N13/5/MATHL/HP2/ENG/TZ0/XX/M

International Baccalaureate®
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MARKSCHEME

November 2013

MATHEMATICS

Higher Level

Paper 2

20 pages

SECTION A

1. $\boldsymbol{AX} = \boldsymbol{B}$

EITHER

$\Rightarrow \boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B}$ ***(M1)***

OR

attempting row reduction:

eg $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -1 & -6 \\ 0 & -2 & 0 & -1 \end{array}\right)$ ***(M1)***

THEN

$\Rightarrow \boldsymbol{X} = \begin{pmatrix} -\frac{7}{2} \\ \frac{1}{2} \\ 5 \end{pmatrix}$ ***A1A1A1***

Total [4 marks]

2. (a) **METHOD 1**

$34 = a + 3d$ and $76 = a + 9d$ ***(M1)***

$d = 7$ ***A1***

$a = 13$ ***A1***

METHOD 2

$76 = 34 + 6d$ ***(M1)***

$d = 7$ ***A1***

$34 = a + 21$

$a = 13$ ***A1***

[3 marks]

(b) $\frac{n}{2}\left(26 + 7(n-1)\right) > 5000$ ***(M1)(A1)***

$n > 36.463...$ ***(A1)***

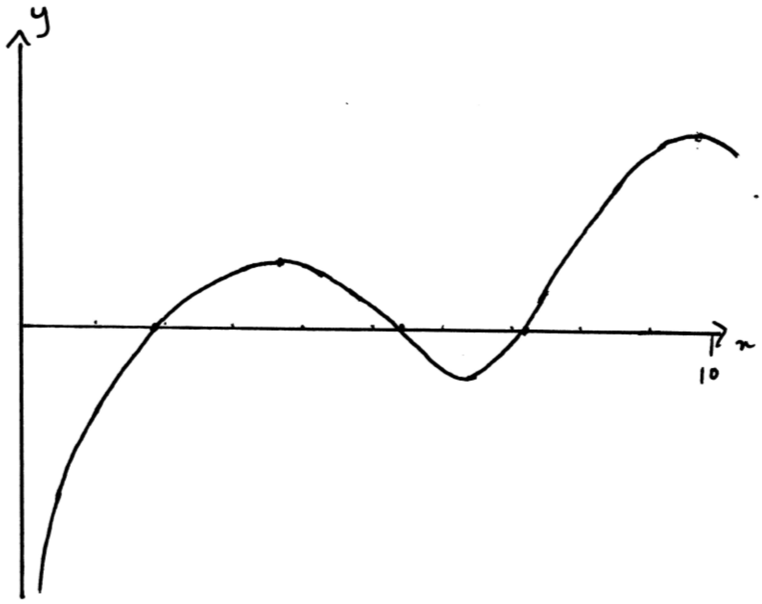
Note: Award ***M1A1A1*** for using either an equation, a graphical approach or a numerical approach.

$n = 37$ ***A1*** ***N3***

[4 marks]

Total [7 marks]

3. (a)



A correct graph shape for $0 < x \le 10$. ***A1***

maxima $(3.78, 0.882)$ and $(9.70, 1.89)$ ***A1***

minimum $(6.22, -0.885)$ ***A1***

x-axis intercepts $(1.97, 0)$, $(5.24, 0)$ and $(7.11, 0)$ ***A2***

Note: Award ***A1*** if two x-axis intercepts are correct.

[5 marks]

(b) $0 < x \le 1.97$ ***A1***

$5.24 \le x \le 7.11$ ***A1***

[2 marks]

Total [7 marks]

4. $\mathrm{P}\left(Z < \frac{780-\mu}{\sigma}\right) = 0.92$ and $\mathrm{P}\left(Z < \frac{755-\mu}{\sigma}\right) = 0.12$ ***(M1)***

use of inverse normal ***(M1)***

$\Rightarrow \frac{780-\mu}{\sigma} = 1.405\ldots$ and $\frac{755-\mu}{\sigma} = -1.174\ldots$ ***(A1)***

solving simultaneously ***(M1)***

Note: Award ***M1*** for attempting to solve an incorrect pair of equations *eg*, inverse normal not used.

