***M1*** for attempt at implicit differentiation, second ***M1*** for use of product rule.

$\left(\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2y^2 + 3x^2}{3y^2 - 2x^3y - 9}\right)$

$\Rightarrow 3x^2 + 3x^2y^2 = 0$ ***(A1)***

$\Rightarrow 3x^2\left(1 + y^2\right) = 0$

$x = 0$ ***A1***

Note: Do not award ***A1*** if extra solutions given *eg* $y = \pm 1$.

substituting $x = 0$ into original equation ***(M1)***

$y^3 - 9y = 0$

$y(y+3)(y-3) = 0$

$y = 0$, $y = \pm 3$

coordinates $(0, 0)$, $(0, 3)$, $(0, -3)$ ***A1***

Total [7 marks]

6. $n=1$: $1^3+11=12$
$=3\times 4$ **or** a multiple of 3 ***A1***
assume the proposition is true for $n=k$ $(ie\ k^3+11k=3\text{m})$ ***M1***

Note: Do not award ***M1*** for statements with "Let $n=k$".

consider $n=k+1$: $(k+1)^3+11(k+1)$ ***M1***

$=k^3+3k^2+3k+1+11k+11$ ***A1***
$=k^3+11k+(3k^2+3k+12)$ ***M1***
$=3(m+k^2+k+4)$ ***A1***

Note: Accept $k^3+11k+3(k^2+k+4)$ or statement that $k^3+11k+(3k^2+3k+12)$ is a multiple of 3.

true for $n=1$, and $n=k$ true $\Rightarrow n=k+1$ true
hence true for all $n\in\mathbb{Z}^+$ ***R1***

Note: Only award the final ***R1*** if at least 4 of the previous marks have been achieved.

Total [7 marks]

7. (a) **METHOD 1**

$a+ar=10$ ***A1***
$a+ar+ar^2+ar^3=30$ ***A1***
$a+ar=10\Rightarrow ar^2+ar^3=10r^2$ **or** $ar^2+ar^3=20$ ***M1***
$10+10r^2=30$ **or** $r^2(a+ar)=20$ ***A1***
$\Rightarrow r^2=2$ ***AG***

METHOD 2

$\frac{a\left(1-r^2\right)}{1-r}=10$ and $\frac{a\left(1-r^4\right)}{1-r}=30$ ***M1A1***
$\Rightarrow \frac{1-r^4}{1-r^2}=3$ ***M1***
leading to either $1+r^2=3$ (or $r^4-3r^2+2=0$) ***A1***
$\Rightarrow r^2=2$ ***AG***

[4 marks]

continued

Question 7 continued

(b) (i) $a + a\sqrt{2} = 10$

$\Rightarrow a = \dfrac{10}{1+\sqrt{2}}$ **or** $a = 10\left(\sqrt{2}-1\right)$ ***A1***

(ii) $S_{10} = \dfrac{10}{1+\sqrt{2}}\left(\dfrac{\sqrt{2}^{10}-1}{\sqrt{2}-1}\right)\left(=10\times 31\right)$ ***M1***

$= 310$ ***A1***

[3 marks]

Total [7 marks]

8. (a) $\sin(x+y)\sin(x-y)$

$= (\sin x\cos y + \cos x\sin y)(\sin x\cos y - \cos x\sin y)$ ***M1A1***

$= \sin^2 x\cos^2 y + \sin x\sin y\cos x\cos y - \sin x\sin y\cos x\cos y - \cos^2 x\sin^2 y$

$= \sin^2 x\cos^2 y - \cos^2 x\sin^2 y$ ***A1***

$= \sin^2 x\left(1-\sin^2 y\right) - \sin^2 y\left(1-\sin^2 x\right)$ ***A1***

$= \sin^2 x - \sin^2 x\sin^2 y - \sin^2 y + \sin^2 x\sin^2 y$

$= \sin^2 x - \sin^2 y$ ***AG***

[4 marks]

(b) $f(x) = \sin^2 x - \dfrac{1}{4}$

range is $f \in \left[-\dfrac{1}{4}, \dfrac{3}{4}\right]$ ***A1A1***

Note: Award ***A1*** for each end point. Condone incorrect brackets.

