Chapter 11

Probability

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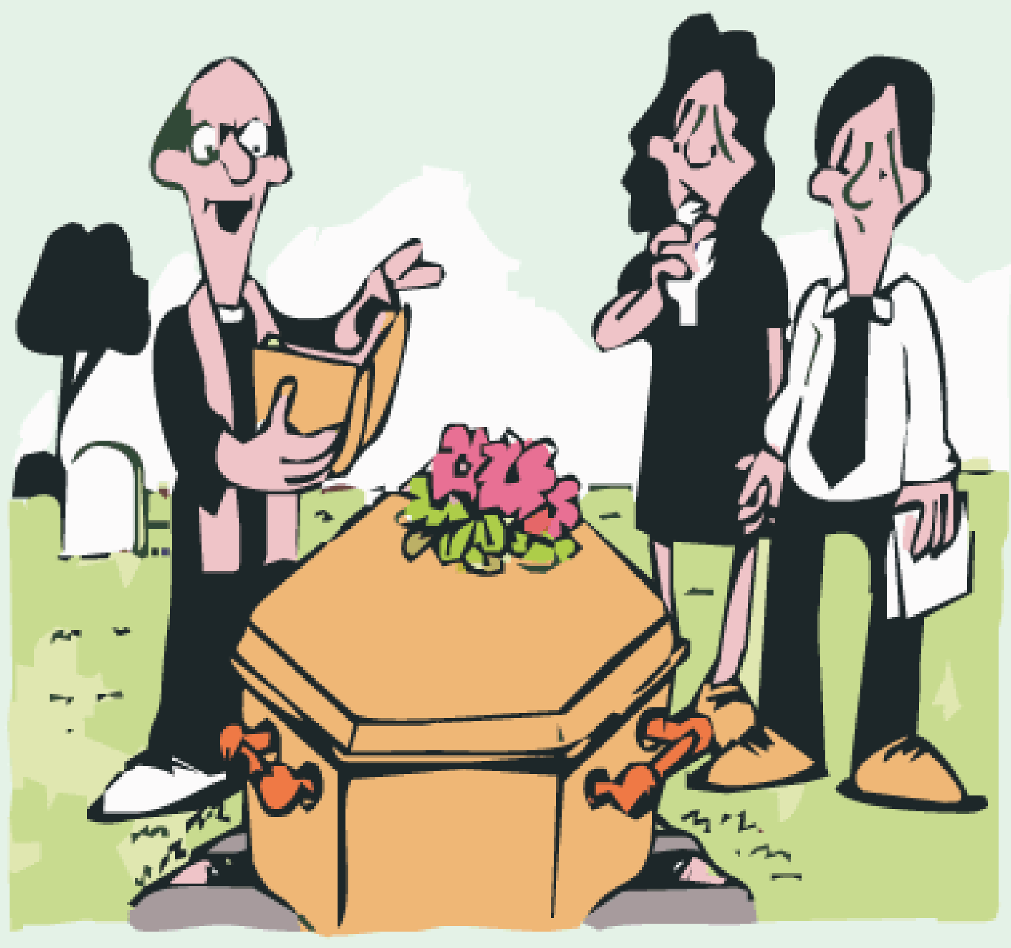
OPENING PROBLEM

In the late 17th century, English mathematicians compiled and analysed mortality tables which showed the number of people who died at different ages. From these tables they could estimate the probability that a person would be alive at a future date. This led to the establishment of the first life insurance company in 1699.

Life insurance companies use statistics on **life expectancy** and **death rates** to calculate the premiums to charge people who insure with them.

The **life table** shown is from Australia. It shows the number of people out of 100 000 births who survive to different ages, and the expected years of remaining life at each age.

For example, we can see that out of 100 000 births, 98 052 males are expected to survive to the age of 20, and from that age the survivors are expected to live a further 54.35 years.



LIFE TABLE

Male			Female		
Age	*Number surviving*	*Expected remaining life*	*Age*	*Number surviving*	*Expected remaining life*
0	100 000	73.03	0	100 000	79.46
5	98 809	68.90	5	99 307	75.15
10	98 698	63.97	10	99 125	70.22
15	98 555	59.06	15	98 956	65.27
20	98 052	54.35	20	98 758	60.40
25	97 325	49.74	25	98 516	55.54
30	96 688	45.05	30	98 278	50.67
35	96 080	40.32	35	98 002	45.80
40	95 366	35.60	40	97 615	40.97
45	94 323	30.95	45	96 997	36.22
50	92 709	26.45	50	95 945	31.59
55	89 891	22.20	55	94 285	27.10
60	85 198	18.27	60	91 774	22.76
65	78 123	14.69	65	87 923	18.64
70	67 798	11.52	70	81 924	14.81
75	53 942	8.82	75	72 656	11.36
80	37 532	6.56	80	58 966	8.38
85	20 998	4.79	85	40 842	5.97
90	8416	3.49	90	21 404	4.12
95	2098	2.68	95	7004	3.00
99	482	2.23	99	1953	2.36

Things to think about:

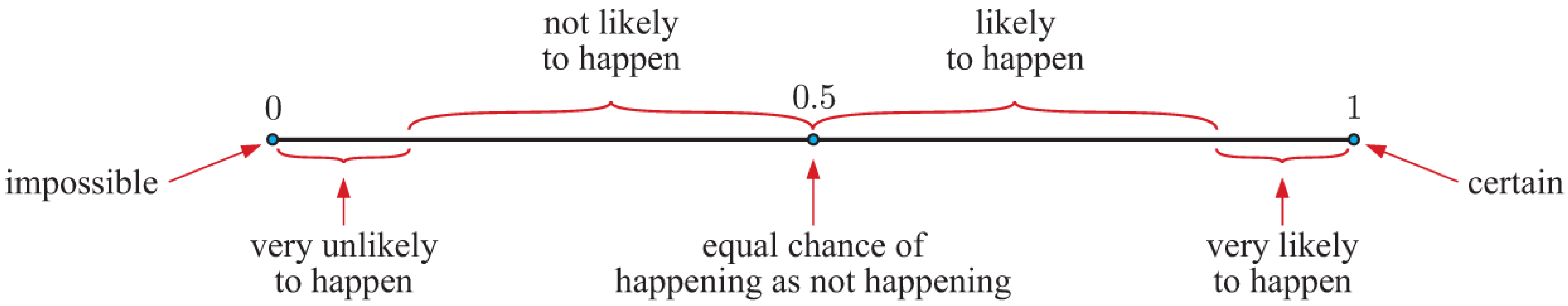
- **a** Can you use the life table to estimate how many years you can expect to live?
- **b** Can you estimate the chance that a new-born boy or girl will reach the age of 15?
- **c** Can the table be used to estimate the chance that:
  - **i** a 15 year old boy *will* reach age 75
  - **ii** a 15 year old girl *will not* reach age 75?
- **d** In general, do males or females live longer?
- **e** An insurance company sells policies to people to insure them against death over a 30-year period. If the person dies during this period, the beneficiaries receive the agreed payout figure. Why are such policies cheaper to take out for a 20 year old than for a 50 year old?
- **f** How many of your classmates would you expect to be alive and able to attend a 30 year class reunion?
- **g** How do you think life tables would compare between countries?

In the real world, we cannot predict with certainty what will happen in the future. Understanding the **chance** or likelihood of something happening is extremely useful for us to make decisions.

In mathematics, the chance of an event occurring is assigned a number between 0 and 1 inclusive. We call this number a **probability**.

> An **impossible** event has 0% chance of happening, and is assigned the probability 0.
>
> A **certain** event has 100% chance of happening, and is assigned the probability 1.
>
> All other events can be assigned a probability between 0 and 1.

This number line shows how we could interpret different probabilities:



We can determine probabilities based on:

- the results of an experiment
- what we theoretically expect to happen.

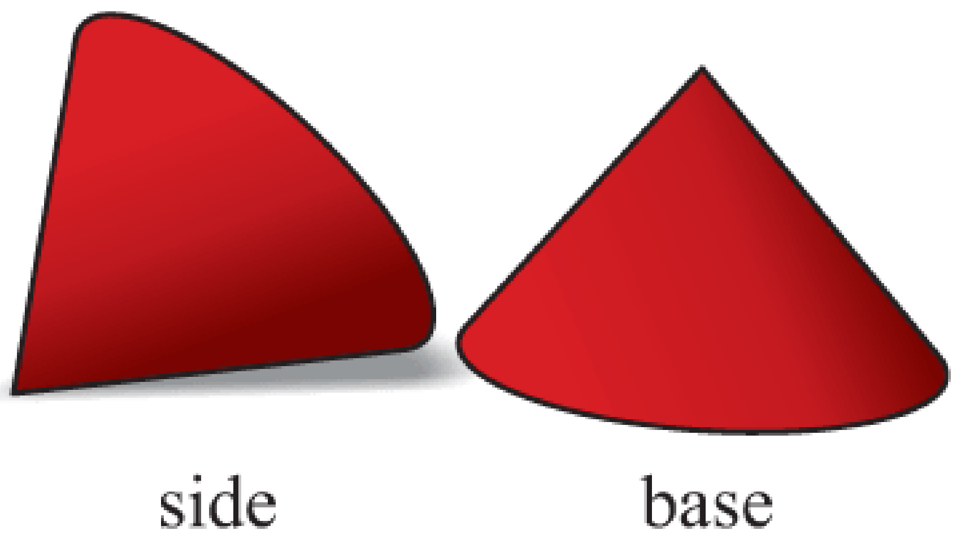
Probability theory is applied in physical and biological sciences, economics, politics, sport, quality control, production planning, and many other areas.

A EXPERIMENTAL PROBABILITY

In experiments involving chance we use the following terms to talk about what we are doing and the results we obtain:

- The **number of trials** is the total number of times the experiment is repeated.
- The **outcomes** are the different results possible for one trial of the experiment.
- The **frequency** of a particular outcome is the number of times that this outcome is observed.
- The **relative frequency** of an outcome is the frequency of that outcome expressed as a fraction or percentage of the total number of trials.

For example, when a small plastic cone was tossed into the air 279 times it fell on its *side* 183 times and on its *base* 96 times.



We say:

- the number of trials is 279
- the outcomes are *side* and *base*
- the frequencies of *side* and *base* are 183 and 96 respectively
- the relative frequencies of *side* and *base* are $\frac{183}{279} \approx 0.656$ and $\frac{96}{279} \approx 0.344$ respectively.

In the absence of any further data, the relative frequency of each event is our best estimate of the probability of that event occurring.

experimental probability = relative frequency

In this case: $\text{P}(side) \approx$ the experimental probability the cone will land on its side when tossed
≈ 0.656
$\text{P}(base) \approx$ the experimental probability the cone will land on its base when tossed
≈ 0.344

INVESTIGATION 1 DICE ROLLING EXPERIMENT

You will need:

At least one normal six-sided die with numbers 1 to 6 on its faces. Several dice would be useful to speed up the experiment.

WORKSHEET

What to do:

1 List the possible outcomes for the uppermost face when the die is rolled.

2 Discuss what you would expect the relative frequency of rolling a 2 to be when a die is rolled many times.

3 Roll a die 20 times, and count the number of times a 2 is rolled. Hence, calculate the relative frequency of rolling a 2.

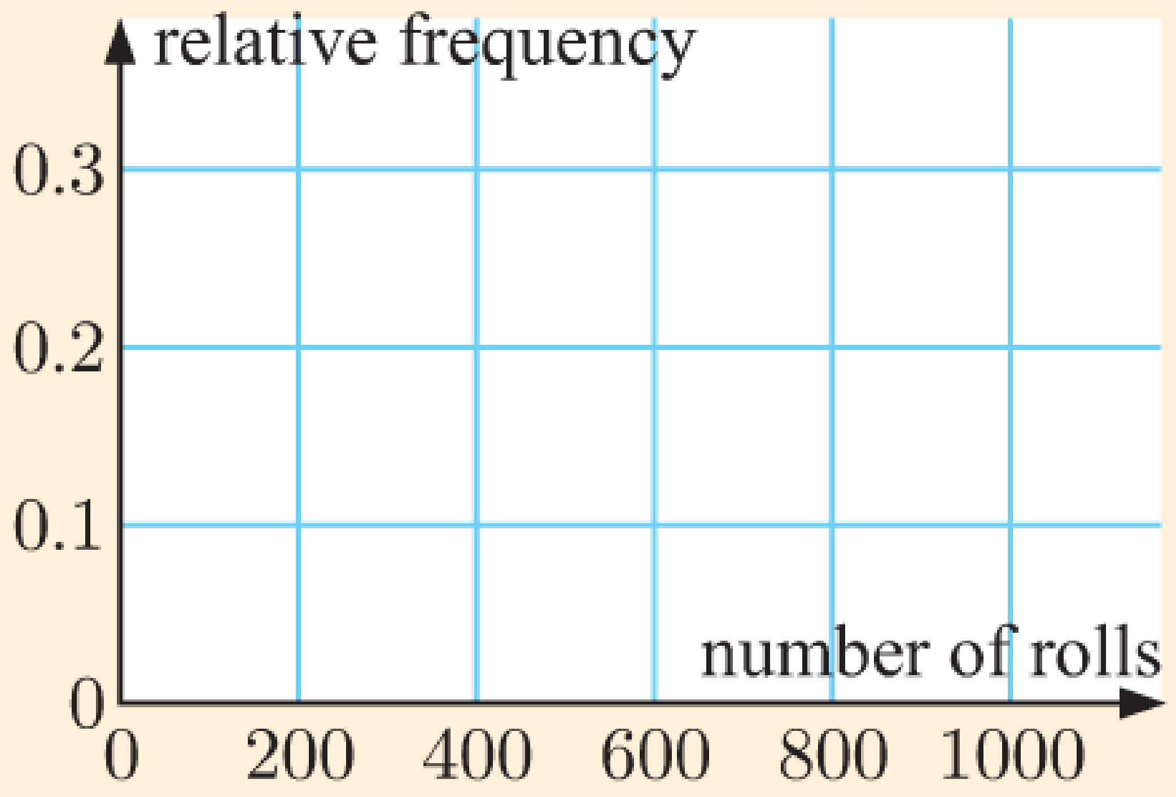
4 Pool your results with another student, so in total you have data for 40 rolls. Calculate the relative frequency of rolling a 2 for 40 rolls.

5 Use the simulation to roll a die 60, 100, 200, 300, 500, and 1000 times. In each case, calculate the relative frequency of rolling a 2.

SIMULATION

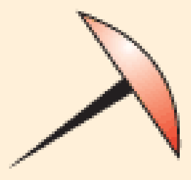
6 Plot a graph of relative frequency against the number of rolls. What do you notice?

7 What do you think will happen to the relative frequency of rolling a 2 as the number of rolls increases?



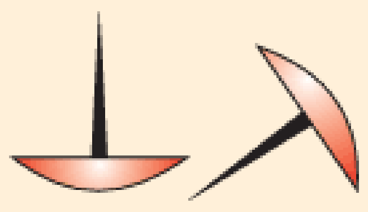
The larger the number of trials, the more confident we are that the estimated probability is accurate.

INVESTIGATION 2 TOSSING DRAWING PINS

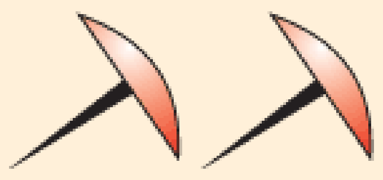
If a drawing pin tossed in the air finishes  we say it has finished on its *back*. If it finishes  we say it has finished on its *side*.

If two drawing pins are tossed simultaneously, the possible results are:

two backs



back and side



two sides

What to do:

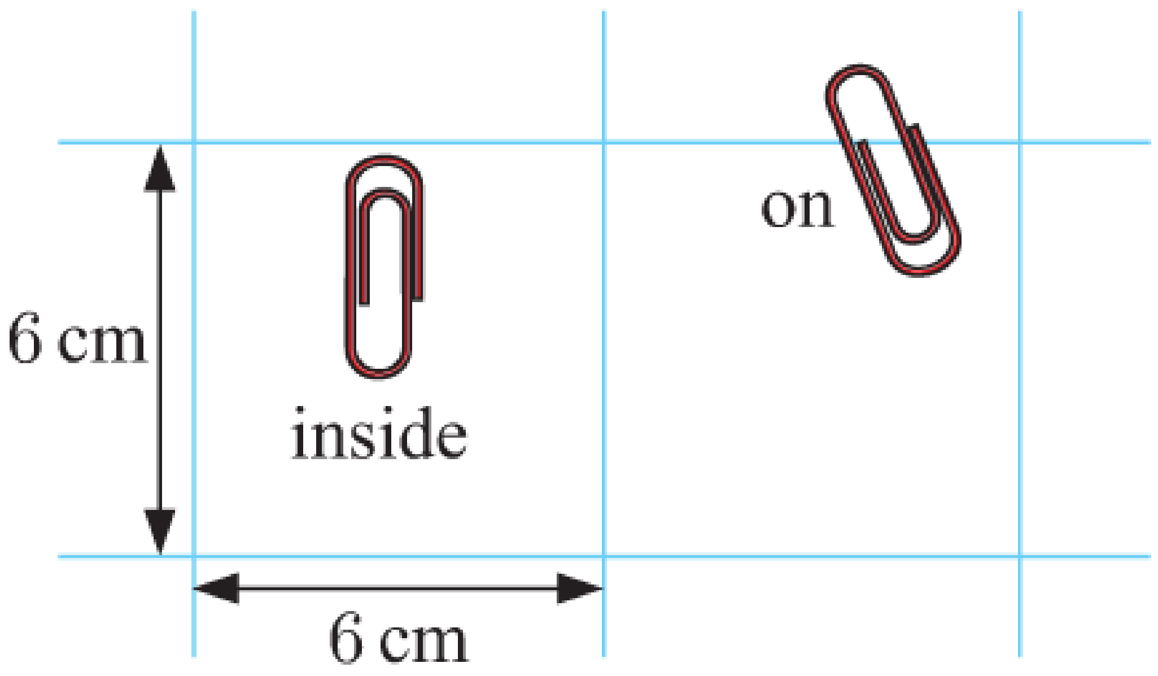
1. Obtain two drawing pins of the same shape and size. Toss the pair 80 times and record the outcomes in a table.
2. Obtain relative frequencies (experimental probabilities) for each of the three outcomes.
3. Pool your results with four other people using drawing pins with the same shape. Hence obtain experimental probabilities from 400 tosses.
4. Which gives the more reliable probability estimates, your results or the whole group's? Explain your answer.

In some situations, such as in the **Investigation** above, experimentation is the only way of obtaining probabilities.

EXERCISE 11A

1 When a batch of 145 paper clips was dropped onto 6 cm by 6 cm squared paper, it was observed that 113 fell completely inside squares and 32 landed on a grid line. Find, to 2 decimal places, the experimental probability of a clip falling:

a inside a square **b** on a line.



