N14/5/MATHL/HP1/ENG/TZ0/XX/M

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MARKSCHEME

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MATHEMATICS

Higher Level

Paper 1

20 pages

SECTION A

1. (a) $g(x) = \frac{1}{x+3} + 1$ ***A1A1***

Note: Award ***A1*** for $x+3$ in the denominator and ***A1*** for the "+1".

[2 marks]

(b) $x = -3$ ***A1***
$y = 1$ ***A1***

[2 marks]

Total [4 marks]

2. (a) using the formulae for the sum and product of roots:

(i) $\alpha + \beta = 4$ ***A1***

(ii) $\alpha\beta = \frac{1}{2}$ ***A1***

Note: Award ***A0A0*** if the above results are obtained by solving the original equation (except for the purpose of checking).

[2 marks]

(b) **METHOD 1**

required quadratic is of the form $x^2 - \left(\frac{2}{\alpha} + \frac{2}{\beta}\right)x + \left(\frac{2}{\alpha}\right)\left(\frac{2}{\beta}\right)$ ***(M1)***

$q = \frac{4}{\alpha\beta}$

$q = 8$ ***A1***

$p = -\left(\frac{2}{\alpha} + \frac{2}{\beta}\right)$

$= -\frac{2(\alpha+\beta)}{\alpha\beta}$ ***M1***

$= -\frac{2 \times 4}{\frac{1}{2}}$

$p = -16$ ***A1***

Note: Accept the use of exact roots

continued …

Question 2 continued

METHOD 2

replacing x with $\frac{2}{x}$ ***M1***

$2\left(\frac{2}{x}\right)^2 - 8\left(\frac{2}{x}\right) + 1 = 0$

$\frac{8}{x^2} - \frac{16}{x} + 1 = 0$ ***(A1)***

$x^2 - 16x + 8 = 0$

$p = -16$ and $q = 8$ ***A1A1***

Note:	Award ***A1A0*** for $x^2 - 16x + 8 = 0$ *ie*, if $p = -16$ and $q = 8$ are not explicitly stated.

[4 marks]

Total [6 marks]

3. **METHOD 1**

$\left|\overrightarrow{OP}\right| = \sqrt{(1+s)^2 + (3+2s)^2 + (1-s)^2} \quad \left(= \sqrt{6s^2 + 12s + 11}\right)$ ***A1***

Note: Award ***A1*** if the square of the distance is found.

EITHER

attempt to differentiate: $\frac{d}{ds}\left|\overrightarrow{OP}\right|^2 \left(= 12s + 12\right)$ ***M1***

attempting to solve $\frac{d}{ds}\left|\overrightarrow{OP}\right|^2 = 0$ for s ***(M1)***

$s = -1$ ***(A1)***

OR

attempt to differentiate: $\frac{d}{ds}\left|\overrightarrow{OP}\right| \quad \left(= \frac{6s + 6}{\sqrt{6s^2 + 12s + 11}}\right)$ ***M1***

attempting to solve $\frac{d}{ds}\left|\overrightarrow{OP}\right| = 0$ for s ***(M1)***

$s = -1$ ***(A1)***

OR

attempt at completing the square: $\left(\left|\overrightarrow{OP}\right|^2 = 6(s+1)^2 + 5\right)$ ***M1***

minimum value ***(M1)***
occurs at $s = -1$ ***(A1)***

THEN

the minimum length of $\overrightarrow{OP}$ is $\sqrt{5}$ ***A1***

METHOD 2

the length of $\overrightarrow{OP}$ is a minimum when $\overrightarrow{OP}$ is perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ ***(R1)***

$\begin{pmatrix} 1+s \\ 3+2s \\ 1-s \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$ ***A1***

attempting to solve $1 + s + 6 + 4s - 1 + s = 0 \ (6s + 6 = 0)$ for s ***(M1)***
$s = -1$ ***(A1)***
$\left|\overrightarrow{OP}\right| = \sqrt{5}$ ***A1***