[2 marks]

(c) $g(x) = \dfrac{1}{\sin^2 x - \dfrac{1}{4}}$

range is $g \in]-\infty, -4] \cup \left[\dfrac{4}{3}, \infty\right[$ ***A1A1***

Note: Award ***A1*** for each part of range. Condone incorrect brackets.

[2 marks]

Total [8 marks]

9. (a) $\log_2(x-2) = \log_4\left(x^2 - 6x + 12\right)$

EITHER

$\log_2(x-2) = \dfrac{\log_2\left(x^2 - 6x + 12\right)}{\log_2 4}$ ***M1***

$2\log_2(x-2) = \log_2\left(x^2 - 6x + 12\right)$

OR

$\dfrac{\log_4(x-2)}{\log_4 2} = \log_4\left(x^2 - 6x + 12\right)$ ***M1***

$2\log_4(x-2) = \log_4\left(x^2 - 6x + 12\right)$

THEN

$(x-2)^2 = x^2 - 6x + 12$ ***A1***

$x^2 - 4x + 4 = x^2 - 6x + 12$

$x = 4$ ***A1*** ***N1***

[3 marks]

(b) $x^{\ln x} = \mathrm{e}^{(\ln x)^3}$

taking ln of both sides or writing $x = \mathrm{e}^{\ln x}$ ***M1***

$(\ln x)^2 = (\ln x)^3$ ***A1***

$(\ln x)^2(\ln x - 1) = 0$ ***(A1)***

$x = 1$, $x = \mathrm{e}$ ***A1A1*** ***N2***

Note: Award second ***(A1)*** only if factorisation seen or if two correct solutions are seen.

[5 marks]

Total [8 marks]

SECTION B

10. (a) (i) $f'(x) = \mathrm{e}^{-x} - x\mathrm{e}^{-x}$ ***M1A1***

(ii) $f'(x) = 0 \Rightarrow x = 1$

coordinates $\left(1,\ \mathrm{e}^{-1}\right)$ ***A1***

[3 marks]

(b) $f''(x) = -\mathrm{e}^{-x} - \mathrm{e}^{-x} + x\mathrm{e}^{-x}\left(= -\mathrm{e}^{-x}(2-x)\right)$ ***A1***

substituting $x = 1$ into $f''(x)$ ***M1***

$f''(1)\left(= -\mathrm{e}^{-1}\right) < 0$ hence maximum ***R1AG***

[3 marks]

(c) $f''(x) = 0 \quad \left(\Rightarrow x = 2\right)$ ***M1***

coordinates $\left(2,\ 2\mathrm{e}^{-2}\right)$ ***A1***

[2 marks]

(d) (i) $g(x) = \frac{x}{2}\mathrm{e}^{-\frac{x}{2}}$ ***A1***

(ii) coordinates of maximum $\left(2,\ \mathrm{e}^{-1}\right)$ ***A1***

(iii) equating $f(x) = g(x)$ and attempting to solve $x\mathrm{e}^{-x} = \frac{x}{2}\mathrm{e}^{-\frac{x}{2}}$

$\Rightarrow x\left(2\mathrm{e}^{\frac{x}{2}} - \mathrm{e}^{x}\right) = 0$ ***(A1)***