2

Length	*Frequency*
0 - 19	17
20 - 39	38
40 - 59	19
60+	4

Jose surveyed the length of TV commercials (in seconds). Find, to 3 decimal places, the experimental probability that the next TV commercial will last:

a 20 to 39 seconds **b** at least one minute

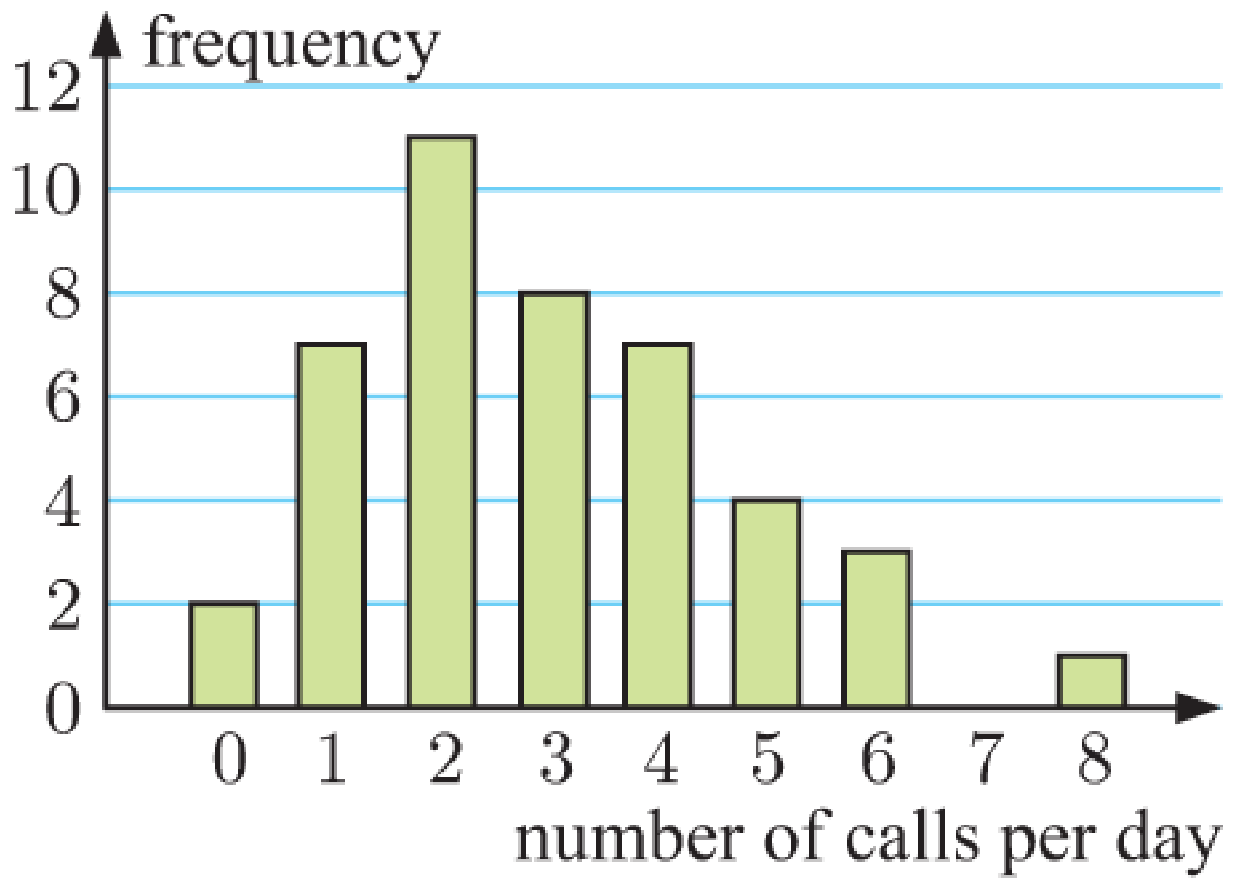
c between 20 and 59 seconds (inclusive).

3 Betul records the number of phone calls she receives over a period of consecutive days.

a For how many days did the survey last?

b Estimate the probability that tomorrow Betul will receive:

  i no phone calls
  ii 5 or more phone calls
  iii less than 3 phone calls.



4 Pat does a lot of travelling in her car, and she keeps records on how often she fills her car with petrol. The table alongside shows the frequencies of the number of days between refills. Estimate the probability that:

a there is a four day gap between refills

b there is at least a four day gap between refills.

Days between refills	*Frequency*
1	37
2	81
3	48
4	17
5	6
6	1

Example 1 **Self Tutor**

The table below shows the number of short-term visitors coming to Australia in the period April - June 2018, and the main reason for their visit.

Short-Term Visitors to Australia

Main reason for journey	*April 2018*	*May 2018*	*June 2018*
Convention/conference	8300	14 800	8800
Business	27 200	33 900	31 900
Visiting friends/relatives	77 500	52 700	59 900
Holiday	159 300	119 300	156 500
Employment	4200	4300	5500
Education	9800	7900	12 500
Other	35 200	28 000	33 200
Total	321 500	260 900	308 300

a Estimate the probability that a person who visits in June is on holiday.

b Estimate the probability that a person who came to Australia in the period April - June 2018 arrived in May.

c Lars arrived in Australia in April, May, or June 2018 to visit his brother. Estimate the probability that he arrived in April.

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a $$\text{P(on holiday in June)} \approx \frac{156\,500}{308\,300} \begin{array}{l} \longleftarrow \text{number on holiday in June} \\ \longleftarrow \text{total number for June} \end{array}$$
$$\approx 0.508$$

b $321\,500 + 260\,900 + 308\,300 = 890\,700$ short-term visitors arrived during the three months.

$$\therefore \quad \text{P(arrives in May)} \approx \frac{260\,900}{890\,700} \approx 0.293$$

c $77\,500 + 52\,700 + 59\,900 = 190\,100$ people came to Australia to visit friends or relatives during this period.

$$\therefore \quad \text{P(arrived in April)} \approx \frac{77\,500}{190\,100} \begin{array}{l} \longleftarrow \text{number visiting friends or relatives in April} \\ \longleftarrow \text{total number visiting friends or relatives over April, May, and June} \end{array}$$
$$\approx 0.408$$

5 The table shows data from a survey conducted at five schools to study the rate of smoking among 15 year old students.

School	*Number of 15 year olds*		*Number of smokers*	
	Male	*Female*	*Male*	*Female*
A	45	51	10	11
B	36	42	9	6
C	52	49	13	13
D	28	33	9	10
E	40	39	7	4
Total	201	214	48	44

a Estimate the probability that a randomly chosen female 15 year old student at school **C** is a smoker.

b Estimate the probability that a randomly chosen 15 year old student at school **E** is *not* a smoker.

c If a 15 year old is chosen at random from the five schools, estimate the probability that he or she is a smoker.

6 This table describes the complaints received by a telecommunications ombudsman concerning internet services over a four year period.

Reason	2014/15	2015/16	2016/17	2017/18
Access	585	1127	2545	1612
Billing	1822	2102	3136	3582
Contracts	242	440	719	836
Credit control	3	44	118	136
Customer Service	12	282	1181	1940
Disconnection	n/a	n/a	n/a	248
Faults	86	79	120	384
Privacy	93	86	57	60
Provision	172	122	209	311
Total	3015	4282	8085	9109

Find the probability that a complaint received:

- **a** in 2016/17 was about customer service
- **b** at any time during the 4 year period was related to billing
- **c** in 2017/18 did *not* relate to either billing or faults.

7 This table provides data on the daily maximum temperatures in Barcelona during summer.

Summer Temperatures in Barcelona	*Month*		
	June	*July*	*Aug*
Mean days max. $\geqslant 40°\text{C}$	0.3	1.2	0.7
Mean days max. $\geqslant 35°\text{C}$	3.0	5.8	5.3
Mean days max. $\geqslant 30°\text{C}$	9.4	12.3	12.0

- **a** Estimate the probability that on an August day in Barcelona, the maximum temperature will be:
  - **i** 35°C or higher
  - **ii** less than 30°C.
- **b** Estimate the probability that on any summer day in Barcelona, the temperature will be 30°C or higher.
- **c** It is a 40°C summer day in Barcelona. Estimate the probability that the month is July.

B TWO-WAY TABLES

Two-way tables are tables which compare two categorical variables.

For example, the teachers at a school were asked which mode of transport they used to travel to school. Their responses are summarised in the table below. The variables are *gender* and *mode of transport*.

	Car	Bicycle	Bus
Male	37	10	10
Female	30	5	13

13 female teachers catch the bus to school.

In the following Example we will see how these tables can be used to estimate probabilities. To help us, we extend the table to include totals for each row and column.

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3 A small hotel in London has kept a record of all room bookings made for the year. The results are summarised in the two-way table.

	Single	*Double*	*Family*
Peak season	225	420	98
Off-peak season	148	292	52

a Estimate the probability that the next randomly selected booking will be:

i in the peak season
ii a single room in the off-peak season
iii a single or a double room
iv during the peak season or a family room.

b A randomly selected booking is in the off-peak season. Estimate the probability that it is a family room.

c A randomly selected booking is *not* a single room. Estimate the probability that it is in the peak season.

C SAMPLE SPACE AND EVENTS

The **sample space** U is the set of all possible outcomes of an experiment.

An **event** is a set of outcomes in the sample space that have a particular property.

You should notice that we are applying the set theory we studied in **Chapter 2**:

- the sample space is the **universal set** U
- the outcomes are the **elements** of the sample space
- events are **subsets** of the sample space.

We can therefore **list** the outcomes in the sample space and in events using **set notation**, and illustrate them with a **Venn diagram**.

COMPLEMENTARY EVENTS

Two events are **complementary** if exactly one of the events *must* occur.
If A is an event, then A' is the complementary event of A, or "not A".

Example 3 **Self Tutor**

A normal six-sided die is rolled once. Let A be the event that a prime number is rolled.

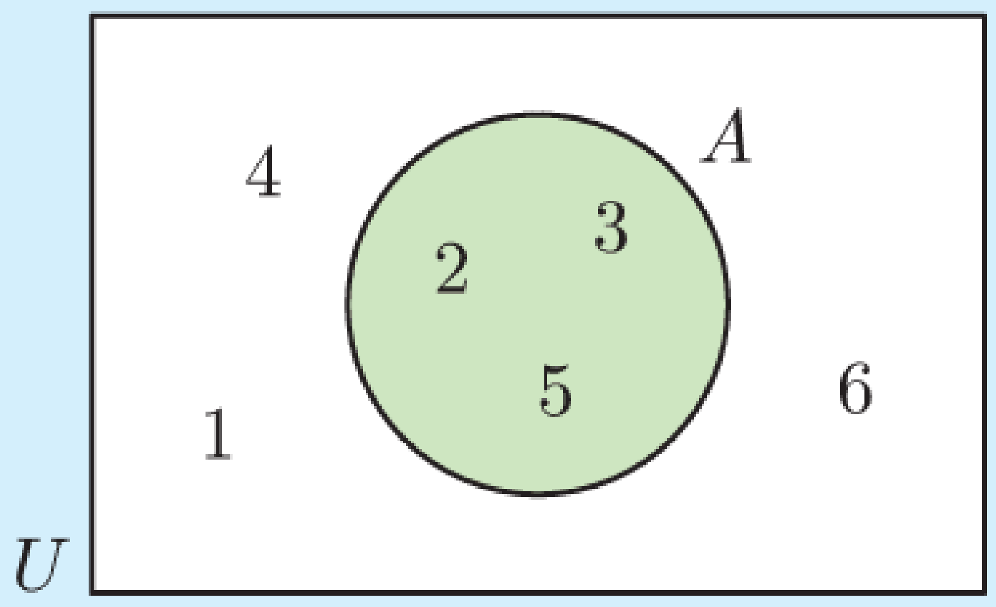
a Use set notation to list the outcomes in:

i the sample space U **ii** A **iii** A'.

b Draw a Venn diagram to illustrate the sample space.

a **i** $U = \{1, 2, 3, 4, 5, 6\}$
ii $A = \{2, 3, 5\}$
iii $A' = \{1, 4, 6\}$

b



2-DIMENSIONAL GRIDS AND TREE DIAGRAMS

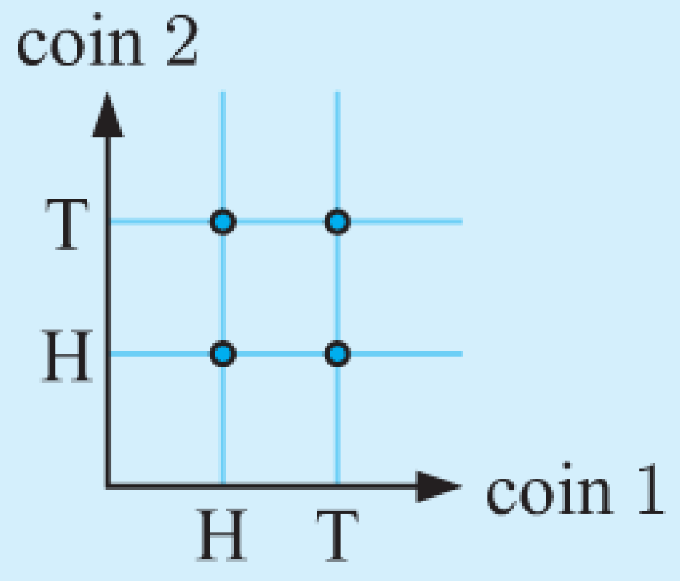
When an experiment involves more than one operation we can still list the sample space. However, it is often more efficient to illustrate the sample space on a **2-dimensional grid** or using a **tree diagram**.

Example 4 Self Tutor

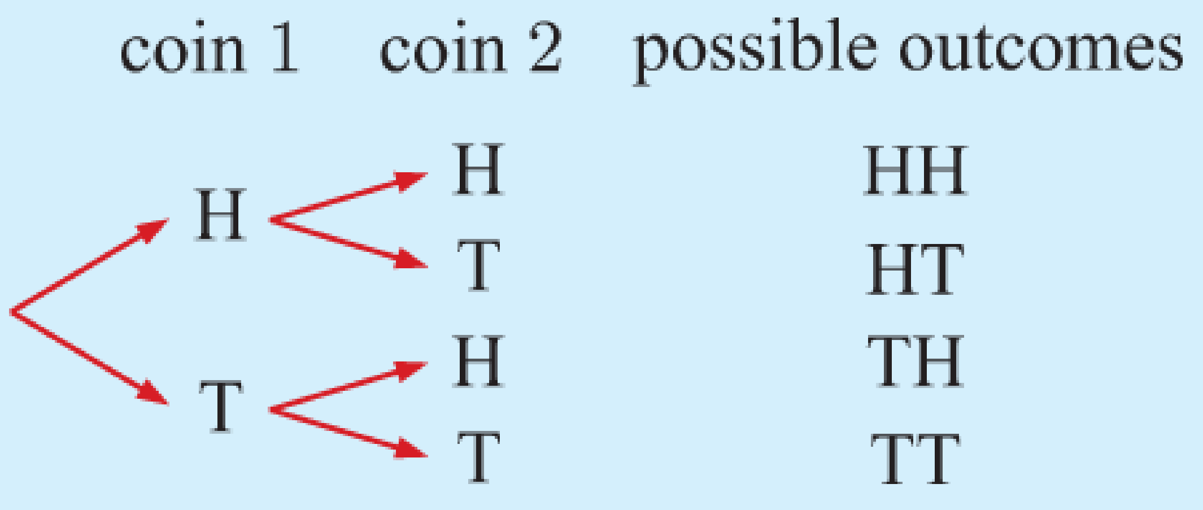
Illustrate the possible outcomes when two coins are tossed using:

a a 2-dimensional grid **b** a tree diagram.

a



b



Notice in the Example that each outcome in the sample space {HH, HT, TH, TT} is represented by:

- a point on the grid
- a “branch” on the tree diagram.

EXERCISE 11C

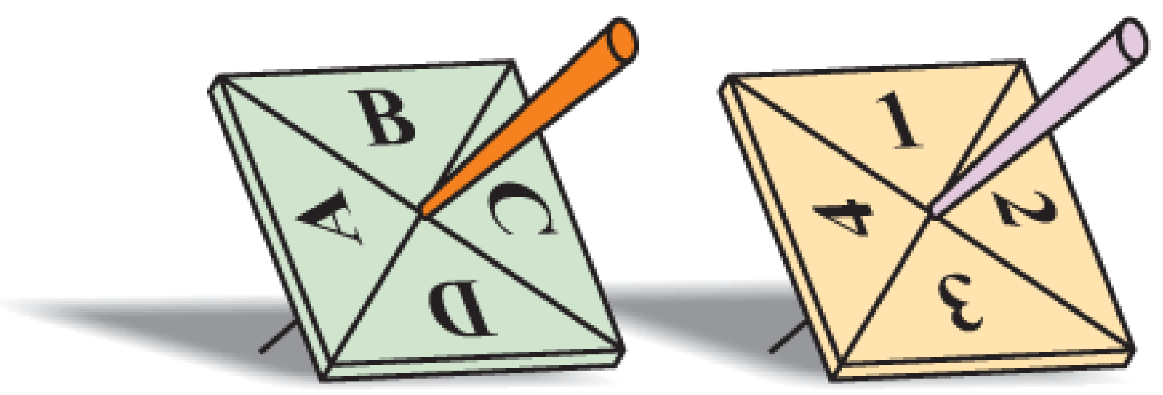
1 List using set notation, the sample space for:

  a twirling a square spinner labelled A, B, C, D
  b spinning a wheel with sectors labelled with the numbers 1 to 8
  c the sexes of a 2-child family.

2 One ticket is drawn from a box containing tickets labelled with the numbers 1 to 16 inclusive.

  a Write down the sample space U.
  b List the outcomes in the following events:
    i $A =$ the ticket's number is a multiple of 4
    ii $B =$ the ticket's number is a perfect square.
  c Draw a Venn diagram to illustrate the sample space U and the events A and B.

3 Illustrate on a 2-dimensional grid the sample space for:



  a rolling a die and tossing a coin simultaneously
  b rolling two dice
  c rolling a die and spinning a spinner with sides A, B, C, D
  d twirling two square spinners, one labelled A, B, C, D and the other 1, 2, 3, 4.

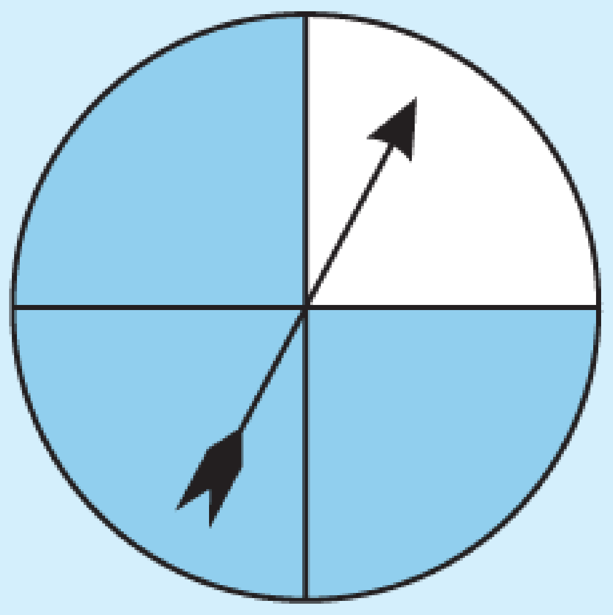
4 Illustrate on a tree diagram the sample space for:

  a tossing a 5-cent and a 10-cent coin simultaneously
  b tossing a coin and twirling an equilateral triangular spinner labelled A, B, C
  c twirling two equilateral triangular spinners labelled 1, 2, 3, and X, Y, Z
  d drawing two tickets from a hat containing a large number of pink, blue, and white tickets.