$\mu = 766.385$

$\sigma = 9.6897$

$\mu = 12$ hrs 46 mins $(= 766$ mins$)$ ***A1***

$\sigma = 10$ mins ***A1***

Total [6 marks]

5. (a) $\mathrm{P}(F) = \left(\frac{1}{7} \times \frac{7}{9}\right) + \left(\frac{6}{7} \times \frac{4}{9}\right)$ ***(M1)(A1)***

Note: Award ***M1*** for the sum of two products.

$= \frac{31}{63}$ $(= 0.4920\ldots)$ ***A1***

[3 marks]

(b) Use of $\mathrm{P}(S \mid F) = \frac{\mathrm{P}(S \cap F)}{\mathrm{P}(F)}$ to obtain $\mathrm{P}(S \mid F) = \frac{\frac{1}{7} \times \frac{7}{9}}{\frac{31}{63}}$. ***M1***

Note: Award ***M1*** only if the numerator results from the product of two probabilities.

$= \frac{7}{31}$ $(= 0.2258\ldots)$ ***A1***

[2 marks]

Total [5 marks]

6. (a) $\frac{a+\mathrm{i}}{a-\mathrm{i}} \times \frac{a+\mathrm{i}}{a+\mathrm{i}}$ ***M1***

$= \frac{a^2-1+2a\mathrm{i}}{a^2+1} \left(= \frac{a^2-1}{a^2+1} + \frac{2a}{a^2+1}\mathrm{i} \right)$ ***A1***

(i) z is real when $a = 0$ ***A1***

(ii) z is purely imaginary when $a = \pm 1$ ***A1***

Note: Award ***M1A0A1A0*** for $\frac{a^2-1+2a\mathrm{i}}{a^2-1} \left(= 1 + \frac{2a}{a^2-1}\mathrm{i} \right)$ leading to $a = 0$ in (i).

[4 marks]

(b) **METHOD 1**

attempting to find either $|z|$ or $|z|^2$ by expanding and simplifying

eg $|z|^2 = \frac{(a^2-1)^2 + 4a^2}{(a^2+1)^2} = \frac{a^4+2a^2+1}{(a^2+1)^2}$ ***M1***

$= \frac{(a^2+1)^2}{(a^2+1)^2}$

$|z|^2 = 1 \Rightarrow |z| = 1$ ***A1***

METHOD 2

$|z| = \frac{|a+\mathrm{i}|}{|a-\mathrm{i}|}$ ***M1***

$|z| = \frac{\sqrt{a^2+1}}{\sqrt{a^2+1}} \Rightarrow |z| = 1$ ***A1***

[2 marks]

Total [6 marks]

7. (a) attempting to form $(3\cos\theta+6)(\cos\theta-2)+7(1+\sin\theta)=0$ ***M1***

$3\cos^2\theta-12+7\sin\theta+7=0$ ***A1***

$3\left(1-\sin^2\theta\right)+7\sin\theta-5=0$ ***M1***

$3\sin^2\theta-7\sin\theta+2=0$ ***AG***

[3 marks]

(b) attempting to solve algebraically (including substitution) or graphically for $\sin\theta$ ***(M1)***

$\sin\theta=\frac{1}{3}$ ***(A1)***

$\theta=0.340\quad(=19.5^\circ)$ ***A1***

[3 marks]

Total [6 marks]

8. (a) $A=\frac{1}{2}\times10^2\times\theta-\frac{1}{2}\times10^2\times\sin\theta$ ***M1A1***

Note: Award ***M1*** for use of area of segment $=$ area of sector $-$ area of triangle.