$\Rightarrow x = 0$ ***A1***

or $2\mathrm{e}^{\frac{x}{2}} = \mathrm{e}^{x}$

$\Rightarrow \mathrm{e}^{\frac{x}{2}} = 2$

$\Rightarrow x = 2\ln 2 \qquad (\ln 4)$ ***A1***

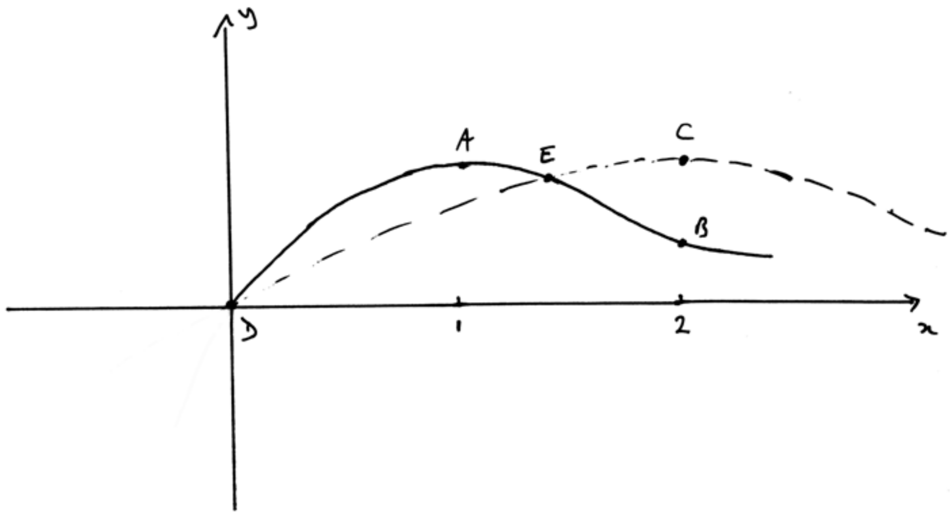
[5 marks]

Note: Award first ***(A1)*** only if factorisation seen or if two correct solutions are seen.

continued ...

Question 10 continued

(e)



A4

> **Note:** Award ***A1*** for shape of f, including domain extending beyond $x = 2$.
> Ignore any graph shown for $x < 0$.
> Award ***A1*** for A and B correctly identified.
> Award ***A1*** for shape of g, including domain extending beyond $x = 2$.
> Ignore any graph shown for $x < 0$. Allow follow through from f.
> Award ***A1*** for C, D and E correctly identified (D and E are interchangeable).

[4 marks]

(f) $A = \int_0^1 \frac{x}{2} e^{-\frac{x}{2}} \, dx$ ***M1***

$= \left[-x e^{-\frac{x}{2}} \right]_0^1 - \int_0^1 -e^{-\frac{x}{2}} \, dx$ ***A1***

> **Note:** Condone absence of limits or incorrect limits.

$= -e^{-\frac{1}{2}} - \left[2e^{-\frac{x}{2}} \right]_0^1$

$= -e^{-\frac{1}{2}} - \left(2e^{-\frac{1}{2}} - 2 \right) = 2 - 3e^{-\frac{1}{2}}$ ***A1***

[3 marks]

Total [20 marks]

11. (a) $\vec{CA} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ ***(A1)***

$\vec{CB} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ ***(A1)***

> **Note:** If $\vec{AC}$ and $\vec{BC}$ found correctly award ***(A1) (A0)***.

$\vec{CA} \times \vec{CB} = \begin{vmatrix} i & j & k \\ 1 & -2 & -1 \\ 2 & 0 & 1 \end{vmatrix}$ ***(M1)***

$\begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix}$ ***A1***

[4 marks]

(b) **METHOD 1**

$\frac{1}{2}\left|\vec{CA} \times \vec{CB}\right|$ ***(M1)***

$= \frac{1}{2}\sqrt{(-2)^2 + (-3)^2 + 4^2}$ ***(A1)***

$= \frac{\sqrt{29}}{2}$ ***A1***

METHOD 2

attempt to apply $\frac{1}{2}|CA||CB|\sin C$ ***(M1)***

$CA.CB = \sqrt{5}.\sqrt{6}\cos C \Rightarrow \cos C = \frac{1}{\sqrt{30}} \Rightarrow \sin C = \frac{\sqrt{29}}{\sqrt{30}}$ ***(A1)***

$\text{area} = \frac{\sqrt{29}}{2}$ ***A1***

[3 marks]

continued ...