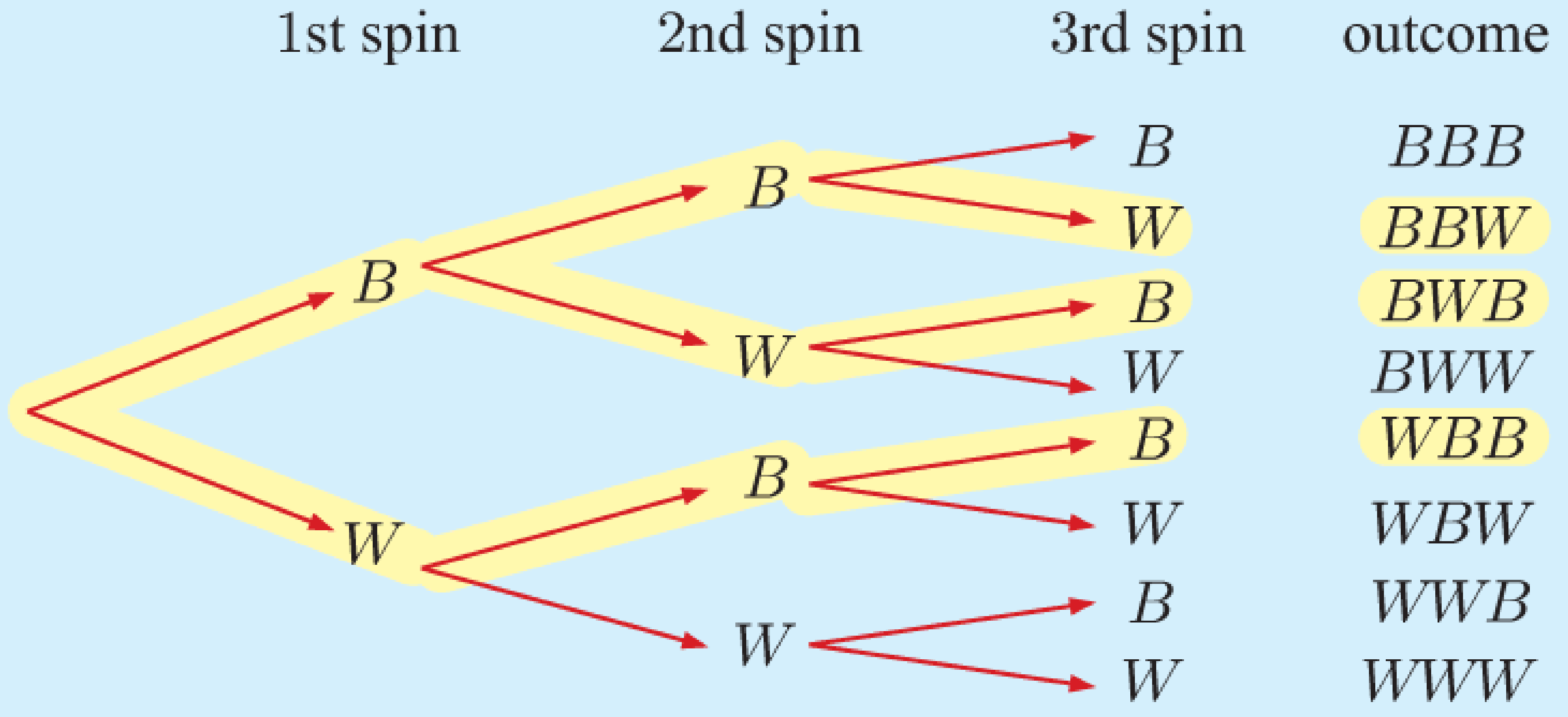
Example 5 **Self Tutor**

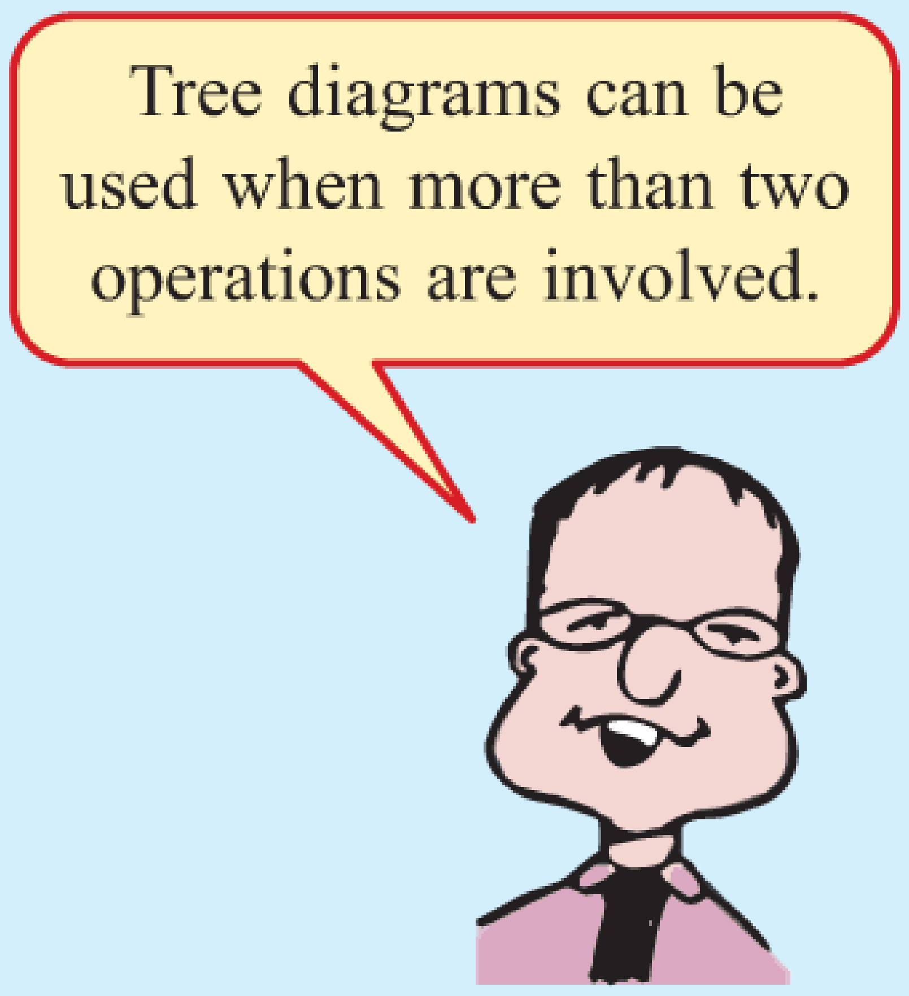
Use a tree diagram to illustrate the possible outcomes when this spinner is spun three times.

Highlight the outcomes corresponding to the event "obtaining blue twice".



Let B represent blue and W represent white.



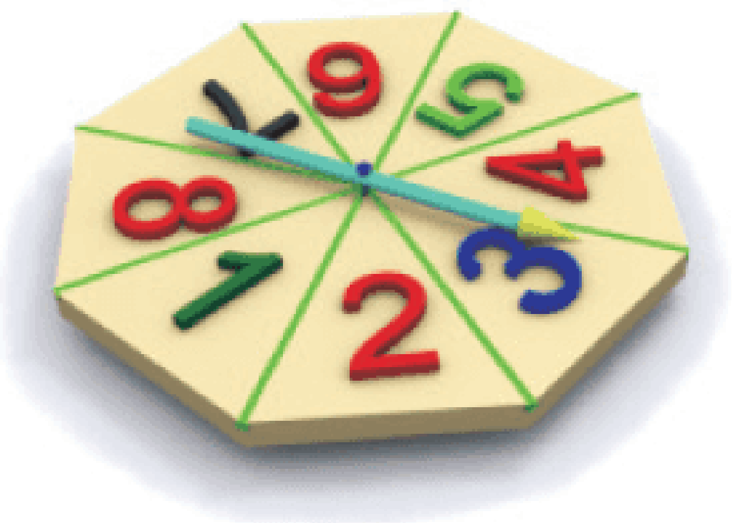


5 From the whole numbers 1 to 7, Adam and Bill each select a number. Illustrate the sample space on a 2-dimensional grid. Circle the outcomes in the event "Adam and Bill's numbers are the same".

6 Suppose three coins are tossed simultaneously. Draw a tree diagram to illustrate the sample space. Highlight the outcomes corresponding to the event "getting at least 1 head".

D THEORETICAL PROBABILITY

The sample space when spinning the octagonal spinner shown is $\{1, 2, 3, 4, 5, 6, 7, 8\}$.



Since the spinner is symmetrical, we expect that each of the eight outcomes will be **equally likely** to occur. We say that the **theoretical probability** of any particular outcome occurring is 1 in 8, or $\frac{1}{8}$.

If a sample space has n outcomes which are **equally likely** to occur when the experiment is performed once, then each outcome has probability $\dfrac{1}{n}$ of occurring.

Consider the event of *spinning a prime number* with the spinner above. Of the 8 possible outcomes, the four outcomes 2, 3, 5, and 7 all correspond to this event. So, the probability of rolling a prime number is 4 in 8, or $\frac{4}{8}$.

When the outcomes of an experiment are equally likely, the probability that an event A occurs is:

$$\mathbf{P}(A) = \frac{\textbf{number of outcomes corresponding to } A}{\textbf{number of outcomes in the sample space}} = \frac{n(A)}{n(U)}$$

Example 6

Self Tutor

A ticket is *randomly selected* from a basket containing 3 green, 4 yellow, and 5 blue tickets. Determine the probability of getting:

a a green ticket

b a green or yellow ticket

c an orange ticket

d a green, yellow, or blue ticket.

There are $3+4+5=12$ tickets which could be selected with equal chance.

a $\text{P}(\text{G}) = \frac{3}{12} = \frac{1}{4}$

b $\text{P}(\text{G or Y}) = \frac{3+4}{12} = \frac{7}{12}$

c $\text{P}(\text{O}) = \frac{0}{12} = 0$

d $\text{P}(\text{G, Y, or B}) = \frac{3+4+5}{12} = 1$

Example 7

Self Tutor

An ordinary six-sided die is rolled once. Determine the chance of:

a getting a 6

b not getting a 6

c getting a 1 or 2

d not getting a 1 or 2.

The sample space is $\{1, 2, 3, 4, 5, 6\}$.

a $\text{P}(6) = \frac{1}{6}$

b $\text{P}(\text{not a } 6) = \text{P}(1, 2, 3, 4, \text{ or } 5) = \frac{5}{6}$

c $\text{P}(1 \text{ or } 2) = \frac{2}{6} = \frac{1}{3}$

d $\text{P}(\text{not a } 1 \text{ or } 2) = \text{P}(3, 4, 5, \text{ or } 6) = \frac{4}{6} = \frac{2}{3}$

In **Example 7**, notice that $\text{P}(6) + \text{P}(\text{not a } 6) = 1$
and that $\text{P}(1 \text{ or } 2) + \text{P}(\text{not a } 1 \text{ or } 2) = 1$.

This is no surprise as "6" and "not a 6" are complementary events. It is certain that exactly one of the events will occur, and impossible for both of them to occur at the same time.

Likewise, "1 or 2" and "not a 1 or 2" are complementary events.

In general, $\mathbf{P(A) + P(A') = 1}$.

EXERCISE 11D

1 A marble is randomly selected from a box containing 5 green, 3 red, and 7 blue marbles. Determine the probability that the marble is:

a red

b green

c blue

d not red

e neither green nor blue

f green or red.

2 A barrel contains tickets numbered from 1 to 200. If a ticket is selected at random, find the probability that the ticket number:

a is even

b is a multiple of 12

c contains a 2.

3 A giant spinner has 36 sectors labelled 1 to 36. Determine the probability that when it is spun, the arrow will land on a sector labelled with:

- **a** a multiple of 4
- **b** a number between 6 and 9 inclusive
- **c** a number greater than 20
- **d** a multiple of 13
- **e** an odd number that is a multiple of 3
- **f** a number containing a 1
- **g** a multiple of both 4 and 6
- **h** a multiple of 4 or 6, or both.

4 Find the probability that a randomly chosen person has his or her next birthday:

- **a** on a Tuesday
- **b** on a weekend
- **c** in July
- **d** in January or February
- **e** in a month containing the letter "a".

5 **a** List the 8 possible 3-child families according to the gender of the children. For example, GGB means "*the first is a girl, the second is a girl, the third is a boy*".

b Assuming that each of these is equally likely to occur, determine the probability that a randomly selected 3-child family consists of:

- **i** all boys
- **ii** all girls
- **iii** boy then girl then girl
- **iv** two girls and a boy
- **v** a girl for the eldest
- **vi** at least one boy.

6 **a** List the 24 different orders in which four people A, B, C, and D may sit in a row.

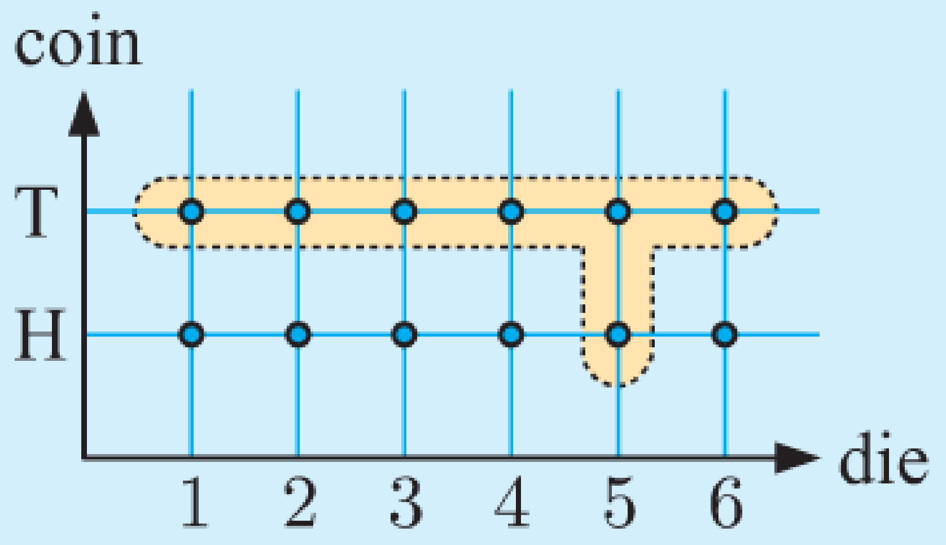
b Determine the probability that when the four people sit at random in a row:

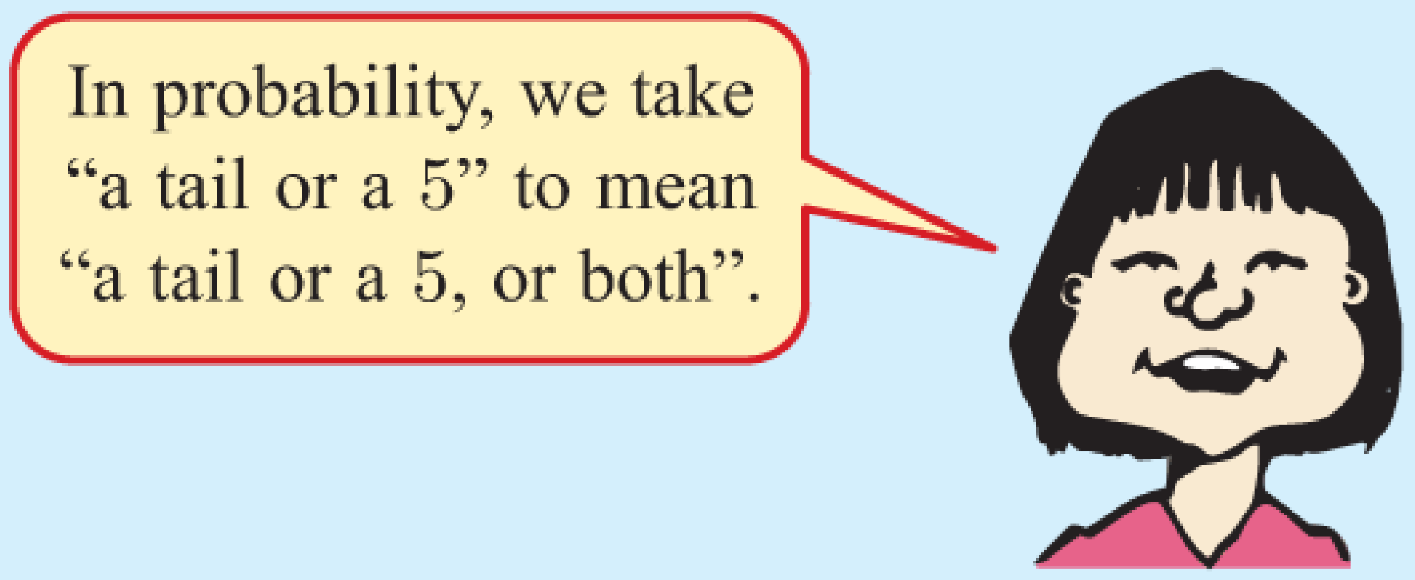
- **i** A sits on one of the end seats
- **ii** B sits on one of the two middle seats
- **iii** A and B are seated together
- **iv** A, B, and C are seated together, not necessarily in that order.

Example 8 **Self Tutor**

Use a 2-dimensional grid to illustrate the sample space for tossing a coin and rolling a die simultaneously. Hence determine the probability of:

- **a** tossing a head
- **b** tossing a tail and rolling a 5
- **c** tossing a tail or rolling a 5.





There are 12 outcomes in the sample space.

a P(head) $= \frac{6}{12} = \frac{1}{2}$ **b** P(tail and a 5) $= \frac{1}{12}$

c P(tail or a 5) $= \frac{7}{12}$ {the points in the shaded region}

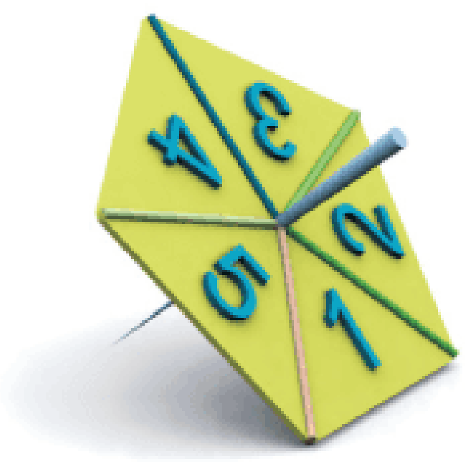
7 A 5-cent and a 20-cent coin are tossed simultaneously.

a Draw the grid of the sample space.

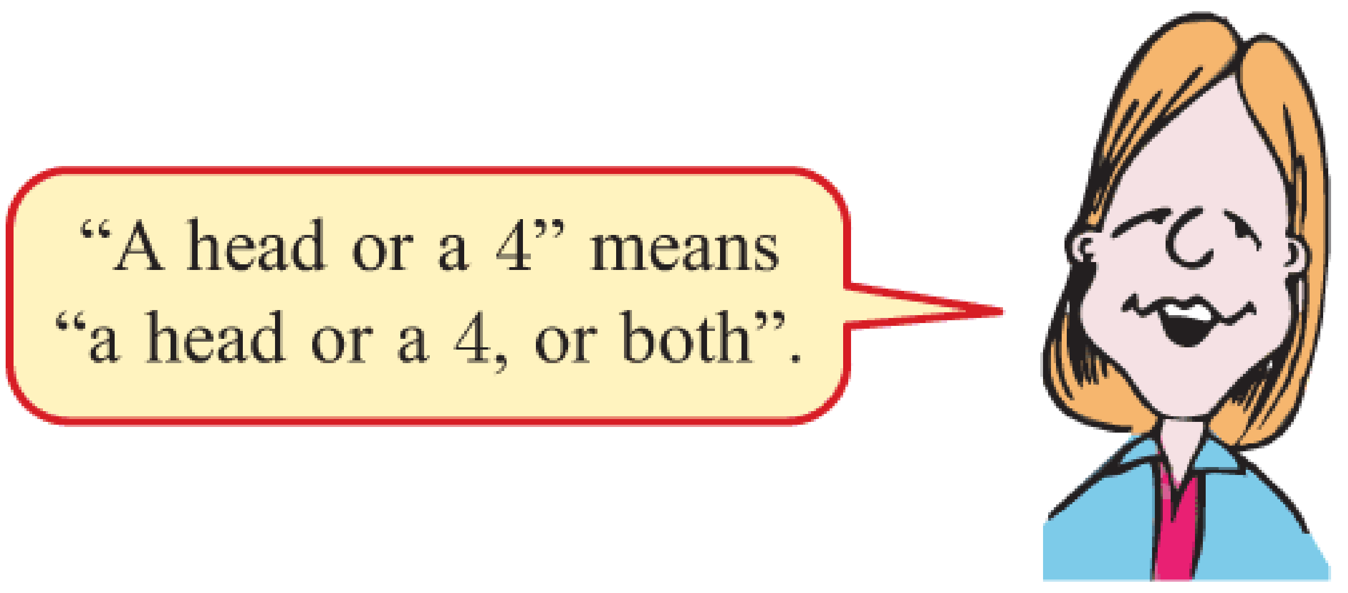
b Hence determine the probability of tossing:

- **i** two heads
- **ii** two tails
- **iii** exactly one tail
- **iv** at most one tail.