Total [5 marks]

4. (a) (i) use of $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$ ***(M1)***

$\mathrm{P}(A \cup B) = 0.2 + 0.5$

$= 0.7$ ***A1***

(ii) use of $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A)\mathrm{P}(B)$ ***(M1)***

$\mathrm{P}(A \cup B) = 0.2 + 0.5 - 0.1$

$= 0.6$ ***A1***

[4 marks]

(b) $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

$\mathrm{P}(A \mid B)$ is a maximum when $\mathrm{P}(A \cap B) = \mathrm{P}(A)$

$\mathrm{P}(A \mid B)$ is a minimum when $\mathrm{P}(A \cap B) = 0$

$0 \le \mathrm{P}(A \mid B) \le 0.4$ ***A1A1A1***

Note: ***A1*** for each endpoint and ***A1*** for the correct inequalities**.**

[3 marks]

Total [7 marks]

5. use of the quotient rule or the product rule ***M1***

$$C'(t) = \frac{(3+t^2) \times 2 - 2t \times 2t}{(3+t^2)^2} \left(= \frac{6-2t^2}{(3+t^2)^2} \right) \text{ or } \frac{2}{3+t^2} - \frac{4t^2}{(3+t^2)^2}$$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator in the quotient rule, and ***A1*** for each correct term in the product rule.

attempting to solve $C'(t) = 0$ for t ***(M1)***

$t = \pm\sqrt{3}$ (minutes) ***A1***

$C\left(\sqrt{3}\right) = \dfrac{\sqrt{3}}{3}$ $\left(\mathrm{mg\,l^{-1}}\right)$ or equivalent. ***A1***

Total [6 marks]

6. $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$ ***A1***

$dx = 2(u-1)\,du$

Note: Award the ***A1*** for any correct relationship between dx and du.

$\displaystyle\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = 2\int \frac{(u-1)^2}{u}\,du$ ***(M1)A1***

Note: Award the ***M1*** for an attempt at substitution resulting in an integral only involving u.

$= 2\displaystyle\int u - 2 + \frac{1}{u}\,du$ ***(A1)***

$= u^2 - 4u + 2\ln u\,(+C)$ ***A1***

$= x - 2\sqrt{x} - 3 + 2\ln\left(1+\sqrt{x}\right)(+C)$ ***A1***

Note: Award the ***A1*** for a correct expression in x, but not necessarily fully expanded/simplified.

Total [6 marks]

7. (a) $p'(3) = f'(3)\,g(3) + g'(3)\,f(3)$ ***(M1)***

Note: Award ***M1*** if the derivative is in terms of x or 3.

$= 2\times 4 + 3\times 1$

$= 11$ ***A1***

[2 marks]

(b) $h'(x) = g'\left(f(x)\right)f'(x)$ ***(M1)(A1)***

$h'(2) = g'(1)\,f'(2)$ ***A1***

$= 4\times 4$

$= 16$ ***A1***

[4 marks]

Total [6 marks]

8. let $\mathrm{P}(n)$ be the proposition that $(2n)! \geq 2^n (n!)^2$, $n \in \mathbb{Z}^+$

consider $\mathrm{P}(1)$:

$2! = 2$ and $2^1 (1!)^2 = 2$ so $\mathrm{P}(1)$ is true ***R1***

assume $\mathrm{P}(k)$ is true *ie* $(2k)! \geq 2^k (k!)^2$, $k \in \mathbb{Z}^+$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n = k$".

consider $\mathrm{P}(k+1)$:

$\left(2(k+1)\right)! = (2k+2)(2k+1)(2k)!$ ***M1***

$\left(2(k+1)\right)! \geq (2k+2)(2k+1)(k!)^2 2^k$ ***A1***

$= 2(k+1)(2k+1)(k!)^2 2^k$

$> 2^{k+1}(k+1)(k+1)(k!)^2$ since $2k+1 > k+1$ ***R1***

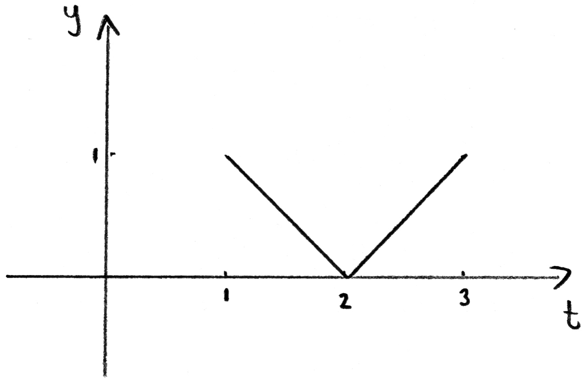
$= 2^{k+1}\left((k+1)!\right)^2$ ***A1***

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true for $n \in \mathbb{Z}^+$ ***R1***

Note: To obtain the final ***R1***, four of the previous marks must have been awarded.

Total [7 marks]

9. (a)



$|2-t|$ correct for $[1, 2]$ ***A1***

$|2-t|$ correct for $[2, 3]$ ***A1***

[2 marks]

(b) **EITHER**

let q_1 be the lower quartile and let q_3 be the upper quartile

let $d = 2 - q_1 \left(= q_3 - 2\right)$ and so $\text{IQR} = 2d$ by symmetry

use of area formulae to obtain $\frac{1}{2}d^2 = \frac{1}{4}$

(or equivalent) ***M1A1***

$d = \frac{1}{\sqrt{2}}$ or the value of at least one q. ***A1***

OR

let q_1 be the lower quartile

consider $\int_1^{q_1} (2-t)\,dt = \frac{1}{4}$ ***M1A1***

obtain $q_1 = 2 - \frac{1}{\sqrt{2}}$ ***A1***

THEN

$\text{IQR} = \sqrt{2}$ ***A1***

Note: Only accept this final answer for the ***A1***.

[4 marks]

Total [6 marks]

10. (a) use of the addition principle with 3 terms *(M1)*
to obtain ${}^{4}C_{3} + {}^{5}C_{3} + {}^{6}C_{3}$ $(= 4 + 10 + 20)$ ***A1***
number of possible selections is 34 ***A1***

[3 marks]

(b) **EITHER**

recognition of three cases: (2 odd and 2 even or 1 odd and 3 even or 0 odd and 4 even) ***(M1)***

$\left({}^{5}C_{2} \times {}^{4}C_{2}\right) + \left({}^{5}C_{1} \times {}^{4}C_{3}\right) + \left({}^{5}C_{0} \times {}^{4}C_{4}\right)$ $(= 60 + 20 + 1)$ ***(M1)A1***

OR

recognition to subtract the sum of 4 odd and 3 odd and 1 even from the total ***(M1)***

${}^{9}C_{4} - {}^{5}C_{4} - \left({}^{5}C_{3} \times {}^{4}C_{1}\right)$ $(= 126 - 5 - 40)$ ***(M1)A1***

THEN

number of possible selections is 81 ***A1***

[4 marks]

Total [7 marks]

SECTION B

11. (a) (i) $x = \mathrm{e}^{3y+1}$ ***M1***

> **Note:** The ***M1*** is for switching variables and can be awarded at any stage. Further marks do not rely on this mark being awarded.

taking the natural logarithm of both sides and attempting to transpose ***M1***

$\left(f^{-1}(x)\right) = \frac{1}{3}(\ln x - 1)$ ***A1***

(ii) $x \in \mathbb{R}^+$ or equivalent, for example $x > 0$. ***A1***

[4 marks]

(b) $\ln x = \frac{1}{3}(\ln x - 1) \Rightarrow \ln x - \frac{1}{3}\ln x = -\frac{1}{3}$ (or equivalent) ***M1A1***

$\ln x = -\frac{1}{2}$ (or equivalent) ***A1***

$x = \mathrm{e}^{-\frac{1}{2}}$ ***A1***

coordinates of P are $\left(\mathrm{e}^{-\frac{1}{2}}, -\frac{1}{2}\right)$ ***A1***

[5 marks]

(c) coordinates of Q are $(1, 0)$ seen anywhere ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$ ***M1***

at Q, $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ ***A1***

$y = x - 1$ ***AG***

[3 marks]

continued ...