$=50\theta-50\sin\theta$ ***AG***

[2 marks]

(b) **METHOD 1**

unshaded area $=\frac{\pi\times10^2}{2}-50(\theta-\sin\theta)$

(or equivalent *eg* $50\pi-50\theta+50\sin\theta$) ***(M1)***

$50\theta-50\sin\theta=\frac{1}{2}(50\pi-50\theta+50\sin\theta)$ ***(A1)***

$3\theta-3\sin\theta-\pi=0$

$\Rightarrow\theta=1.969$ (rad) ***A1***

METHOD 2

$50\theta-50\sin\theta=\frac{1}{3}\left(\frac{\pi\times10^2}{2}\right)$ ***(M1)(A1)***

$3\theta-3\sin\theta-\pi=0$

$\Rightarrow\theta=1.969$ (rad) ***A1***

[3 marks]

Total [5 marks]

9. (a) **METHOD 1**

for P on L_1, $\vec{\text{OP}} = \begin{pmatrix} -5-\lambda \\ -3+2\lambda \\ 2+2\lambda \end{pmatrix}$

require $\begin{pmatrix} -5-\lambda \\ -3+2\lambda \\ 2+2\lambda \end{pmatrix} . \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 0$ ***M1***

$5+\lambda-6+4\lambda+4+4\lambda=0$ (or equivalent) ***A1***

$\lambda = -\frac{1}{3}$ ***A1***

$\therefore \vec{\text{OP}} = \begin{pmatrix} -\frac{14}{3} \\ -\frac{11}{3} \\ \frac{4}{3} \end{pmatrix}$ ***A1***

$L_2 : \boldsymbol{r} = \mu \begin{pmatrix} -14 \\ -11 \\ 4 \end{pmatrix}$ ***A1***

Note: Do not award the final ***A1*** if $\boldsymbol{r} =$ is not seen.

[5 marks]

METHOD 2

Calculating either $\left|\vec{\text{OP}}\right|$ or $\left|\vec{\text{OP}}\right|^2$ *eg*

$\left|\vec{\text{OP}}\right| = \sqrt{(-5-\lambda)^2 + (-3+2\lambda)^2 + (2+2\lambda)^2}$

$= \sqrt{9\lambda^2 + 6\lambda + 38}$ ***A1***

Solving either $\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\left|\vec{\text{OP}}\right|\right) = 0$ or $\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\left|\vec{\text{OP}}\right|^2\right) = 0$ for λ. ***M1***

$\lambda = -\frac{1}{3}$ ***A1***

$\vec{\text{OP}} = \begin{pmatrix} -\frac{14}{3} \\ -\frac{11}{3} \\ \frac{4}{3} \end{pmatrix}$ ***A1***

$L_2 : \boldsymbol{r} = \mu \begin{pmatrix} -14 \\ -11 \\ 4 \end{pmatrix}$ ***A1***

Note: Do not award the final ***A1*** if $\boldsymbol{r} =$ is not seen. ***[5 marks]***

continued…

Question 9 continued

(b) **METHOD 1**

$$\left|\overrightarrow{\mathrm{OP}}\right| = \sqrt{\left(-\frac{14}{3}\right)^2 + \left(-\frac{11}{3}\right)^2 + \left(\frac{4}{3}\right)^2}$$ ***(M1)***

$= 6.08 \quad \left(= \sqrt{37}\right)$ ***A1***

METHOD 2

$$\text{shortest distance} = \frac{\left|\begin{pmatrix}-1\\2\\2\end{pmatrix} \times \begin{pmatrix}-5\\-3\\2\end{pmatrix}\right|}{\left|\begin{pmatrix}-1\\2\\2\end{pmatrix}\right|}$$ ***(M1)***

$$= \frac{|10\boldsymbol{i} + 8\boldsymbol{j} + 13\boldsymbol{k}|}{\sqrt{1+4+4}}$$

$= 6.08 \left(= \sqrt{37}\right)$ ***A1***

[2 marks]

Total [7 marks]

10. **EITHER**

$\frac{\mathrm{d}x}{\mathrm{d}u} = 2\sec^2 u$ ***A1***

$\int \frac{2\sec^2 u \, \mathrm{d}u}{4\tan^2 u\sqrt{4+4\tan^2 u}}$ ***(M1)***

$= \int \frac{2\sec^2 u \, \mathrm{d}u}{4\tan^2 u \times 2\sec u} \quad \left(= \int \frac{\mathrm{d}u}{4\sin^2 u\sqrt{\tan^2 u + 1}} \text{ or } = \int \frac{2\sec^2 u \, \mathrm{d}u}{4\tan^2 u\sqrt{4\sec^2 u}}\right)$ ***A1***