8 A coin and a pentagonal spinner with sectors 1, 2, 3, 4, and 5 are tossed and spun respectively.

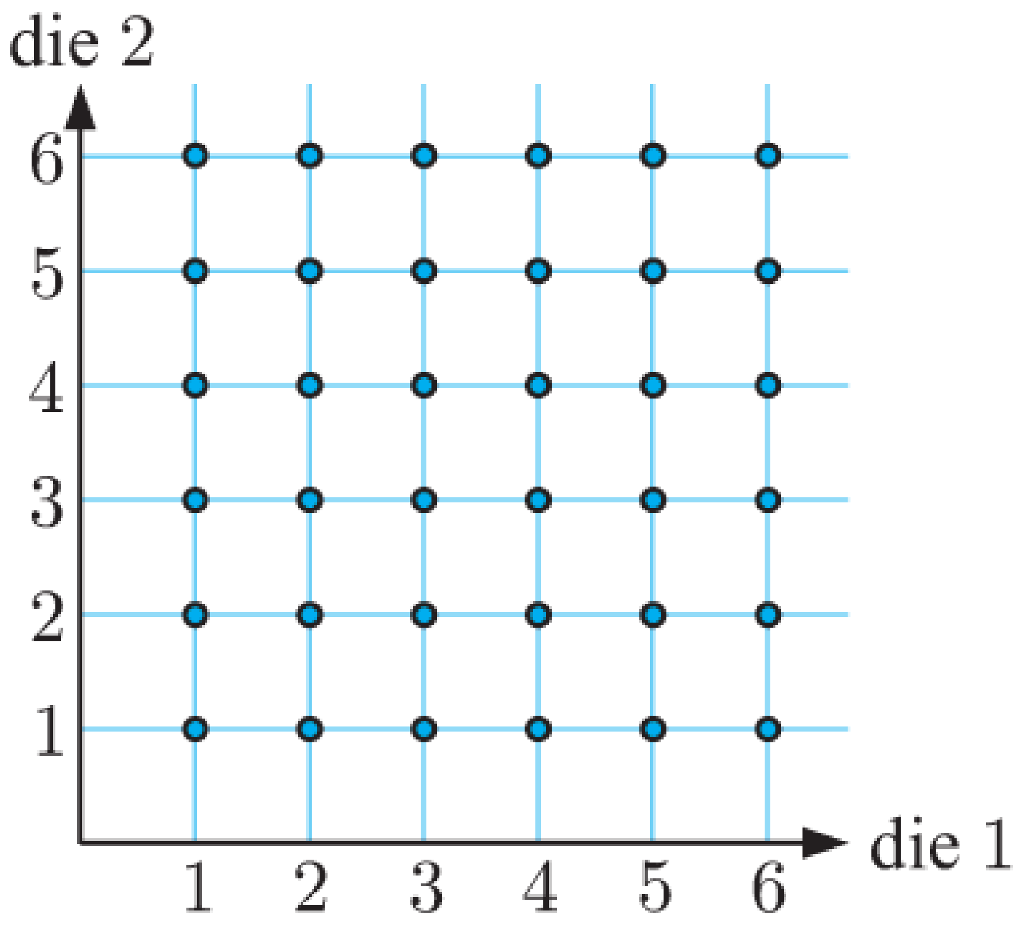


- **a** Draw a grid to illustrate the sample space of possible outcomes.
- **b** Use your grid to determine the chance of getting:
  - **i** a head and a 5
  - **ii** a tail and a prime number
  - **iii** an even number
  - **iv** a head or a 4.



9 The 36 different possible results from rolling two dice are illustrated on the 2-dimensional grid.
Use the grid to find the probability of rolling:

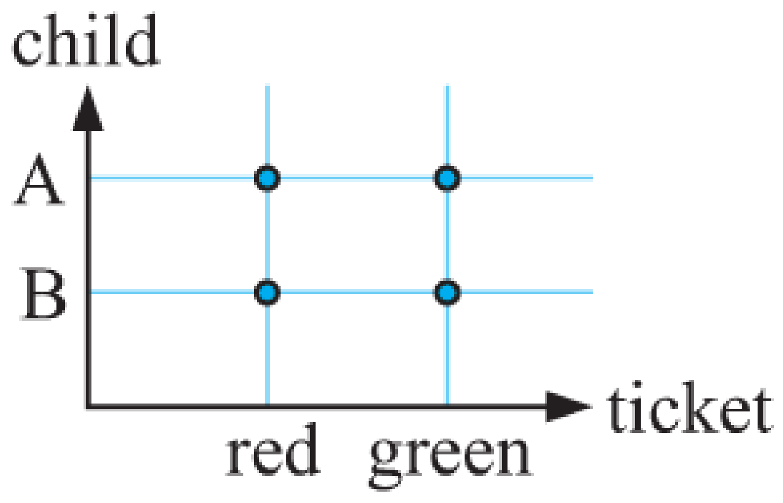
- **a** two 3s
- **b** a 5 and a 6
- **c** a 5 or a 6 (or both)
- **d** at least one 6
- **e** exactly one 6
- **f** no sixes.



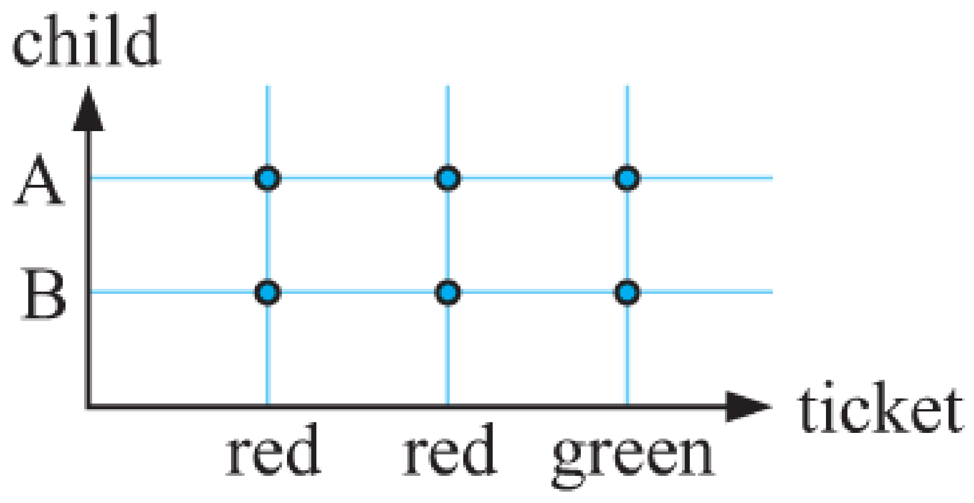
10 Two children A and B toss a coin to determine which of them will select a ticket from a bag. The bag contains two red tickets and one green ticket.

- **a** Which of these grids shows the sample space correctly? Discuss your answer.

A



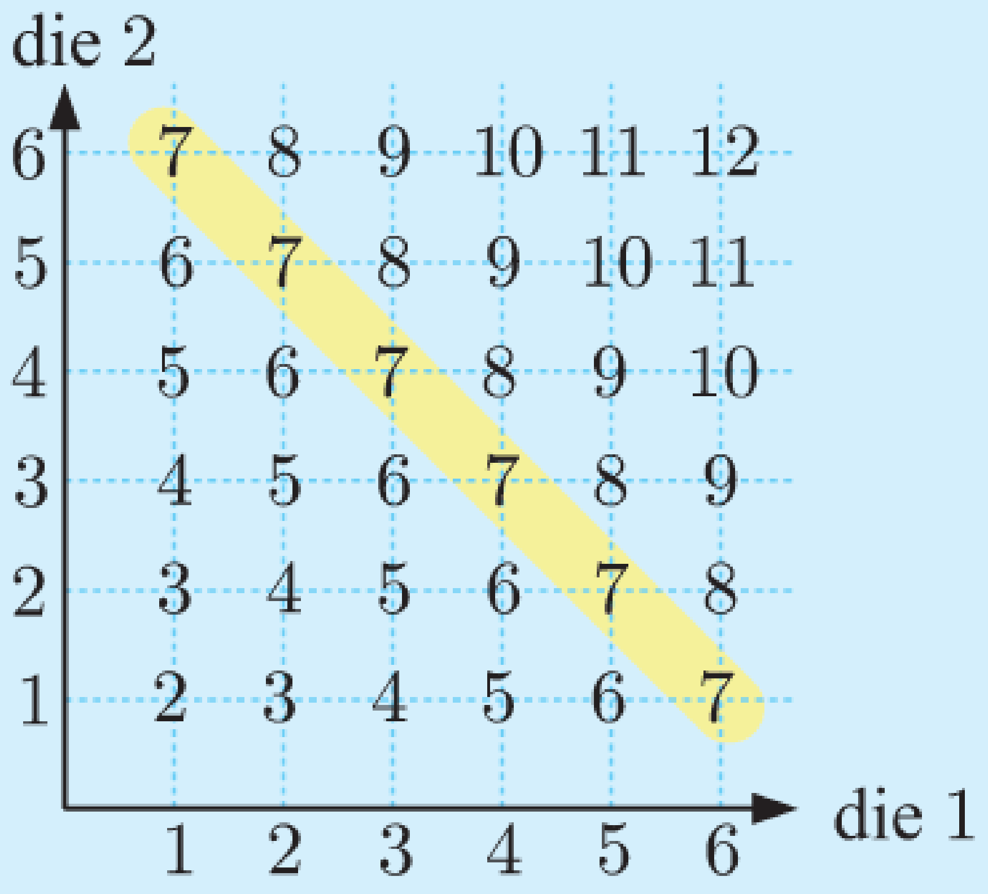
B



- **b** Use the appropriate grid to find the probability that child B will select a green ticket.

Example 9

Display the possible results when two dice are rolled and the scores are added together.
Hence find the probability that the sum of the dice is 7.



6	7	8	9	10	11	12
5	6	7	8	9	10	11
4	5	6	7	8	9	10
3	4	5	6	7	8	9
2	3	4	5	6	7	8
1	2	3	4	5	6	7
	1	2	3	4	5	6

Of the 36 possible combinations of scores from the two dice, six have the sum 7.

$\therefore$ the probability $= \frac{6}{36} = \frac{1}{6}$

11
- **a** Display the possible results when two dice are rolled and the scores are added together.
- **b** Hence find the probability that the sum of the dice is:
  - **i** 11
  - **ii** 6
  - **iii** 8 or 9
  - **iv** less than 6
  - **v** greater than 8
  - **vi** no more than 8.

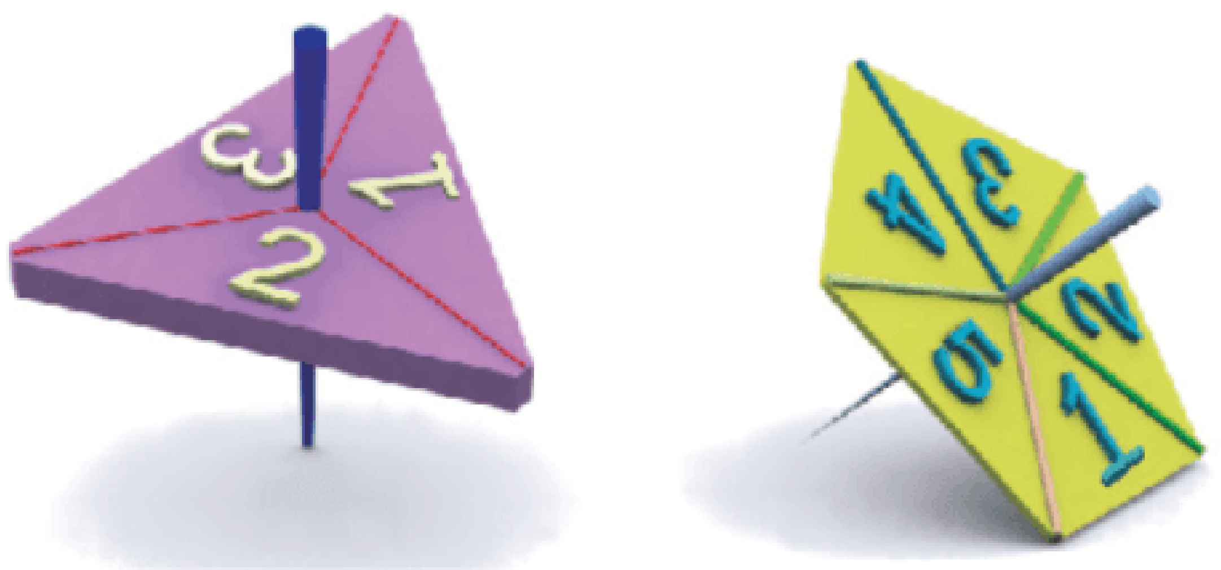
12 **a** Display the possible results when two dice are rolled and the difference between the numbers is found.

b Hence find the probability that the resulting value is:

i 0 **ii** 2 **iii** 1 or 2

iv more than 3 **v** less than 3.

13 The spinners alongside are spun, and the scores are multiplied together.



a Display the possible results.

b Hence find the probability that the result is:

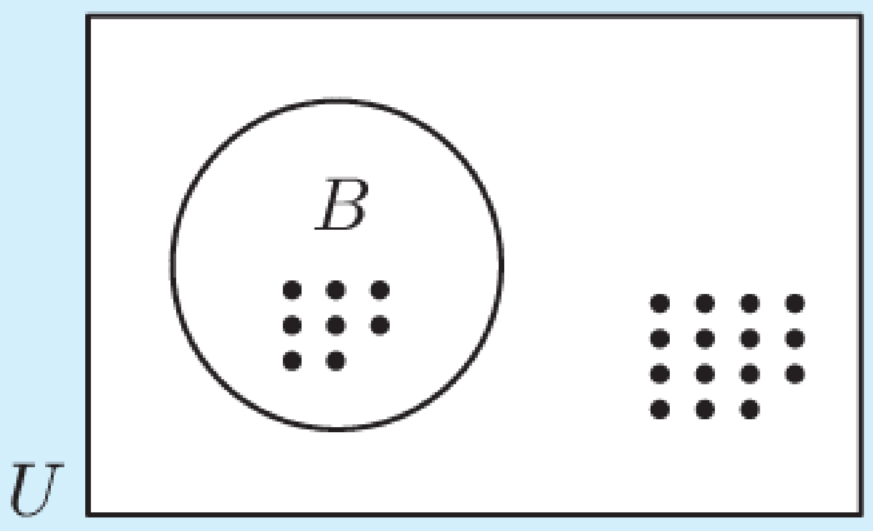
i 6 **ii** less than 5 **iii** odd.

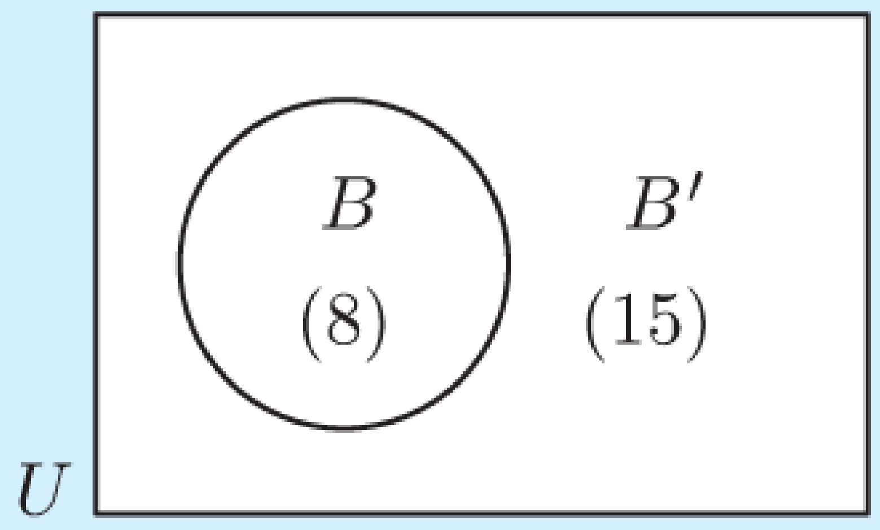
Example 10

Self Tutor

In this Venn diagram, the universal set U is the children in a class. Each dot represents a student. The event B is that a student has blue eyes. Find the probability that a randomly selected child:

a has blue eyes **b** does not have blue eyes.





$n(U) = 23, \quad n(B) = 8$

a $\text{P(blue eyes)} = \dfrac{n(B)}{n(U)} = \frac{8}{23}$

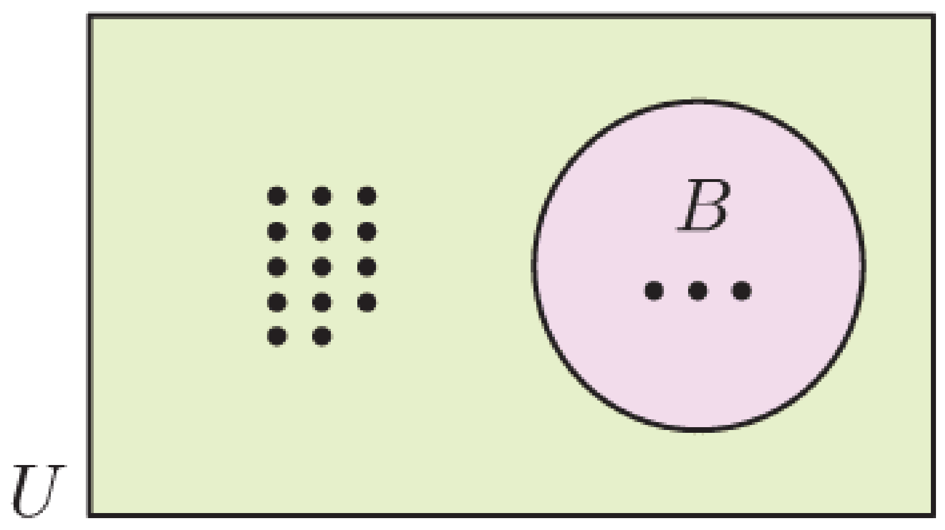
b $\text{P(not blue eyes)} = \dfrac{n(B')}{n(U)} = \frac{15}{23}$

or $\text{P(not blue)} = 1 - \text{P(blue eyes)} = 1 - \frac{8}{23} = \frac{15}{23}$

14 In this Venn diagram, the universal set U is the sheep in a pen. Each dot represents a sheep. The event B is that a sheep has black wool.