Question 11 continued

(d) let the required area be A

$A = \int_1^e x - 1\mathrm{d}x - \int_1^e \ln x\mathrm{d}x$ ***M1***

Note: The ***M1*** is for a difference of integrals. Condone absence of limits here.

attempting to use integration by parts to find $\int \ln x\mathrm{d}x$ ***(M1)***

$= \left[\frac{x^2}{2} - x\right]_1^{\mathrm{e}} - [x\ln x - x]_1^{\mathrm{e}}$ ***A1A1***

Note: Award ***A1*** for $\frac{x^2}{2} - x$ and ***A1*** for $x\ln x - x$.

Note: The second ***M1*** and second ***A1*** are independent of the first ***M1*** and the first ***A1***.

$= \frac{\mathrm{e}^2}{2} - \mathrm{e} - \frac{1}{2}\left(= \frac{\mathrm{e}^2 - 2\mathrm{e} - 1}{2}\right)$ ***A1***

[5 marks]

(e) (i) **METHOD 1**

consider for example $h(x) = x - 1 - \ln x$

$h(1) = 0$ and $h'(x) = 1 - \frac{1}{x}$ ***(A1)***

as $h'(x) \geq 0$ for $x \geq 1$, then $h(x) \geq 0$ for $x \geq 1$ ***R1***

as $h'(x) \leq 0$ for $0 < x \leq 1$, then $h(x) \geq 0$ for $0 < x \leq 1$ ***R1***

so $g(x) \leq x - 1,\ x \in \mathbb{R}^+$ ***AG***

METHOD 2

$g''(x) = -\frac{1}{x^2}$ ***A1***

$g''(x) < 0$ (concave down) for $x \in \mathbb{R}^+$ ***R1***

the graph of $y = g(x)$ is below its tangent ($y = x - 1$ at $x = 1$) ***R1***

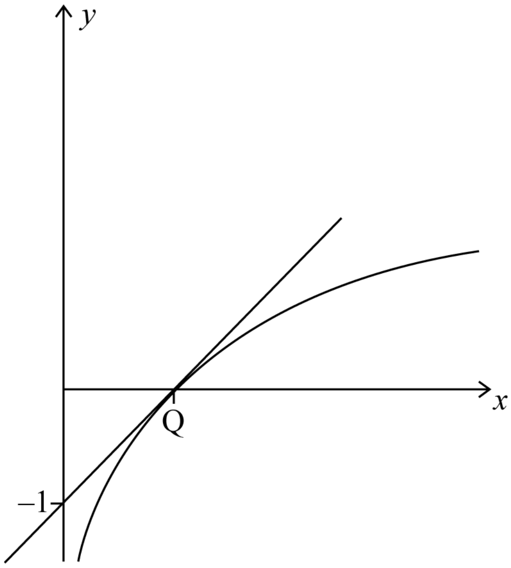
so $g(x) \leq x - 1,\ x \in \mathbb{R}^+$ ***AG***

Note: The reasoning may be supported by drawn graphical arguments.

continued ...

Question 11 continued

METHOD 3



clear correct graphs of $y = x - 1$ and $\ln x$ for $x > 0$ ***A1A1***

statement to the effect that the graph of $\ln x$ is below the graph of its tangent at $x = 1$ ***R1AG***

(ii) replacing x by $\frac{1}{x}$ to obtain $\ln\left(\frac{1}{x}\right) \le \frac{1}{x} - 1 \left(= \frac{1-x}{x}\right)$ ***M1***