OR

$u = \arctan \frac{x}{2}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{x^2+4}$ ***A1***

$\int \frac{\sqrt{4\tan^2 u + 4}\,\mathrm{d}u}{2 \times 4\tan^2 u}$ ***(M1)***

$\int \frac{2\sec u \, \mathrm{d}u}{2 \times 4\tan^2 u}$ ***A1***

THEN

$= \frac{1}{4}\int \frac{\sec u \, \mathrm{d}u}{\tan^2 u}$

$= \frac{1}{4}\int \operatorname{cosec} u \cot u \, \mathrm{d}u \quad \left(= \frac{1}{4}\int \frac{\cos u}{\sin^2 u}\mathrm{d}u\right)$ ***A1***

$= -\frac{1}{4}\operatorname{cosec} u \quad (+C) \quad \left(= -\frac{1}{4\sin u} \quad (+C)\right)$ ***A1***

use of either $\tan u = \frac{x}{2}$ or an appropriate trigonometric identity ***M1***

either $\sin u = \frac{x}{\sqrt{x^2+4}}$ or $\operatorname{cosec} u = \frac{\sqrt{x^2+4}}{x}$ (or equivalent) ***A1***

$= \frac{-\sqrt{x^2+4}}{4x}(+C)$ ***AG***

Total [7 marks]

SECTION B

11. (a) (i) $X \sim \text{Po}(0.6)$

$\text{P}(X = 0) = 0.549 \ \left(= \text{e}^{-0.6}\right)$ ***A1***

(ii) $\text{P}(X \geq 3) = 1 - \text{P}(X \leq 2)$ ***(M1)(A1)***

$= 1 - \left(\text{e}^{-0.6} + \text{e}^{-0.6} \times 0.6 + \text{e}^{-0.6} \times \frac{0.6^2}{2}\right)$

$= 0.0231$ ***A1***

(iii) $Y \sim \text{Po}(2.4)$ ***(M1)***

$\text{P}(Y \leq 5) = 0.964$ ***A1***

(iv) $Z \sim \text{B}(12,\ 0.451...)$ ***(M1)(A1)***

Note: Award ***M1*** for recognising binomial and ***A1*** for using correct parameters.

$\text{P}(Z = 4) = 0.169$ ***A1***

[9 marks]

(b) (i) $k\int_1^3 \ln x \,\text{d}x = 1$ ***(M1)***

$(k \times 1.2958... = 1)$

$k = 0.771702$ ***A1***

(ii) $\text{E}(X) = \int_1^3 kx \ln x \,\text{d}x$ ***(A1)***

attempting to evaluate their integral ***(M1)***

$= 2.27$ ***A1***

(iii) $x = 3$ ***A1***

(iv) $\int_1^m k \ln x \,\text{d}x = 0.5$ ***(M1)***

$k[x \ln x - x]_1^m = 0.5$

attempting to solve for m ***(M1)***

$m = 2.34$ ***A1***

[9 marks]

Total [18 marks]

12. (a) (i) **METHOD 1**

$v = \int 3\cos\frac{t}{4}\,\mathrm{d}t$ ***M1***

$= 12\sin\frac{t}{4} + c$ ***A1***

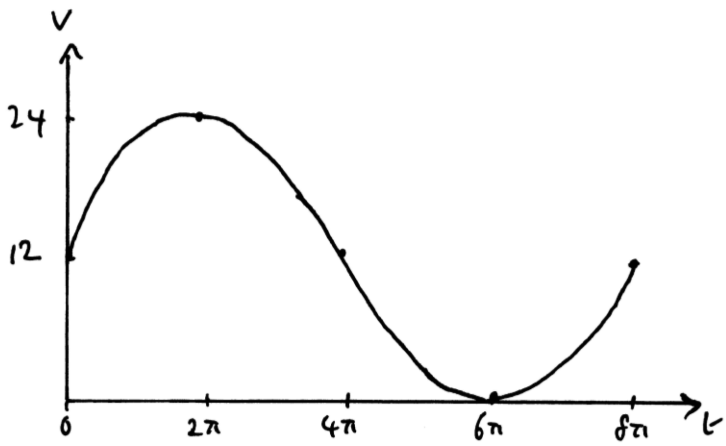
$t = 0,\ v = 12 \Rightarrow v = 12\sin\frac{t}{4} + 12$ ***A1***