Find the probability that a randomly selected sheep:

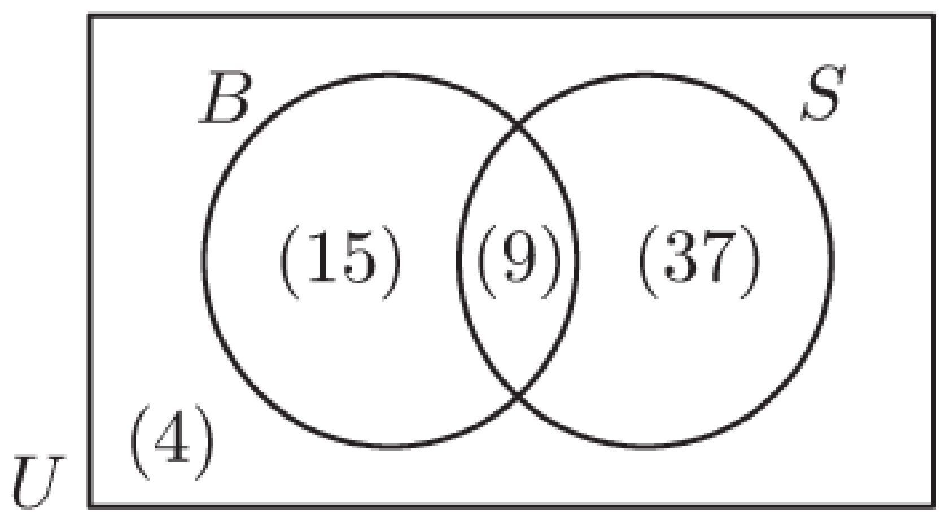
a has black wool **b** does not have black wool.



15 In a survey at an alpine resort, people were asked whether they liked skiing (S) or snowboarding (B). The results are shown in the Venn diagram. Find the probability that a randomly chosen person at the resort likes:

a both activities **b** neither activity

c exactly one activity.

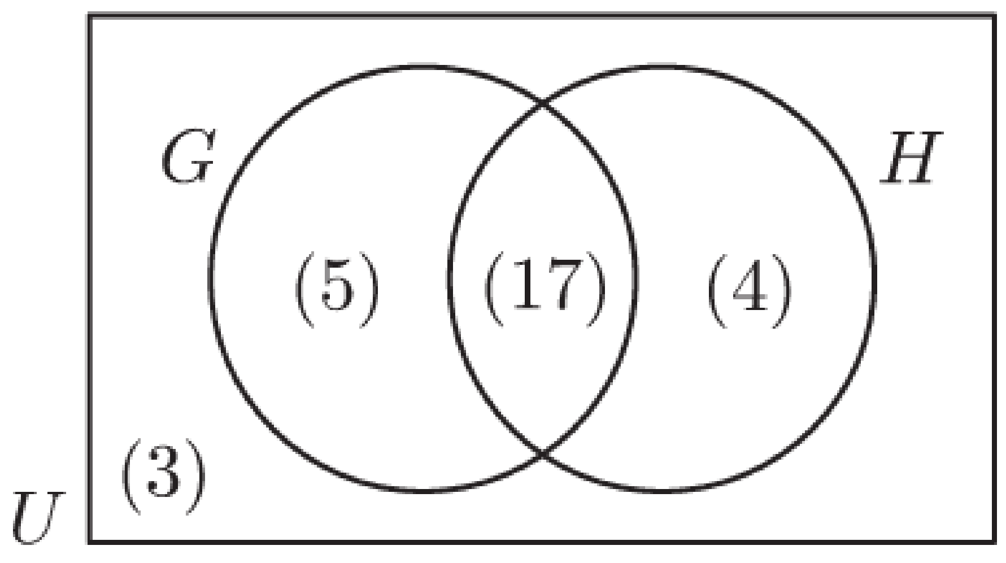


16 This Venn diagram shows the number of students in a particular class who study Geography (G) and History (H). Find the probability that a randomly selected student in the class studies:

a both of these subjects

b at least one of these subjects

c only Geography.



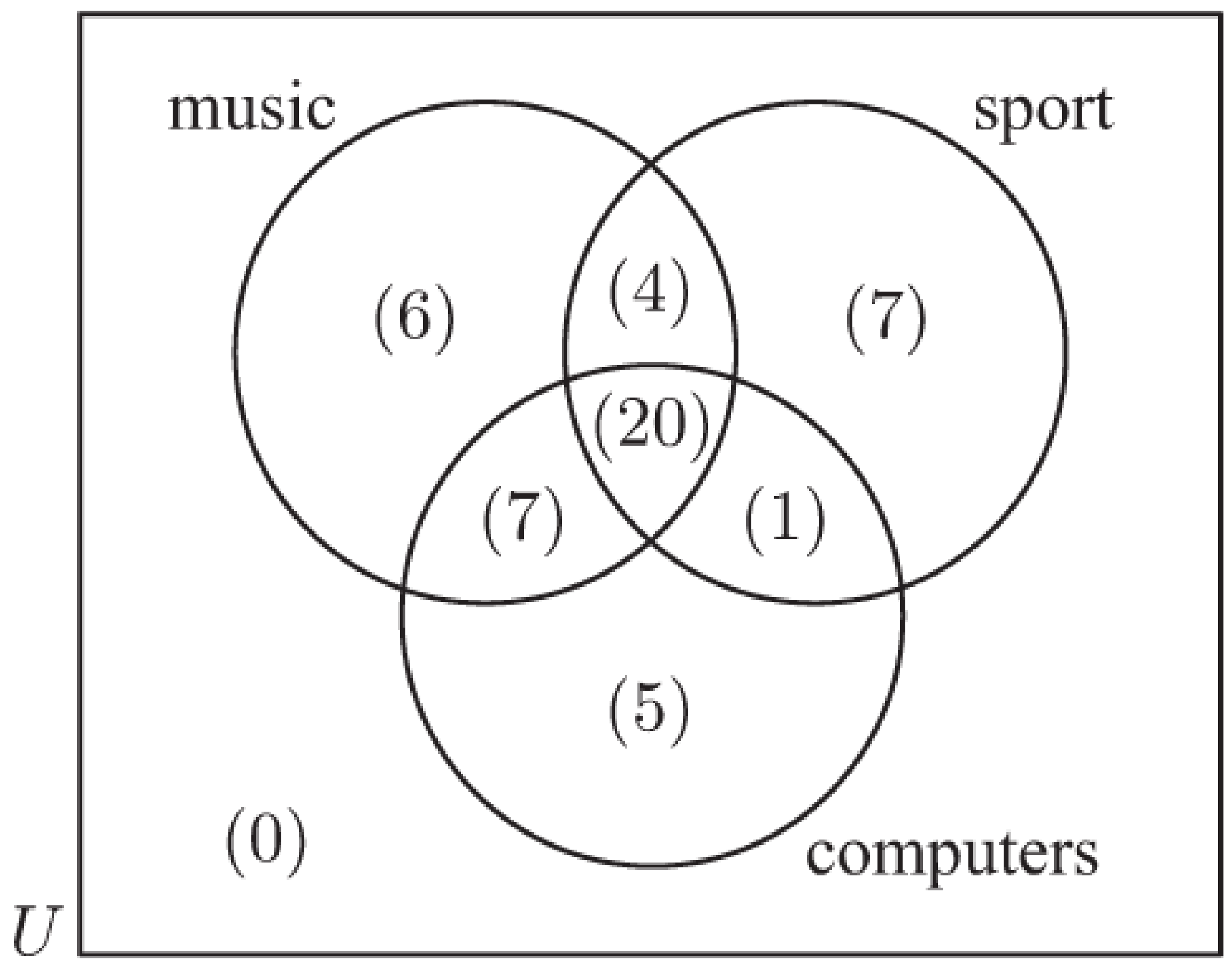
17 A group of 50 employees were surveyed regarding their interest in music, sport, and computers. The number of employees interested in each area is shown in the Venn diagram.

If an employee is selected at random, determine the probability that they are:

a interested in music

b interested in music, sport, and computers

c not interested in computers.



Example 11 **Self Tutor**

In a class of 30 students, 19 study Physics, 17 study Chemistry, and 15 study both of these subjects.

a Display this information on a Venn diagram.

b Hence determine the probability that a randomly selected student from this class studies:

i both subjects

ii at least one of the subjects

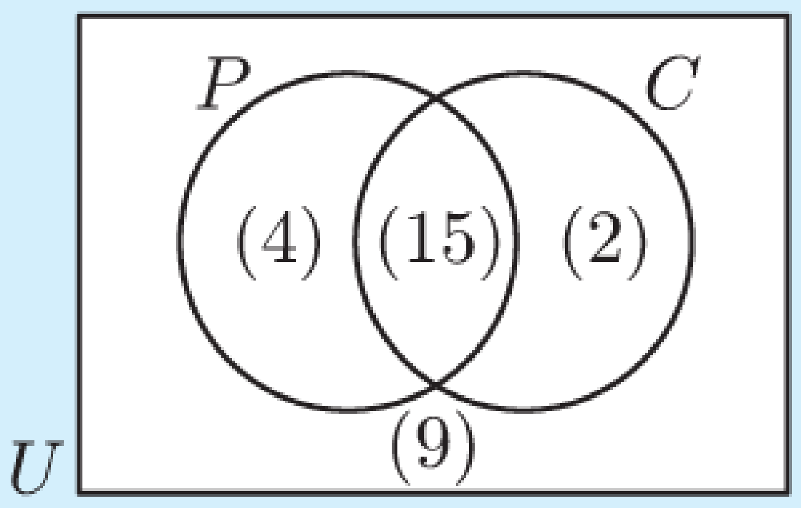
iii Physics but not Chemistry

iv exactly one of the subjects

v neither subject.

a Let P represent the event "the student studies Physics" and C represent the event "the student studies Chemistry".

$$n(P \cap C) = 15$$
$$\therefore \quad n(P \cap C') = 19 - 15 = 4$$
$$\text{and} \quad n(P' \cap C) = 17 - 15 = 2$$
$$\therefore \quad n(P' \cap C') = 30 - 15 - 4 - 2 = 9$$



b **i** P(studies both)
$= \frac{15}{30}$
$= \frac{1}{2}$

ii P(studies at least one subject)
$= \frac{4+15+2}{30}$
$= \frac{7}{10}$

iii P(P but not C)
$= \frac{4}{30}$
$= \frac{2}{15}$

iv P(studies exactly one)
$= \frac{4+2}{30}$
$= \frac{1}{5}$

v P(studies neither)
$= \frac{9}{30}$
$= \frac{3}{10}$

18 50 married men were asked whether they gave their wife flowers or chocolates for her last birthday. 31 gave chocolates, 12 gave flowers, and 5 gave both chocolates and flowers.

a Display this information on a Venn diagram.

b If one of the married men was chosen at random, determine the probability that he gave his wife:

i chocolates or flowers

ii chocolates but not flowers

iii neither chocolates nor flowers.

19 In a class of 40 students, 19 play tennis, 20 play netball, and 8 play neither of these sports. A student is randomly chosen from the class. Determine the probability that the student:

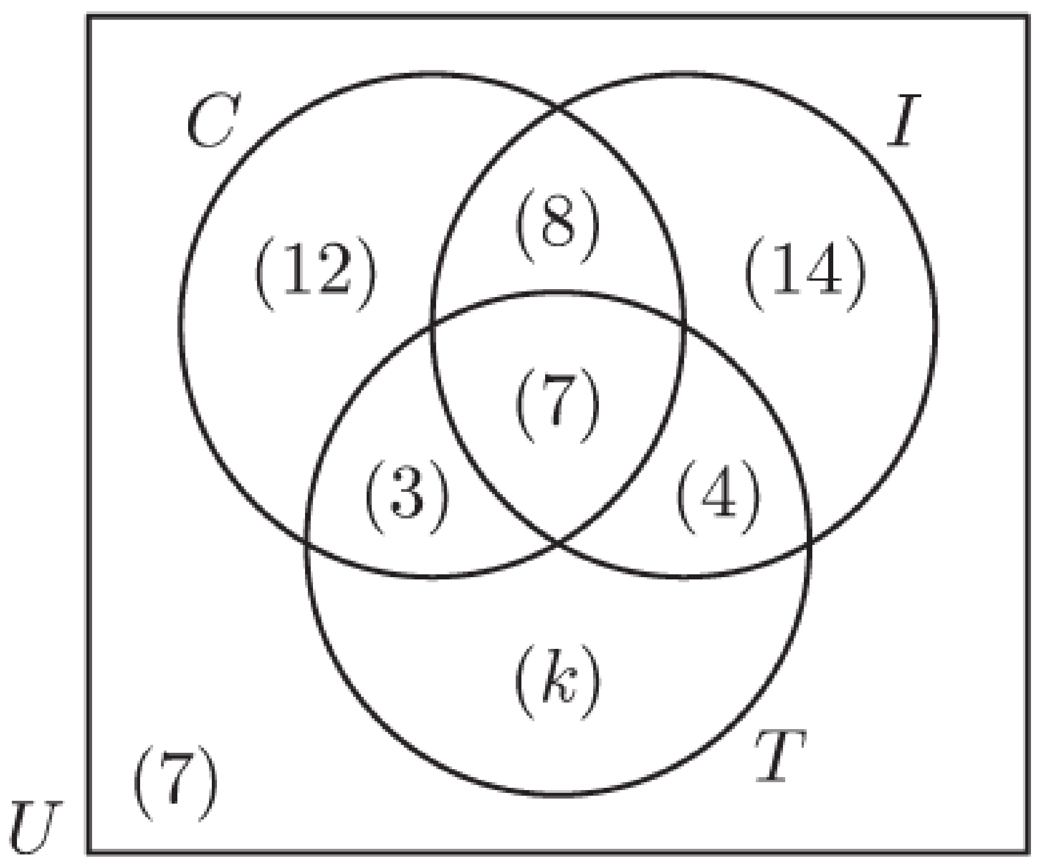
- **a** plays tennis
- **b** does not play netball
- **c** plays at least one of the sports
- **d** plays exactly one of the sports
- **e** plays netball but not tennis.

20 The medical records for a class of 30 children showed that 24 had previously had measles, 12 had previously had measles and mumps, and 26 had previously had at least one of measles or mumps. If one child from the class is selected at random, determine the probability that he or she has had:

- **a** mumps
- **b** mumps but not measles
- **c** neither mumps nor measles.

21 In this Venn diagram, U is the set of all 60 members of a club.

The members indicate their liking for Chinese (C), Italian (I), and Thai (T) food.



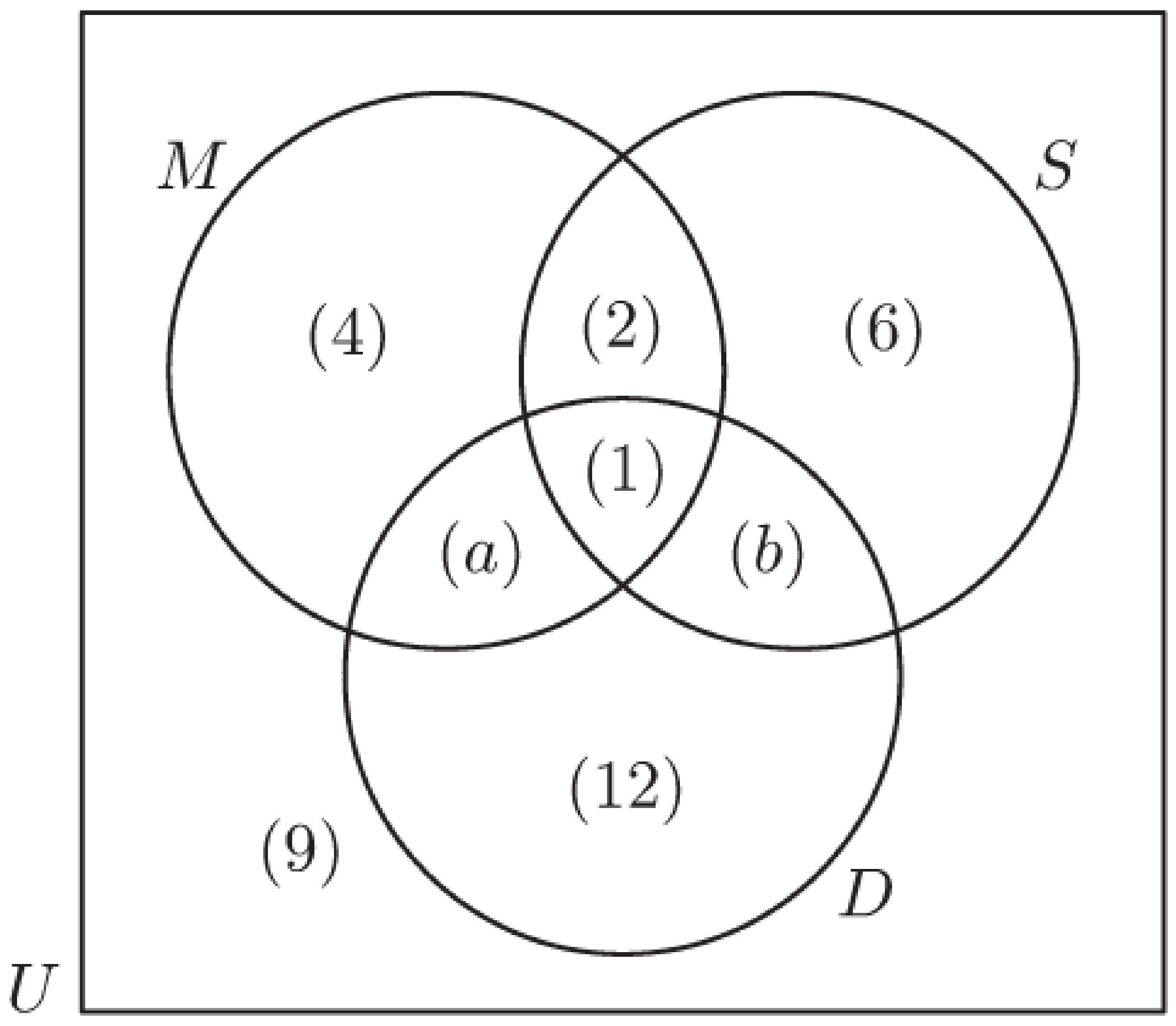
- **a** Find the value of k.
- **b** Find the probability that a randomly chosen member likes:
  - **i** only Italian
  - **ii** Italian and Thai
  - **iii** none of these foods
  - **iv** at least one of these foods
  - **v** all of these foods
  - **vi** Chinese and Italian, but not Thai
  - **vii** Thai or Italian
  - **viii** exactly one of these foods.

22 As a group bonding project, 50 delegates at a European conference were asked what languages they had conversations in at lunch time. The data collected is summarised alongside.

Languages	*Delegates*
English only	17
French only	7
Spanish only	12
English and French only	3
English and Spanish only	6
French and Spanish only	4
English, French, and Spanish	1

- **a** Construct a Venn diagram to display the information.
- **b** Find the probability that a randomly selected delegate had a conversation in:
  - **i** English
  - **ii** French
  - **iii** Spanish, but not in English
  - **iv** French, but not in Spanish
  - **v** French, and also one in English.

23 The Venn diagram opposite indicates the types of programs a group of 40 individuals watched on television last night. M represents movies, S represents sports, and D represents dramas.



- **a** Given that 10 people watched a movie last night, calculate a and b.
- **b** Find the probability that one of these individuals, selected at random, watched:
  - **i** sport
  - **ii** drama and sport
  - **iii** a movie but not sport
  - **iv** drama but not a movie
  - **v** drama or a movie.

DISCUSSION

Three children have been tossing a coin in the air and recording the outcomes. They have done this 10 times and have recorded 10 tails. Before the next toss they make these statements:

Jackson: "It's got to be a head next time!"