Question 11 continued

(c) **METHOD 1**

$\mathbf{r}.\begin{pmatrix}-2\\-3\\4\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}.\begin{pmatrix}-2\\-3\\4\end{pmatrix}$ ***M1A1***

$\Rightarrow -2x-3y+4z=-2$ ***A1***

$\Rightarrow 2x+3y-4z=2$ ***AG***

METHOD 2

$-2x-3y+4z=\text{d}$

substituting a point in the plane ***M1A1***

$\text{d}=-2$ ***A1***

$\Rightarrow -2x-3y+4z=-2$

$\Rightarrow 2x+3y-4z=2$ ***AG***

Note: Accept verification that all 3 vertices of the triangle lie on the given plane.

[3 marks]

continued ...

Question 11 continued

(d) **METHOD 1**

$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -4 \\ 4 & -1 & -1 \end{vmatrix} = \begin{pmatrix} -7 \\ -14 \\ -14 \end{pmatrix}$ ***M1A1***

$\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$z = 0 \Rightarrow y = 0,\ x = 1$ ***(M1)(A1)***

$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ ***A1***

Note: Do not award the final ***A1*** if $\mathbf{r} =$ is not seen.

METHOD 2

eliminate 1 of the variables, *eg x* ***M1***
$-7y + 7z = 0$ ***(A1)***
introduce a parameter ***M1***
$\Rightarrow z = \lambda,$

$y = \lambda,\ x = 1 + \frac{\lambda}{2}$ ***(A1)***

$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ or equivalent ***A1***

Note: Do not award the final ***A1*** if $\mathbf{r} =$ is not seen.

METHOD 3

$z = t$ ***M1***
write x and y in terms of $t \Rightarrow 4x - y = 4 + t,\ 2x + 3y = 2 + 4t$ or equivalent ***A1***
attempt to eliminate x or y ***M1***
x, y, z expressed in parameters
$\Rightarrow z = t,$

$y = t,\ x = 1 + \frac{t}{2}$ ***A1***

$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ or equivalent ***A1***

Note: Do not award the final ***A1*** if $\mathbf{r} =$ is not seen.

[5 marks]

continued ...

Question 11 continued

(e) **METHOD 1**

direction of the line is perpendicular to the normal of the plane

$$\begin{pmatrix}16\\ \alpha\\ -3\end{pmatrix}.\begin{pmatrix}1\\ 2\\ 2\end{pmatrix}=0$$ ***M1A1***

$16+2\alpha-6=0 \Rightarrow \alpha=-5$ ***A1***

METHOD 2

solving line/plane simultaneously

$16(1+\lambda)+2\alpha\lambda-6\lambda=\beta$ ***M1A1***

$16+(10+2\alpha)\lambda=\beta$

$\Rightarrow \alpha=-5$ ***A1***

METHOD 3

$$\begin{vmatrix}2 & 3 & -4\\ 4 & -1 & -1\\ 16 & \alpha & -3\end{vmatrix}=0$$ ***M1***

$2(3+\alpha)-3(-12+16)-4(4\alpha+16)=0$ ***A1***

$\Rightarrow \alpha=-5$ ***A1***

METHOD 4

attempt to use row reduction on augmented matrix ***M1***

to obtain $\left(\begin{array}{ccc|c}2 & 3 & -4 & 2\\ 0 & -1 & 1 & 0\\ 0 & 0 & \alpha+5 & \beta-16\end{array}\right)$ ***A1***

$\Rightarrow \alpha=-5$ ***A1***

[3 marks]

(f) $\alpha=-5$ ***A1***

$\beta\neq 16$ ***A1***

[2 marks]

Total [20 marks]

12. (a) $z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$ ***M1***

$= \cos n\theta + \cos n\theta + i\sin n\theta - i\sin n\theta$ ***A1***

$= 2\cos n\theta$ ***AG***

[2 marks]

(b) $\left(z + z^{-1}\right)^4 = z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) + 4z\left(\frac{1}{z^3}\right) + \frac{1}{z^4}$ ***A1***

> **Note:** Accept $\left(z + z^{-1}\right)^4 = 16\cos^4\theta$.