METHOD 2

$v - 12 = \int_0^t 3\cos\frac{t}{4}\,\mathrm{d}t$ ***M1A1***

$v = 12\sin\frac{t}{4} + 12$ ***A1***

(ii)



A1A1A1

Note: Award ***A1*** for shape and domain $0 \le t \le 8\pi$.
Award ***A1*** for $(0, 12)$ and $(6\pi, 0)$ $\left((18.8, 0)\right)$.
Award ***A1*** for $(2\pi, 24)$ $\left((6.28, 24)\right)$.

(iii) **METHOD 1**

$\int_0^{6\pi}\left(12\sin\frac{t}{4} + 12\right)\mathrm{d}t$ ***M1***

$= 274$ (m) $\left(= 72\pi + 48 \text{ (m)}\right)$ ***A1***

METHOD 2

$s = \int 12\sin\frac{t}{4} + 12\ \mathrm{d}t$

$= -48\cos\frac{t}{4} + 12t + c$ ***M1***

When $t = 0$, $s = 0$ and so $c = 48$.

When $t = 6\pi$, $s = 274$ (m) $\left(= 72\pi + 48 \text{ (m)}\right)$. ***A1***

[8 marks]

continued …

Question 12 continued

(b) (i) **METHOD 1**

$\dfrac{\mathrm{d}v}{\mathrm{d}t} = -\left(v^2 + 4\right)$ ***(A1)***

$\displaystyle\int \frac{\mathrm{d}v}{v^2 + 4} = -\int \mathrm{d}t$ ***M1***

$\dfrac{1}{2}\arctan\left(\dfrac{v}{2}\right) = -t + c$ ***A1***

EITHER

$t = 0,\ v = 2 \Rightarrow c = \dfrac{\pi}{8}$ ***M1***

$\arctan\left(\dfrac{v}{2}\right) = \dfrac{\pi}{4} - 2t$ ***A1***

OR

$v = 2\tan\left(2c - 2t\right)$ ***A1***

$t = 0,\ v = 2 \Rightarrow \tan 2c = 1$ and so $c = \dfrac{\pi}{8}$ ***M1***

THEN

$v = 2\tan\left(\dfrac{\pi}{4} - 2t\right)$ ***A1***

$v = 2\tan\left(\dfrac{\pi - 8t}{4}\right)$ ***AG***

METHOD 2

$\dfrac{dv}{dt} = -4\sec^2\left(\dfrac{\pi - 8t}{4}\right)$ ***M1A1***

Substituting $v = 2\tan\left(\dfrac{\pi - 8t}{4}\right)$ into $\dfrac{dv}{dt} = -\left(v^2 + 4\right)$:

$\dfrac{dv}{dt} = -\left(4\tan^2\left(\dfrac{\pi - 8t}{4}\right) + 4\right)$ ***M1***

$= -4\left(\tan^2\left(\dfrac{\pi - 8t}{4}\right) + 1\right)$ ***(A1)***

$= -4\sec^2\left(\dfrac{\pi - 8t}{4}\right)$ ***A1***

Verifying that $v = 2$ when $t = 0$. ***A1***

continued …

(ii) **METHOD 1**

$v\frac{\mathrm{d}v}{\mathrm{d}s} = -\left(v^2+4\right)$ ***A1***

$\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}s} = -\frac{\left(v^2+4\right)}{v}$ ***AG***

METHOD 2

$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{\mathrm{d}v}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}s}$ ***A1***

$\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}s} = -\frac{\left(v^2+4\right)}{v}$ ***AG***

(iii) **METHOD 1**

When $v = 0$, $t = \frac{\pi}{8}$ $(t = 0.392...)$. ***(M1)A1***

$s = \int_0^{\frac{\pi}{8}} 2\tan\left(\frac{\pi - 8t}{4}\right)\mathrm{d}t$ ***(M1)***