Sally: "No, it always has an equal chance of being a head or a tail. The coin cannot remember what the outcomes have been."

Amy: "Actually, I think it will probably be a tail again, because I think the coin must be biased. It might be weighted so it is more likely to give a tail."

Discuss the statements of each child. Who do you think is correct?

E MAKING PREDICTIONS USING PROBABILITY

In **Section A** we saw that if we perform an experiment a number of times, then the experimental probability of an event occurring is

$$\begin{aligned}\text{experimental probability} &= \text{relative frequency of event}\\ &= \frac{\text{number of times event occurs}}{\text{number of trials}}.\end{aligned}$$

So, we start with an experiment and use it to generate a probability. In the study of expectation, we go the other way.

Rearranging the equation, we obtain:

$$\text{number of times event occurs} = \text{experimental probability} \times \text{number of trials}.$$

However in this case, we use a theoretical probability to predict the results.

If there are n trials of an experiment, and an event has probability p of occurring in each of the trials, then the number of times we *expect* the event to occur is np.

DISCUSSION

In most cases, the expected value np will not be an integer. It will therefore be impossible to actually get the "expected" value. Is this a problem?

Example 12 **Self Tutor**

In his basketball career, Michael Jordan made 83.53% of shots from the free throw line. If he had played one more game and had 18 attempts from the free throw line, how many shots would you expect him to have made?

$n = 18$ throws

$p = \text{P(successfully makes free throw)} = 0.8353$

We would expect him to have made $np = 18 \times 0.8353 \approx 15$ shots.

EXERCISE 11E

1 A goalkeeper has probability $\frac{3}{10}$ of saving a penalty attempt. How many goals would he expect to save from 90 attempts?

2 The coach of a lacrosse team has calculated that Brayden scores on about 23% of his attempts at goal. If Brayden has 68 attempts to score this season, how many times would you expect him to score?

3 **a** If 2 coins are tossed, what is the chance that they both fall heads?

b If the 2 coins are tossed 200 times, on how many occasions would you expect them to both fall heads?

4 During the snow season there is a $\frac{3}{7}$ probability of snow falling on any particular day. If Udo skis for five weeks, on how many days could he expect to see snow falling?

5 If two dice are rolled simultaneously 180 times, on how many occasions would you expect to get a double?

6 In a pre-election poll, residents indicated their voting intentions. The number of voters that favoured each candidate A, B, and C are shown alongside.

A	B	C
165	87	48

a Estimate the probability that a randomly chosen voter in the electorate will vote for:

i A **ii** B **iii** C.

b If 7500 people vote in the election, how many do you expect to vote for:

i A **ii** B **iii** C?

F THE ADDITION LAW OF PROBABILITY

We now more carefully consider **compound events** where there is more than one event in our sample space. This may be because the experiment has more than one process, or because we are interested in more than one property of the outcome.

Suppose there are two events A and B in a sample space U. Following set notation:

- The event that both A **and** B occur is written $\boldsymbol{A \cap B}$, and read as "A intersection B".
- The event that A **or** B **or both** occur is written $\boldsymbol{A \cup B}$, and read as "A union B".

INVESTIGATION 3 THE ADDITION LAW OF PROBABILITY

In this Investigation we look for a formula connecting the probabilities for $P(A \cap B)$ and $P(A \cup B)$.

What to do:

1 Suppose $U = \{x \mid x \text{ is a positive integer less than } 100\}$.

Let $A = \{\text{multiples of 7 in } U\}$ and $B = \{\text{multiples of 5 in } U\}$.

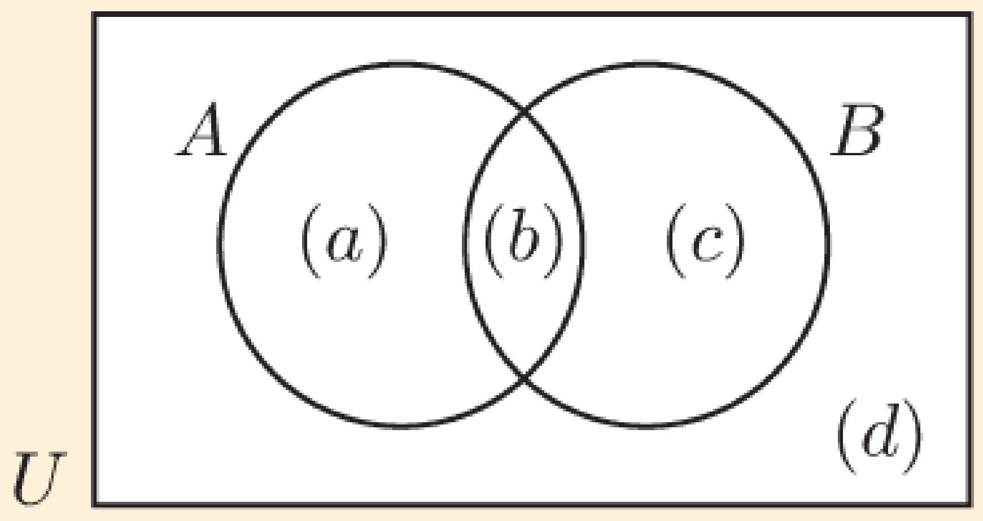
a How many elements are there in:

i A **ii** B **iii** $A \cap B$ **iv** $A \cup B$?

b Show that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

2 By comparing regions of the Venn diagram, verify that for all sets A and B in a universal set U:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$



3 By dividing both sides of the above equation by $n(U)$, establish the connection between $\mathrm{P}(A \cup B)$ and $\mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$.

From the **Investigation** you should have discovered the **addition law of probability**:

For two events A and B, $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

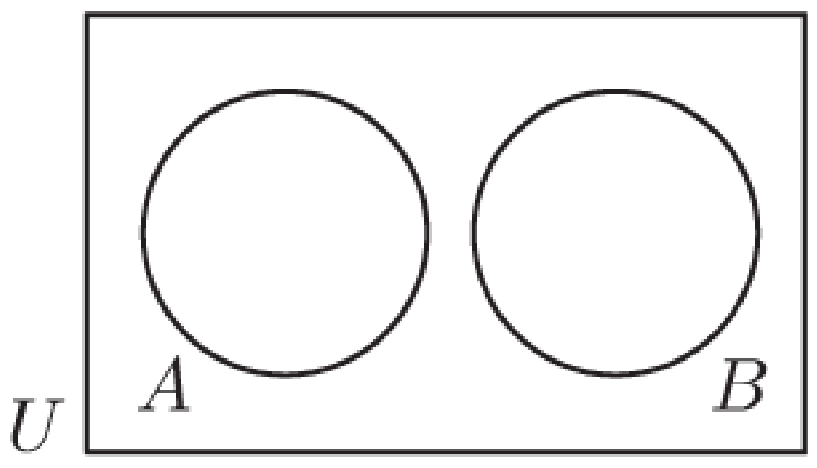
which means: P(**either** A **or** B **or** both) $= \mathrm{P}(A) + \mathrm{P}(B) -$ P(**both** A **and** B).

Example 13 **Self Tutor**

If $\mathrm{P}(A) = 0.6$, $\mathrm{P}(A \cup B) = 0.7$, and $\mathrm{P}(A \cap B) = 0.3$, find $\mathrm{P}(B)$.

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$
$$\therefore\ 0.7 = 0.6 + \mathrm{P}(B) - 0.3$$
$$\therefore\ \mathrm{P}(B) = 0.4$$

If A and B are disjoint **mutually exclusive** events then $\mathrm{P}(A \cap B) = 0$ and so the addition law becomes $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$.



Example 14 **Self Tutor**

30 students were given a History test. 7 students scored an A and 11 students scored a B.

A student is randomly selected. Let A be the event that the student scored an A, and B be the event that the student scored a B.

a Are A and B mutually exclusive?

b Find:

i $\mathrm{P}(A)$ **ii** $\mathrm{P}(B)$ **iii** $\mathrm{P}(A \cap B)$ **iv** $\mathrm{P}(A \cup B)$

a It is impossible for a student to score both an A and a B for the test.

$\therefore$ A and B are mutually exclusive.

b **i** $\mathrm{P}(A) = \frac{7}{30}$

ii $\mathrm{P}(B) = \frac{11}{30}$

iii $\mathrm{P}(A \cap B) = 0$

{A and B are mutually exclusive}

iv
$$\begin{aligned}\mathrm{P}(A \cup B) &= \mathrm{P}(A) + \mathrm{P}(B)\\ &= \tfrac{7}{30} + \tfrac{11}{30}\\ &= \tfrac{3}{5}\end{aligned}$$

EXERCISE 11F

1 If $P(A) = 0.2$, $P(B) = 0.4$, and $P(A \cap B) = 0.05$, find $P(A \cup B)$.

2 If $P(A) = 0.4$, $P(A \cup B) = 0.9$, and $P(A \cap B) = 0.1$, find $P(B)$.

3 If $P(X) = 0.6$, $P(Y) = 0.5$, and $P(X \cup Y) = 0.9$, find $P(X \cap Y)$.

4 Suppose $P(A) = 0.25$, $P(B) = 0.45$, and $P(A \cup B) = 0.7$.

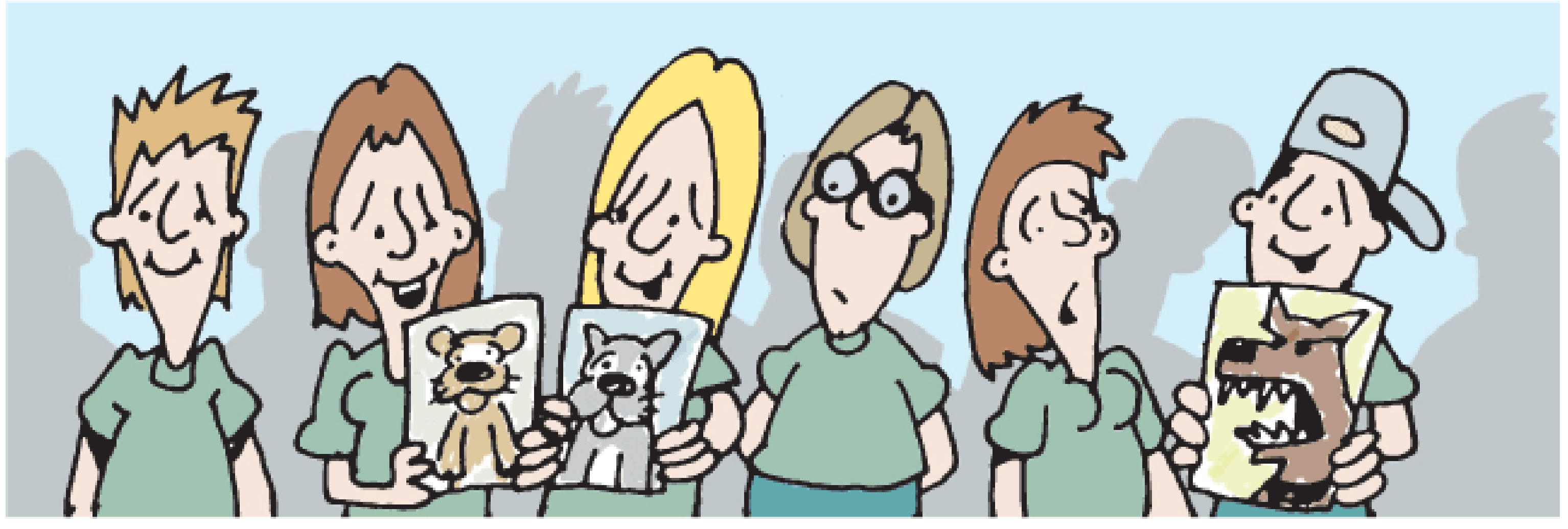
- **a** Find $P(A \cap B)$.
- **b** What can you say about A and B?

5 A and B are mutually exclusive events.
If $P(B) = 0.45$ and $P(A \cup B) = 0.8$, find $P(A)$.

6 Tickets numbered 1 to 15 are placed in a hat, and one ticket is chosen at random. Let A be the event that the number drawn is greater than 11, and B be the event that the number drawn is less than 8.

- **a** Are A and B mutually exclusive?
- **b** Find: **i** $P(A)$ **ii** $P(B)$ **iii** $P(A \cup B)$.

7 A class consists of 25 students.



11 students are fifteen years old (F).
12 students are sixteen years old (S).
8 students own a dog (D).
7 students own a cat (C).
4 students do not own any pets (N).

A student is chosen at random. If possible, find:

- **a** $P(F)$
- **b** $P(S)$
- **c** $P(D)$
- **d** $P(C)$
- **e** $P(N)$
- **f** $P(F \cup S)$
- **g** $P(F \cup D)$
- **h** $P(C \cup N)$
- **i** $P(C \cup D)$
- **j** $P(D \cup N)$

8 Suppose A and B are mutually exclusive, and that A' and B' are mutually exclusive. Find $P(A \cup B)$.

G INDEPENDENT EVENTS

Two events are **independent** if the occurrence of each event does not affect the occurrence of the other.

INVESTIGATION 4 INDEPENDENT EVENTS

WORKSHEET

In this Investigation we seek a rule for calculating $P(A \cap B)$ for two independent events A and B.

What to do:

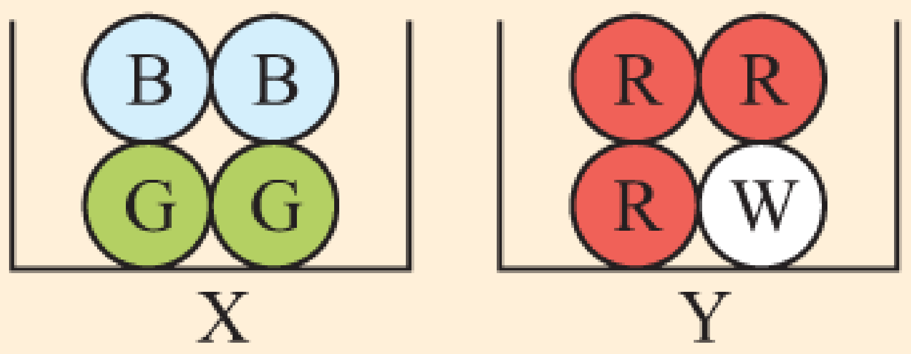
1 Suppose a coin is tossed and a die is rolled at the same time.

- **a** Does the outcome of the coin toss affect the outcome of the die roll, or vice versa?
- **b** Draw a 2-dimensional grid to show the possible outcomes of the experiment.

c Copy and complete this table of probabilities for different events A and B:

	A	B	$\mathrm{P}(A)$	$\mathrm{P}(B)$	$\mathrm{P}(A \cap B)$
i	head	4			
ii	head	odd number			
iii	tail	number greater than 1			
iv	tail	number less than 3			

2 Consider randomly selecting a ball from each of the boxes alongside.



a Does the outcome of the draw from either box affect the occurrence of the other?

b Draw a 2-dimensional grid to show the possible outcomes of the experiment.

c Copy and complete this table of probabilities for different events A and B:

	A	B	$\mathrm{P}(A)$	$\mathrm{P}(B)$	$\mathrm{P}(A \cap B)$
i	green from box X	red from box Y			
ii	green from box X	white from box Y			
iii	blue from box X	red from box Y			
iv	blue from box X	white from box Y			

3 For independent events A and B, what is the connection between $\mathrm{P}(A \cap B)$, $\mathrm{P}(A)$, and $\mathrm{P}(B)$?

From the **Investigation** you should have concluded that:

If A and B are independent events, then $\mathbf{P}(\boldsymbol{A} \cap \boldsymbol{B}) = \mathbf{P}(\boldsymbol{A}) \times \mathbf{P}(\boldsymbol{B})$.

This rule can be extended for any number of independent events.

For example: If A, B, and C are all independent events, then $\mathrm{P}(A \cap B \cap C) = \mathrm{P}(A) \times \mathrm{P}(B) \times \mathrm{P}(C)$.

Example 15 **Self Tutor**

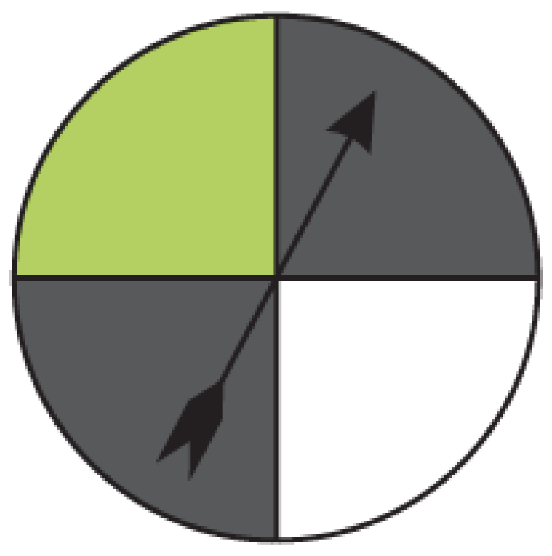
A coin is tossed and a die is rolled simultaneously. Determine the probability of getting a head and a 3, without using a grid.

$$\begin{aligned} \mathrm{P}(\text{a head} \cap \text{a } 3) &= \mathrm{P}(\mathrm{H}) \times \mathrm{P}(3) \qquad \{\text{events are independent}\} \\ &= \tfrac{1}{2} \times \tfrac{1}{6} \\ &= \tfrac{1}{12} \end{aligned}$$

EXERCISE 11G

1 Each of these spinners is spun once. Find the probability of spinning:

- **a** green with spinner 1 and white with spinner 2
- **b** black with both spinners.


Spinner 1


Spinner 2

2 A coin is tossed 3 times. Determine the probability of getting the following sequences of results:

- **a** head, head, head
- **b** tail, head, tail.

3 A school has two photocopiers. On any given day, machine A has an 8% chance of malfunctioning and machine B has a 12% chance of malfunctioning. Determine the probability that on any given day, both machines will:

- **a** malfunction
- **b** work effectively.

4 Two marksmen fire at a target simultaneously. Jiri hits the target 70% of the time and Benita hits it 80% of the time. Determine the probability that:

- **a** they both hit the target
- **b** they both miss the target
- **c** Jiri hits but Benita misses
- **d** Benita hits but Jiri misses.

5 An archer hits the bullseye on average 2 out of every 5 shots. If 3 arrows are fired at the target, determine the probability that the bullseye is hit:

- **a** every time
- **b** the first two times, but not on the third shot
- **c** on no occasion.