$-\ln x \le \frac{1}{x} - 1 \left(= \frac{1-x}{x}\right)$ ***(A1)***

$\ln x \ge 1 - \frac{1}{x} \left(= \frac{x-1}{x}\right)$ ***A1***

so $\frac{x-1}{x} \le g(x),\ x \in \mathbb{R}^+$ ***AG***

[6 marks]

Total [23 marks]

12. (a) (i) $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC}$ ***(M1)***

$= \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$ ***A1***

(ii) $\overrightarrow{BM} = \overrightarrow{BA} + \overrightarrow{AM}$ ***M1***

$= \boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$ ***A1***

$\overrightarrow{BM} = \frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$ ***AG***

[4 marks]

(b) (i) $\overrightarrow{RA} = \frac{1}{3}\overrightarrow{BA}$

$= \frac{1}{3}(\boldsymbol{a} - \boldsymbol{b})$ ***A1***

(ii) $\overrightarrow{RT} = \frac{2}{3}\overrightarrow{RS}$

$= \frac{2}{3}\left(\overrightarrow{RA} + \overrightarrow{AS}\right)$ ***(M1)***

$= \frac{2}{3}\left(\frac{1}{3}(\boldsymbol{a} - \boldsymbol{b}) + \frac{2}{3}(\boldsymbol{c} - \boldsymbol{a})\right)$ or equivalent. ***A1A1***

$= \frac{2}{9}(\boldsymbol{a} - \boldsymbol{b}) + \frac{4}{9}(\boldsymbol{c} - \boldsymbol{a})$ ***A1***

$\overrightarrow{RT} = -\frac{2}{9}\boldsymbol{a} - \frac{2}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$ ***AG***

[5 marks]

(c) $\overrightarrow{BT} = \overrightarrow{BR} + \overrightarrow{RT}$

$= \frac{2}{3}\overrightarrow{BA} + \overrightarrow{RT}$ ***(M1)***

$= \frac{2}{3}\boldsymbol{a} - \frac{2}{3}\boldsymbol{b} - \frac{2}{9}\boldsymbol{a} - \frac{2}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$ ***A1***

$\overrightarrow{BT} = \frac{8}{9}\left(\frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}\right)$ ***A1***

point B is common to $\overrightarrow{BT}$ and $\overrightarrow{BM}$ and $\overrightarrow{BT} = \frac{8}{9}\overrightarrow{BM}$ ***R1R1***

so T lies on [BM] ***AG***

[5 marks]

Total [14 marks]

13. (a) (i) METHOD 1

$(1+\mathrm{i}\tan\theta)^n+(1-\mathrm{i}\tan\theta)^n=(1+\mathrm{i}\frac{\sin\theta}{\cos\theta})^n+(1-\mathrm{i}\frac{\sin\theta}{\cos\theta})^n$ ***M1***

$=\left(\frac{\cos\theta+i\sin\theta}{\cos\theta}\right)^n+\left(\frac{\cos\theta-i\sin\theta}{\cos\theta}\right)^n$ ***A1***

by de Moivre's theorem ***(M1)***

$\left(\frac{\cos\theta+i\sin\theta}{\cos\theta}\right)^n=\frac{\cos n\theta+i\sin n\theta}{\cos^n\theta}$ ***A1***

recognition that $\cos\theta-i\sin\theta$ is the complex conjugate of $\cos\theta+i\sin\theta$ ***(R1)***

use of the fact that the operation of complex conjugation commutes with the operation of raising to an integer power:

$\left(\frac{\cos\theta-i\sin\theta}{\cos\theta}\right)^n=\frac{\cos n\theta-i\sin n\theta}{\cos^n\theta}$ ***A1***

$(1+\mathrm{i}\tan\theta)^n+(1-\mathrm{i}\tan\theta)^n=\frac{2\cos n\theta}{\cos^n\theta}$ ***AG***

METHOD 2

$(1+\mathrm{i}\tan\theta)^n+(1-\mathrm{i}\tan\theta)^n=(1+\mathrm{i}\tan\theta)^n+(1+i\tan(-\theta))^n$ ***(M1)***