$s = 0.347$ (m) $\left(s = \frac{1}{2}\ln 2 \text{ (m)}\right)$ ***A2***

METHOD 2

$\int \frac{v}{4+v^2}\mathrm{d}v = -\int \mathrm{d}s$ ***M1***

EITHER

$\frac{1}{2}\ln\left(v^2+4\right) = -s + c$ (or equivalent) ***A1***

$v = 2,\ s = 0 \Rightarrow c = \frac{1}{2}\ln 8$ ***M1***

$s = -\frac{1}{2}\ln\left(v^2+4\right) + \frac{1}{2}\ln 8$ $\left(s = \frac{1}{2}\ln\left(\frac{8}{v^2+4}\right)\right)$ ***(A1)***

$v = 0 \Rightarrow s = \frac{1}{2}\ln 2$ (m) $\left(s = 0.347 \text{ (m)}\right)$ ***A1***

OR

$-\int_2^0 \frac{v}{4+v^2}\mathrm{d}v = s$ (or equivalent) ***M1A1***

Note: Award ***M1*** for setting up a definite integral and award ***A1*** for stating correct limits.

$s = 0.347$ (m) $\left(s = \frac{1}{2}\ln 2 \text{ (m)}\right)$ ***A2***

[12 marks]

Total [20 marks]

13. (a) (i) either counterexample or sketch or
recognising that $y = k$ $(k > 1)$ intersects the graph of $y = f(x)$ twice ***M1***
function is not 1–1 (does not obey horizontal line test) ***R1***
so f^{-1} does not exist ***AG***

(ii) $f'(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$ ***(A1)***

$f'(\ln 3) = \frac{4}{3}$ $(=1.33)$ ***(A1)***

$m = -\frac{3}{4}$ ***M1***

$f(\ln 3) = \frac{5}{3}$ $(=1.67)$ ***A1***

EITHER

$\frac{y - \frac{5}{3}}{x - \ln 3} = -\frac{3}{4}$ ***M1***

$4y - \frac{20}{3} = -3x + 3\ln 3$ ***A1***

OR

$\frac{5}{3} = -\frac{3}{4}\ln 3 + c$ ***M1***

$c = \frac{5}{3} + \frac{3}{4}\ln 3$

$y = -\frac{3}{4}x + \frac{5}{3} + \frac{3}{4}\ln 3$ ***A1***

$12y = -9x + 20 + 9\ln 3$

THEN

$9x + 12y - 9\ln 3 - 20 = 0$ ***AG***

(iii) The tangent at $(a, f(a))$ has equation $y - f(a) = f'(a)(x - a)$. ***(M1)***

$f'(a) = \frac{f(a)}{a}$ (or equivalent) ***(A1)***

$e^a - e^{-a} = \frac{e^a + e^{-a}}{a}$ (or equivalent) ***A1***

attempting to solve for a ***(M1)***
$a = \pm 1.20$ ***A1A1***

[14 marks]

continued …

Question 13 continued

(b) (i) $2y = e^{x} + e^{-x}$

$e^{2x} - 2ye^{x} + 1 = 0$ ***M1A1***

> **Note:** Award ***M1*** for either attempting to rearrange or interchanging *x* and *y*.

$e^{x} = \frac{2y \pm \sqrt{4y^2 - 4}}{2}$ ***A1***

$e^{x} = y \pm \sqrt{y^2 - 1}$

$x = \ln\left(y \pm \sqrt{y^2 - 1}\right)$ ***A1***

$f^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$ ***A1***

> **Note:** Award ***A1*** for correct notation and for stating the positive "branch".

(ii) $V = \pi\int_{1}^{5}\left(\ln\left(y + \sqrt{y^2 - 1}\right)\right)^2 \mathrm{d}y$ ***(M1)(A1)***

> **Note:** Award ***M1*** for attempting to use $V = \pi\int_{c}^{d} x^2 \mathrm{d}y$.

$= 37.1 \ (\text{units}^3)$ ***A1***

[8 marks]

Total [22 marks]