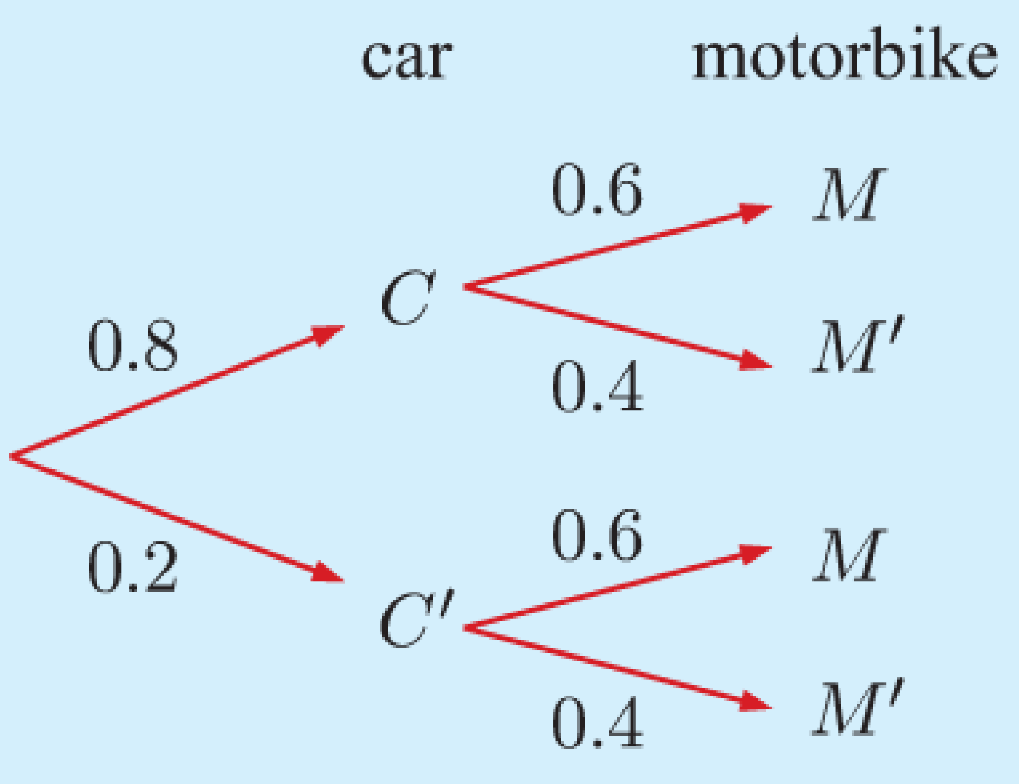
Example 16

Self Tutor

Carl is not having much luck lately. His car will only start 80% of the time and his motorbike will only start 60% of the time, independently of one another.

- **a** Draw a tree diagram to illustrate this situation.
- **b** Use the tree diagram to determine the chance that on the next attempt:
  - **i** both will start
  - **ii** Carl can only use his car.

a Let C be the event that Carl's car starts, and M be the event that his motorbike starts.



outcome	probability
C and M	$0.8 \times 0.6 = 0.48$
C and M'	$0.8 \times 0.4 = 0.32$
C' and M	$0.2 \times 0.6 = 0.12$
C' and M'	$0.2 \times 0.4 = 0.08$
	total 1.00

The probability of each outcome is obtained by **multiplying** the probabilities along its branch.

b **i** P(both start)
$= \text{P}(C \cap M)$
$= 0.8 \times 0.6$
$= 0.48$

ii P(car starts and motorbike does not)
$= \text{P}(C \cap M')$
$= 0.8 \times 0.4$
$= 0.32$

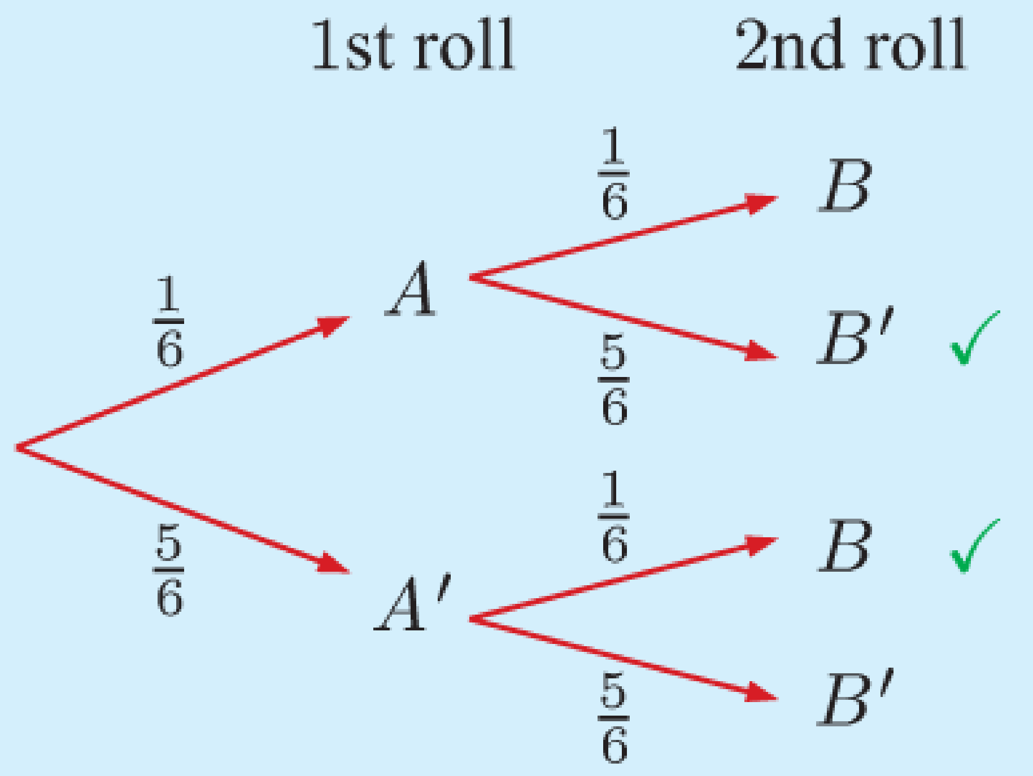
6 For a particular household, there is a 90% chance that at the end of the week the rubbish bin is full, and a 50% chance that the recycling bin is full, independently of one another.

- **a** Draw a tree diagram to illustrate this situation.
- **b** Find the probability that at the end of the week:
  - **i** both bins are full
  - **ii** the recycling bin is full but the rubbish bin is not.

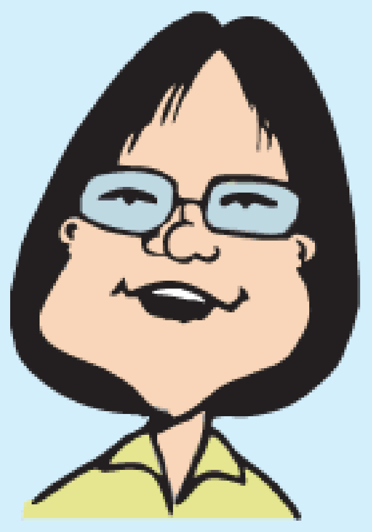
Example 17 **Self Tutor**

Liam rolls a six-sided die twice. Determine the probability that exactly 1 four is rolled.

Let A be the event that a four is rolled on the first roll, and B be the event that a four is rolled on the second roll.



$$\begin{aligned} &\mathrm{P}(1 \text{ four}) \\ &= \mathrm{P}(A \cap B') + \mathrm{P}(A' \cap B) \\ &= \tfrac{1}{6} \times \tfrac{5}{6} + \tfrac{5}{6} \times \tfrac{1}{6} \quad \{\text{branches marked } \checkmark\} \\ &= \tfrac{5}{36} + \tfrac{5}{36} \\ &= \tfrac{10}{36} \\ &= \tfrac{5}{18} \end{aligned}$$

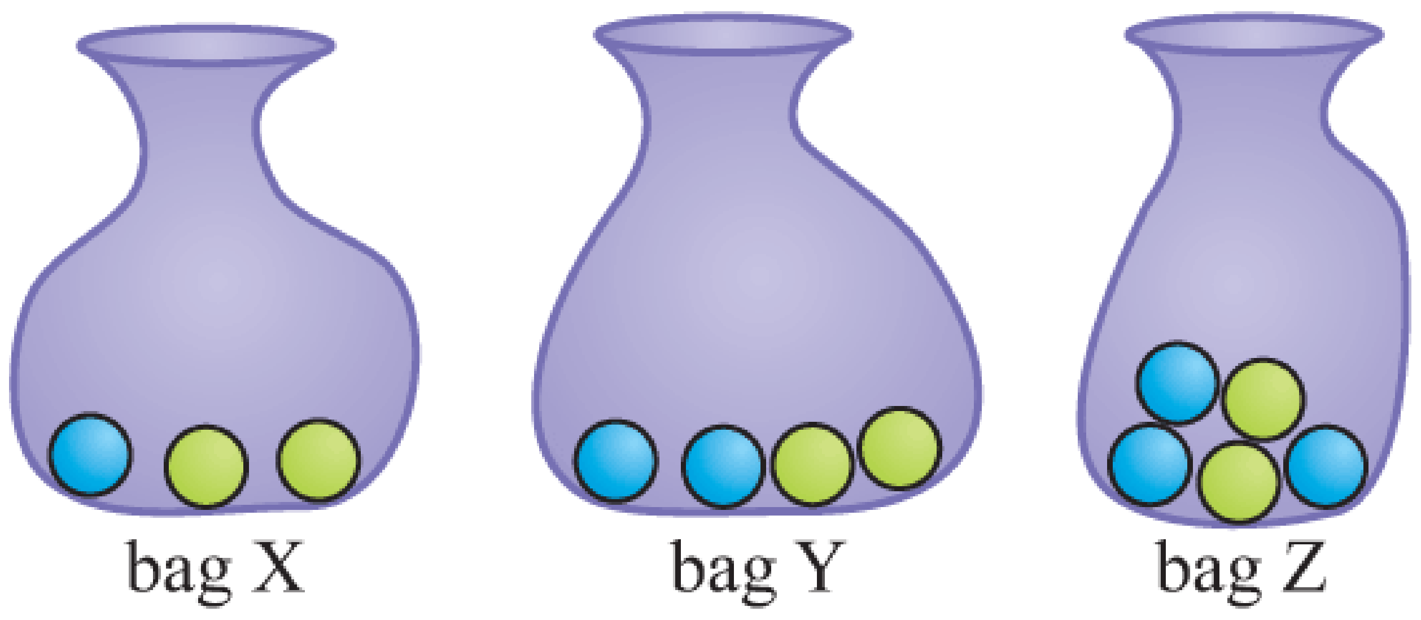


If more than one outcome corresponds to an event, **add** the probabilities of these outcomes.

7 Two baskets each contain 5 red apples and 2 green apples. Celia chooses an apple at random from each basket.

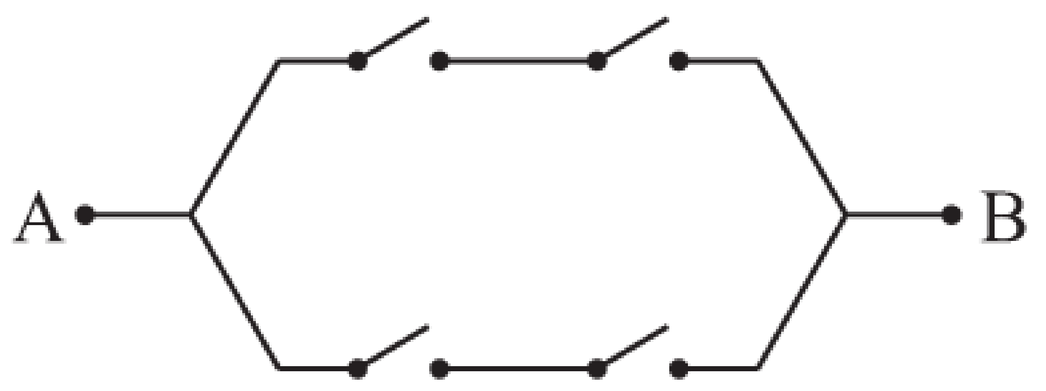
- **a** Draw a tree diagram to illustrate the possible outcomes.
- **b** Find the probability that Celia chooses:
  - **i** two red apples
  - **ii** one red and one green apple.

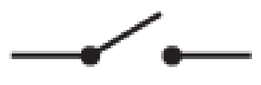
8 One ball is drawn from each of the bags shown.



- **a** Draw a tree diagram to illustrate this situation.
- **b** Find the probability that:
  - **i** 3 blue balls are drawn
  - **ii** green balls are drawn from bags Y and Z
  - **iii** at least one blue ball is drawn.

9 The diagram shows a simple electrical network.



Each symbol  represents a switch.

All four switches operate independently, and the probability of each one of them being closed is p.

- **a** In terms of p, find the probability that the current flows from A to B.
- **b** Find the least value of p for which the probability of current flow is at least 0.5 .

10 Kane plays 3 matches in a darts challenge, alternating between Penny and Quentin as his opponents. Kane must win 2 matches in a row to win the challenge. He is allowed to choose which opponent he plays first. He knows that Penny is the better darts player.
Which strategy should Kane use to maximise his chances of winning the challenge? Justify your answer.

H DEPENDENT EVENTS

Suppose a hat contains 5 red and 3 blue tickets. One ticket is randomly chosen, its colour is noted, and it is then put aside and so *not* put back in the hat. A second ticket is then randomly selected. What is the chance that it is red?

If the first ticket was red, $\text{P(second is red)} = \frac{4}{7} \begin{array}{l} \leftarrow \text{4 reds remaining} \\ \leftarrow \text{7 to choose from} \end{array}$

If the first ticket was blue, $\text{P(second is red)} = \frac{5}{7} \begin{array}{l} \leftarrow \text{5 reds remaining} \\ \leftarrow \text{7 to choose from} \end{array}$

The probability of the second ticket being red *depends* on what colour the first ticket was. We therefore have **dependent events**.

> Two or more events are **dependent** if the occurrence of one of the events *does affect* the occurrence of the other events.
>
> Events are **dependent** if they are **not independent**.
>
> If A and B are dependent events then
>
> $$\text{P}(A \cap B) = \text{P}(A) \times \text{P}(B \text{ given that } A \text{ has occurred}).$$

In general, when an experiment involves sampling:

- **without replacement** we have dependent events
- **with replacement** we have independent events.

Not all scenarios we study involve sampling. However, they may still involve dependent events.

For example, the event *Pahal walks to school today* is dependent on the event *it will rain today*.

Example 18 **Self Tutor**

A box contains 4 red and 2 yellow tickets. Two tickets are randomly selected from the box one by one *without* replacement. Find the probability that:

a both are red **b** the first is red and the second is yellow.

a P(both red)

$= \text{P(first selected is red} \cap \text{second is red)}$

$= \text{P(first selected is red)} \times \text{P(second is red given that the first is red)}$

$= \frac{4}{6} \times \frac{3}{5} \begin{array}{l} \leftarrow \text{4 reds out of a total of 6 tickets} \\ \leftarrow \text{If a red is drawn first, 3 reds remain out of a total of 5.} \end{array}$

$= \frac{2}{5}$

b $\text{P(first is red} \cap \text{second is yellow)}$

$= \text{P(first is red)} \times \text{P(second is yellow given that the first is red)}$

$= \frac{4}{6} \times \frac{2}{5} \begin{array}{l} \leftarrow \text{4 reds out of a total of 6 tickets} \\ \leftarrow \text{If a red is drawn first, 2 yellows remain out of a total of 5.} \end{array}$

$= \frac{4}{15}$

EXERCISE 11H

1 A box contains 7 red and 3 green balls. Two balls are drawn one after another from the box without replacement. Determine the probability that:

a both are red **b** the first is green and the second is red.

2 A bag contains 4 blue and 6 white tokens. Two tokens are drawn from the bag one after another, without replacement. Find the probability that:

a both are blue **b** the first is blue and the second is white.

Example 19 **Self Tutor**

A hat contains 20 tickets numbered 1, 2, 3, ..., 20. If three tickets are drawn from the hat without replacement, determine the probability that they are all prime numbers.

$\{2, 3, 5, 7, 11, 13, 17, 19\}$ are primes.

$\therefore$ 8 of the 20 numbers are primes.

$\therefore$ P(3 primes)

$=$ P(1st drawn is prime $\cap$ 2nd is prime $\cap$ 3rd is prime)

$= \frac{8}{20}$ ← 8 primes out of 20 numbers

$\times \frac{7}{19}$ ← 7 primes out of 19 numbers after a successful first draw

$\times \frac{6}{18}$ ← 6 primes out of 18 numbers after two successful draws

≈ 0.0491

In each fraction, the numerator is the number of outcomes in the event. The denominator is the total number of possible outcomes.

3 A box contains 12 identically shaped chocolates of which 8 are strawberry creams. Three chocolates are selected simultaneously from the box. Determine the probability that:

a they are all strawberry creams

b none of them are strawberry creams.

Drawing three chocolates *simultaneously* implies there is no replacement.

4 A lottery has 100 tickets which are placed in a barrel. Three tickets are drawn at random from the barrel, without replacement, to decide 3 prizes. If John has 3 tickets in the lottery, determine his probability of winning:

a first prize **b** first and second prize **c** all 3 prizes **d** none of the prizes.

5 A hat contains 7 names of players in a tennis squad including the captain and the vice captain. If a team of three is chosen at random by drawing the names from the hat, determine the probability that it does *not* contain:

a the captain **b** the captain or the vice captain.

6 Two students are chosen at random from a group of two girls and five boys, all of different ages. Find the probability that the two students chosen will be:

a two boys **b** the eldest two students.

Example 20

Self Tutor

Two boxes each contain 6 petunia plants. Box A contains 2 plants with purple flowers and 4 plants with white flowers. Box B contains 5 plants with purple flowers and 1 plant with white flowers. A box is selected by tossing a coin, and one plant is removed at random from it.

Find the probability that the plant:

a was taken from box A and has white flowers

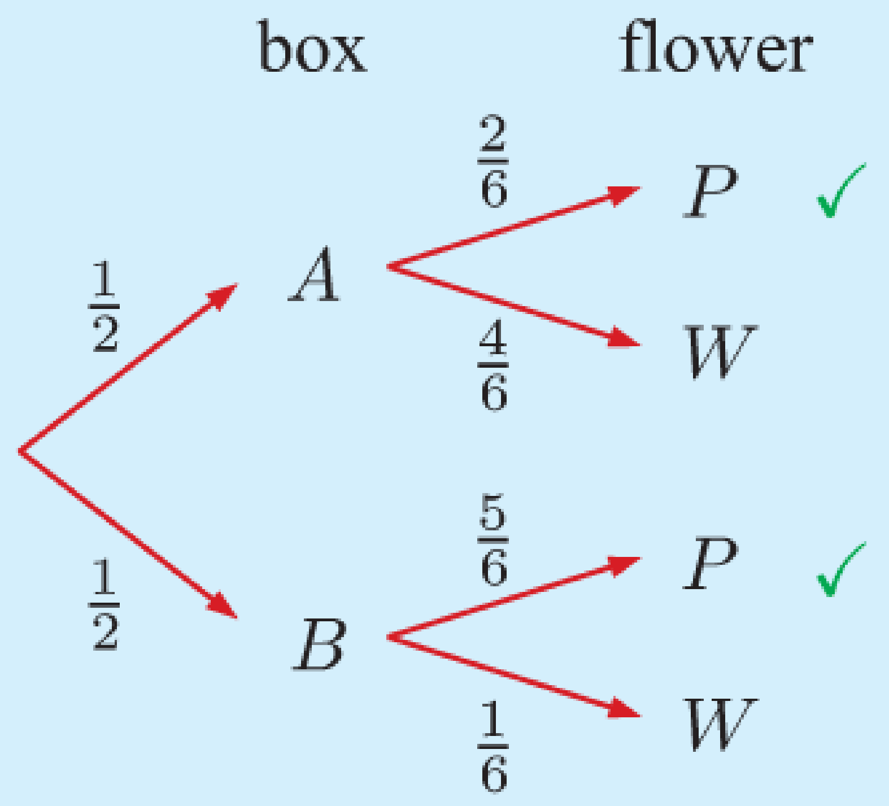
b has purple flowers.