$=\frac{(\cos\theta+i\sin\theta)^n}{\cos^n\theta}+\frac{(\cos(-\theta)+i\sin(-\theta))^n}{\cos^n\theta}$ ***M1A1***

Note: Award ***M1*** for converting to cosine and sine terms.

use of de Moivre's theorem ***(M1)***

$=\frac{1}{\cos^n\theta}(\cos n\theta+\mathrm{i}\sin n\theta+\cos(-n\theta)+\mathrm{i}\sin(-n\theta))$ ***A1***

$=\frac{2\cos n\theta}{\cos^n\theta}$ as $\cos(-n\theta)=\cos n\theta$ and $\sin(-n\theta)=-\sin n\theta$ ***R1AG***

continued ...

Question 13 continued

(ii) $\left(1+\mathrm{i}\tan\frac{3\pi}{8}\right)^4+\left(1-\mathrm{i}\tan\frac{3\pi}{8}\right)^4=\frac{2\cos\left(4\times\frac{3\pi}{8}\right)}{\cos^4\frac{3\pi}{8}}$ ***(A1)***

$=\frac{2\cos\frac{3\pi}{2}}{\cos^4\frac{3\pi}{8}}$ ***A1***

$=0$ as $\cos\frac{3\pi}{2}=0$ ***R1***

Note: The above working could involve theta and the solution of $\cos(4\theta)=0$.

so $\mathrm{i}\tan\frac{3\pi}{8}$ is a root of the equation ***AG***

(iii) either $-\mathrm{i}\tan\frac{3\pi}{8}$ or $-\mathrm{i}\tan\frac{\pi}{8}$ or $\mathrm{i}\tan\frac{\pi}{8}$ ***A1***

Note: Accept $\mathrm{i}\tan\frac{5\pi}{8}$ or $\mathrm{i}\tan\frac{7\pi}{8}$.
Accept $-\left(1+\sqrt{2}\right)\mathrm{i}$ or $\left(1-\sqrt{2}\right)\mathrm{i}$ or $\left(-1+\sqrt{2}\right)\mathrm{i}$.

[10 marks]

(b) (i) $\tan\frac{\pi}{4}=\frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$ ***(M1)***

$\tan^2\frac{\pi}{8}+2\tan\frac{\pi}{8}-1=0$ ***A1***

let $t=\tan\frac{\pi}{8}$

attempting to solve $t^2+2t-1=0$ for t ***M1***

$t=-1\pm\sqrt{2}$ ***A1***

$\frac{\pi}{8}$ is a first quadrant angle and tan is positive in this quadrant, so

$\tan\frac{\pi}{8}>0$ ***R1***

$\tan\frac{\pi}{8}=\sqrt{2}-1$ ***AG***

continued …

Question 13 continued

(ii) $\cos 4x = 2\cos^2 2x - 1$ ***A1***

$= 2\left(2\cos^2 x - 1\right)^2 - 1$ ***M1***

$= 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1$ ***A1***

$= 8\cos^4 x - 8\cos^2 x + 1$ ***AG***

Note: Accept equivalent complex number derivation.

(iii) $\int_0^{\frac{\pi}{8}} \frac{2\cos 4x}{\cos^2 x}\,dx = 2\int_0^{\frac{\pi}{8}} \frac{8\cos^4 x - 8\cos^2 x + 1}{\cos^2 x}\,dx$

$= 2\int_0^{\frac{\pi}{8}} 8\cos^2 x - 8 + \sec^2 x\,dx$ ***M1***

Note: The ***M1*** is for an integrand involving no fractions.

use of $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$ ***M1***

$= 2\int_0^{\frac{\pi}{8}} 4\cos 2x - 4 + \sec^2 x\,dx$ ***A1***

$= \left[4\sin 2x - 8x + 2\tan x\right]_0^{\frac{\pi}{8}}$ ***A1***

$= 4\sqrt{2} - \pi - 2$ (or equivalent) ***A1***

[13 marks]

Total [23 marks]