[1 mark]

(c) **METHOD 1**

$\left(z + z^{-1}\right)^4 = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$ ***M1***

$(2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$ ***A1A1***

> **Note:** Award ***A1*** for RHS, ***A1*** for LHS independent of the ***M1***.

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ ***A1***

$\left(\text{or } p = \frac{1}{8},\ q = \frac{1}{2},\ r = \frac{3}{8}\right)$

METHOD 2

$\cos^4\theta = \left(\frac{\cos 2\theta + 1}{2}\right)^2$ ***M1***

$= \frac{1}{4}(\cos^2 2\theta + 2\cos 2\theta + 1)$ ***A1***

$= \frac{1}{4}\left(\frac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1\right)$ ***A1***

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ ***A1***

$\left(\text{or } p = \frac{1}{8},\ q = \frac{1}{2},\ r = \frac{3}{8}\right)$

[4 marks]

continued …

Question 12 continued

(d) $\left(z+z^{-1}\right)^6 = z^6 + 6z^5\left(\frac{1}{z}\right) + 15z^4\left(\frac{1}{z^2}\right) + 20z^3\left(\frac{1}{z^3}\right) + 15z^2\left(\frac{1}{z^4}\right) + 6z\left(\frac{1}{z^5}\right) + \frac{1}{z^6}$ ***M1***

$\left(z+z^{-1}\right)^6 = \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$

$(2\cos\theta)^6 = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$ ***A1A1***

> **Note:** Award ***A1*** for RHS, ***A1*** for LHS, independent of the ***M1***.

$\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$ ***AG***

> **Note:** Accept a purely trigonometric solution as for (c).

[3 marks]

(e) $\int_0^{\frac{\pi}{2}} \cos^6\theta\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \left(\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}\right)d\theta$

$= \left[\frac{1}{192}\sin 6\theta + \frac{3}{64}\sin 4\theta + \frac{15}{64}\sin 2\theta + \frac{5}{16}\theta\right]_0^{\frac{\pi}{2}}$ ***M1A1***

$= \frac{5\pi}{32}$ ***A1***

[3 marks]

(f) $V = \pi\int_0^{\frac{\pi}{2}} \sin^2 x\cos^4 x\,\mathrm{d}x$ ***M1***

$= \pi\int_0^{\frac{\pi}{2}} \cos^4 x\,\mathrm{d}x - \pi\int_0^{\frac{\pi}{2}} \cos^6 x\,\mathrm{d}x$ ***M1***

$\int_0^{\frac{\pi}{2}} \cos^4 x\,\mathrm{d}x = \frac{3\pi}{16}$ ***A1***

$V = \frac{3\pi^2}{16} - \frac{5\pi^2}{32} = \frac{\pi^2}{32}$ ***A1***

> **Note:** Follow through from an incorrect *r* in (c) provided the final answer is positive.

[4 marks]

continued ...

Question 12 continued

(g) (i) constant term $=\binom{2k}{k}=\frac{(2k)!}{k!k!}=\frac{(2k)!}{(k!)^2}$ (accept C_k^{2k}) ***A1***

(ii) $2^{2k}\int_0^{\frac{\pi}{2}}\cos^{2k}\theta\,\mathrm{d}\theta=\frac{(2k)!}{(k!)^2}\frac{\pi}{2}$ ***A1***

$\int_0^{\frac{\pi}{2}}\cos^{2k}\theta\,\mathrm{d}\theta=\frac{(2k)!\pi}{2^{2k+1}(k!)^2}$ $\left(\text{or } \frac{\binom{2k}{k}\pi}{2^{2k+1}}\right)$ ***A1***

[3 marks]

Total [20 marks]