Box A

P	W	W
W	P	W

Box B

P	P	P
W	P	P



a P(from box A ∩ white flowers)

$= \mathrm{P}(A \cap W)$

$= \frac{1}{2} \times \frac{4}{6}$

$= \frac{1}{3}$

b P(purple flowers)

$= \mathrm{P}(A \cap P) + \mathrm{P}(B \cap P)$

$= \frac{1}{2} \times \frac{2}{6} + \frac{1}{2} \times \frac{5}{6}$ {branches marked ✓}

$= \frac{7}{12}$

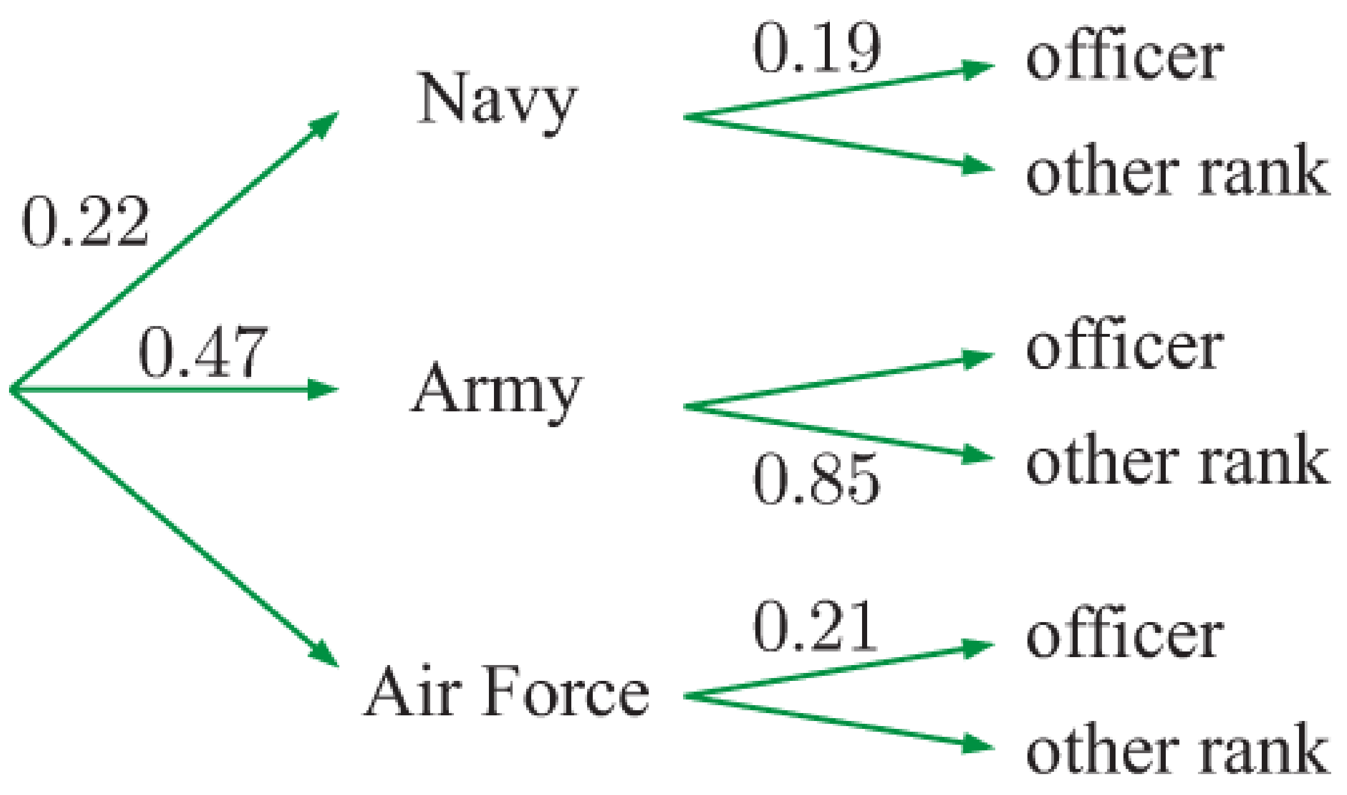
7 **a** Copy and complete this tree diagram about people in the armed forces.

b Find the probability that a member of the armed forces:

i is an officer

ii is not an officer in the navy

iii is not an army or air force officer.



8 Of the students in a class playing musical instruments, 60% are female. 20% of the females and 30% of the males play the violin.
Find the probability that a randomly selected student:

a is male and does not play the violin **b** plays the violin.

9 The probability of rain tomorrow is $\frac{1}{5}$. If it rains, Mudlark will start favourite in the horse race, with probability $\frac{1}{2}$ of winning. If it is fine, Mudlark only has a 1 in 20 chance of winning.

a Display the sample space of possible results for the horse race on a tree diagram.

b Hence determine the probability that Mudlark will win tomorrow.

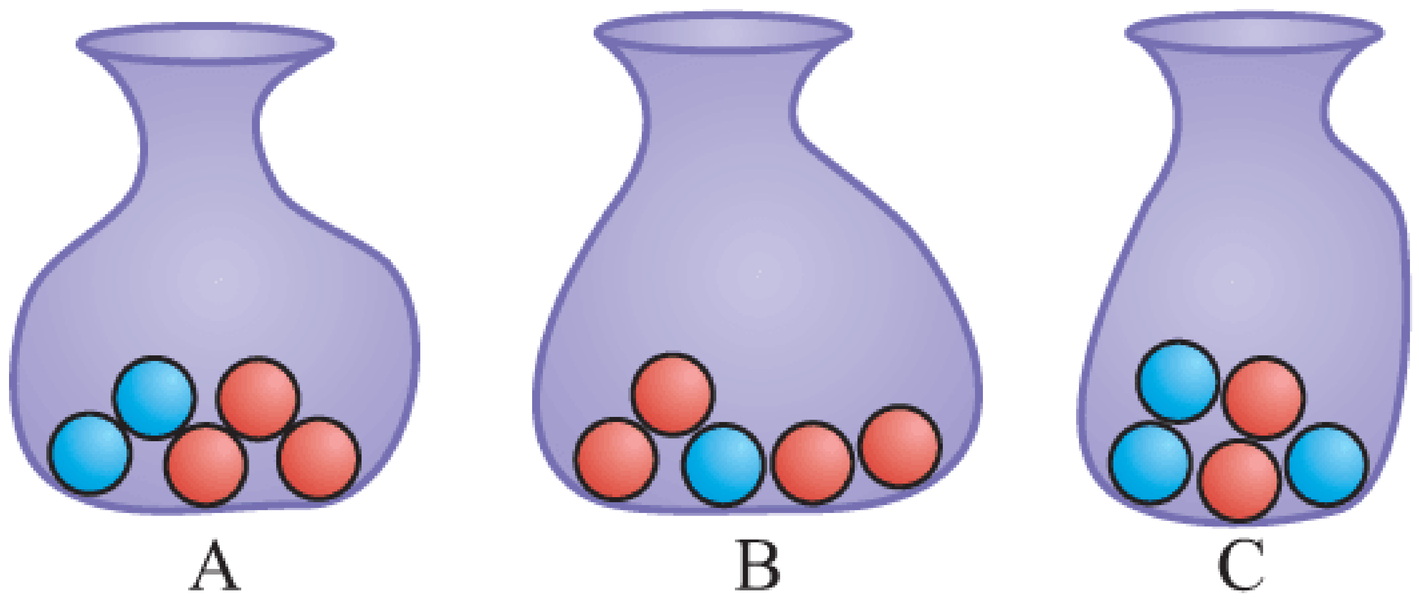
10 Machine A makes 40% of the bottles produced at a factory. Machine B makes the rest. Machine A spoils 5% of its product, while machine B spoils only 2%. Using an appropriate tree diagram, determine the probability that the next bottle inspected at this factory is spoiled.

11 The English Premier League consists of 20 teams. Tottenham is currently in 8th place on the table. It has 20% chance of winning and 50% chance of losing against any team placed above it. If a team is placed below it, Tottenham has a 50% chance of winning and a 30% chance of losing. Find the probability that Tottenham will draw its next game.

12 Three bags contain different numbers of blue and red marbles.

A bag is selected using a die which has three A faces, two B faces, and one C face. One marble is then randomly selected from the bag.

Determine the probability that the marble is:



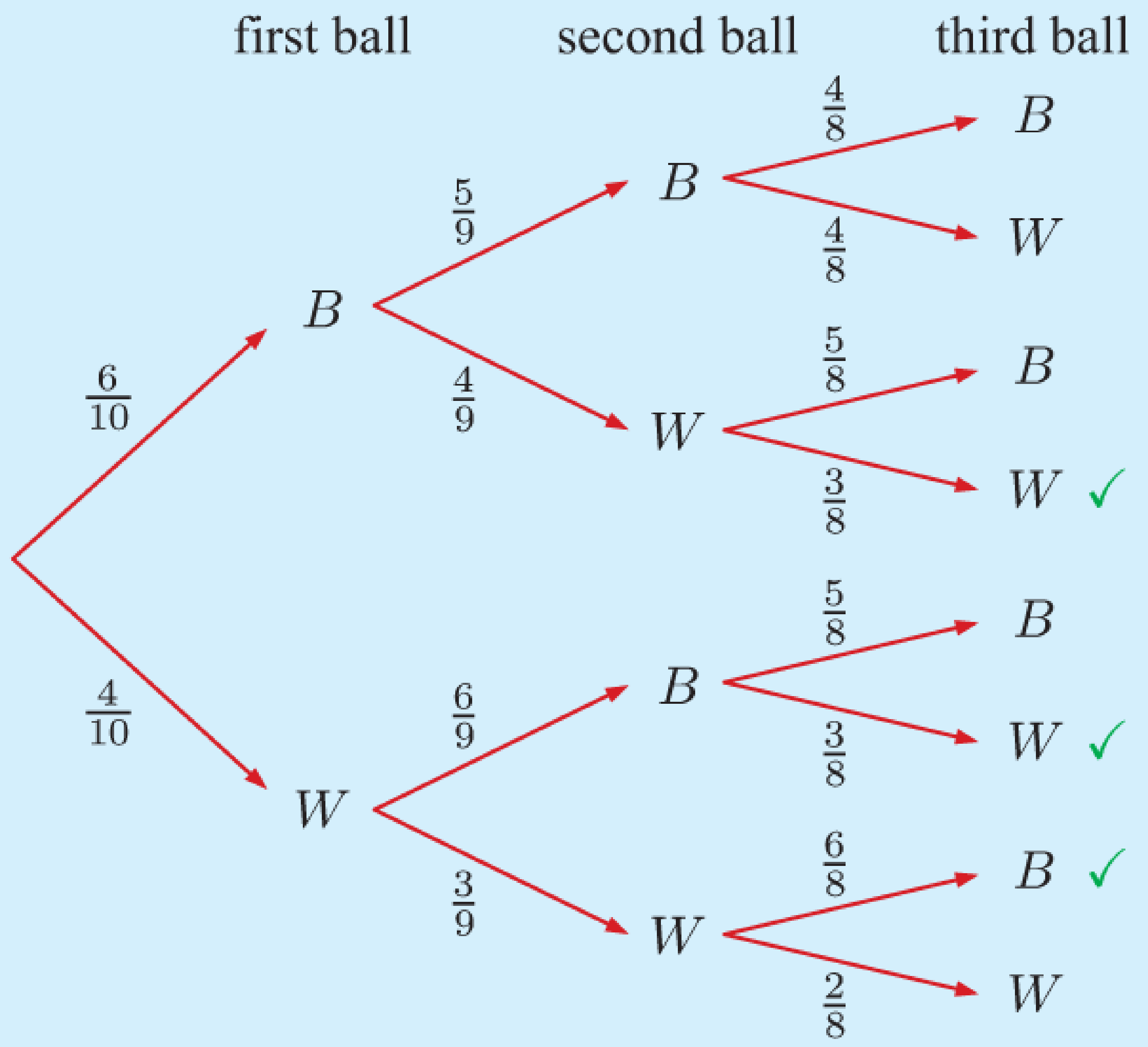
a blue **b** red.

Example 21

Self Tutor

A bag contains 10 balls. 6 balls are black and 4 balls are white. Three balls are drawn from the bag without replacement. Determine the probability that 1 black ball is drawn.

Let B represent drawing a black ball and W represent drawing a white ball.



$$\begin{aligned}&\text{P}(1 \text{ black ball})\\ &= \text{P}(BWW) + \text{P}(WBW) + \text{P}(WWB)\\ &= \left(\tfrac{6}{10}\right)\left(\tfrac{4}{9}\right)\left(\tfrac{3}{8}\right) + \left(\tfrac{4}{10}\right)\left(\tfrac{6}{9}\right)\left(\tfrac{3}{8}\right) + \left(\tfrac{4}{10}\right)\left(\tfrac{3}{9}\right)\left(\tfrac{6}{8}\right)\\ &\qquad\qquad\qquad\qquad \{\text{branches marked } \checkmark\}\\ &= \tfrac{1}{10} + \tfrac{1}{10} + \tfrac{1}{10}\\ &= \tfrac{3}{10}\end{aligned}$$

13 In a class of 25 students, 11 students participate in extra-curricular activities. Suppose 3 students are randomly selected to be on the student representative council. Find the probability that at least two students selected for the council also participate in extra-curricular activities.

14 A standard deck of playing cards contains 52 cards. Four cards are drawn from a well-shuffled deck without replacement. Find the probability that:

a two red cards are drawn **b** at least one black card is drawn.

ACTIVITY 1 PÓLYA'S URN

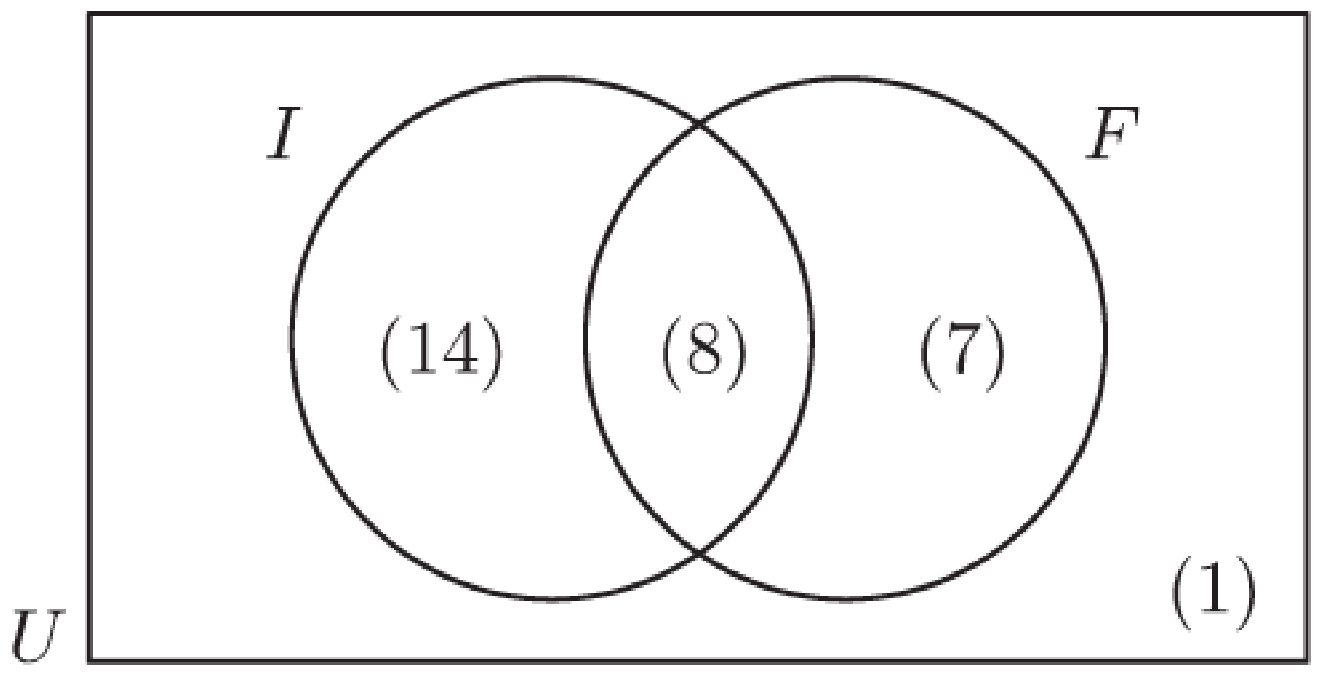
Pólya's urn is a statistical model devised by the Hungarian mathematician **George Pólya** (1887 - 1985).

Click on the icon to obtain this Activity.

PÓLYA'S URN

I CONDITIONAL PROBABILITY

This Venn diagram shows the numbers of students in a class who study Italian (I) and French (F).



Suppose a student is randomly selected from the class and it is found that the student studies French.

We can determine the probability that this student also studies Italian. We call this a **conditional probability** because it is the probability of I occurring on the *condition* that F has occurred.

$$\text{P}(I \text{ given that } F \text{ has occurred}) = \frac{8}{15} \begin{array}{l} \leftarrow \text{number of students who study Italian and French} \\ \leftarrow \text{number of students who study French} \end{array}$$

For events A and B, we use the notation "$A \mid B$" to represent the event "*A given that B has occurred*".

$$\mathbf{P}(\boldsymbol{A} \mid \boldsymbol{B}) = \frac{\boldsymbol{n}(\boldsymbol{A} \cap \boldsymbol{B})}{\boldsymbol{n}(\boldsymbol{B})}$$

If the outcomes in each of the events are equally likely, notice that:

$$\frac{n(A \cap B)}{n(B)} = \frac{\frac{n(A \cap B)}{n(U)}}{\frac{n(B)}{n(U)}} = \frac{\text{P}(A \cap B)}{\text{P}(B)}$$

This gives us the **conditional probability formula**:

$$\mathbf{P}(\boldsymbol{A} \mid \boldsymbol{B}) = \frac{\mathbf{P}(\boldsymbol{A} \cap \boldsymbol{B})}{\mathbf{P}(\boldsymbol{B})}$$

EXERCISE 11I

1 Find $\text{P}(A \mid B)$ if:

- **a** $\text{P}(A \cap B) = 0.1$ and $\text{P}(B) = 0.4$
- **b** $\text{P}(A) = 0.3$, $\text{P}(B) = 0.4$, and $\text{P}(A \cup B) = 0.5$
- **c** A and B are mutually exclusive.

2 The probability that it is cloudy on a particular day is 0.4. The probability that it is cloudy *and* rainy on a particular day is 0.2. Find the probability that it will be rainy on a day when it is cloudy.

3 In a group of 50 students, 40 study Mathematics, 32 study Physics, and each student studies at least one of these subjects.

- **a** Use a Venn diagram to find how many students study both subjects.
- **b** If a student from this group is randomly selected, find the probability that he or she:
  - **i** studies Mathematics but not Physics
  - **ii** studies Physics given that he or she studies Mathematics.

4 Out of 40 boys, 23 have dark hair, 18 have brown eyes, and 26 have dark hair, brown eyes, or both.

- **a** Draw a Venn diagram to display this information.
- **b** One of the boys is selected at random. Determine the probability that he has:
  - **i** dark hair and brown eyes
  - **ii** brown eyes given that he has dark hair.

5 50 hikers participated in an orienteering event during summer. 23 were sunburnt, 22 were bitten by ants, and 5 were both sunburnt and bitten by ants.

- **a** Draw a Venn diagram to display this information.
- **b** Determine the probability that a randomly selected hiker:
  - **i** avoided being bitten
  - **ii** was bitten or sunburnt (or both)
  - **iii** was bitten given that he or she was sunburnt
  - **iv** was sunburnt given that he or she was not bitten.

Example 22

Self Tutor

In a town, 25% of the residents own a cat, 55% own a dog, and 30% do not own either animal.

- **a** Draw a Venn diagram to describe the situation.
- **b** Find the probability that a randomly selected resident:
  - **i** owns a cat given that they own a dog
  - **ii** does not own a dog given that they own a cat.

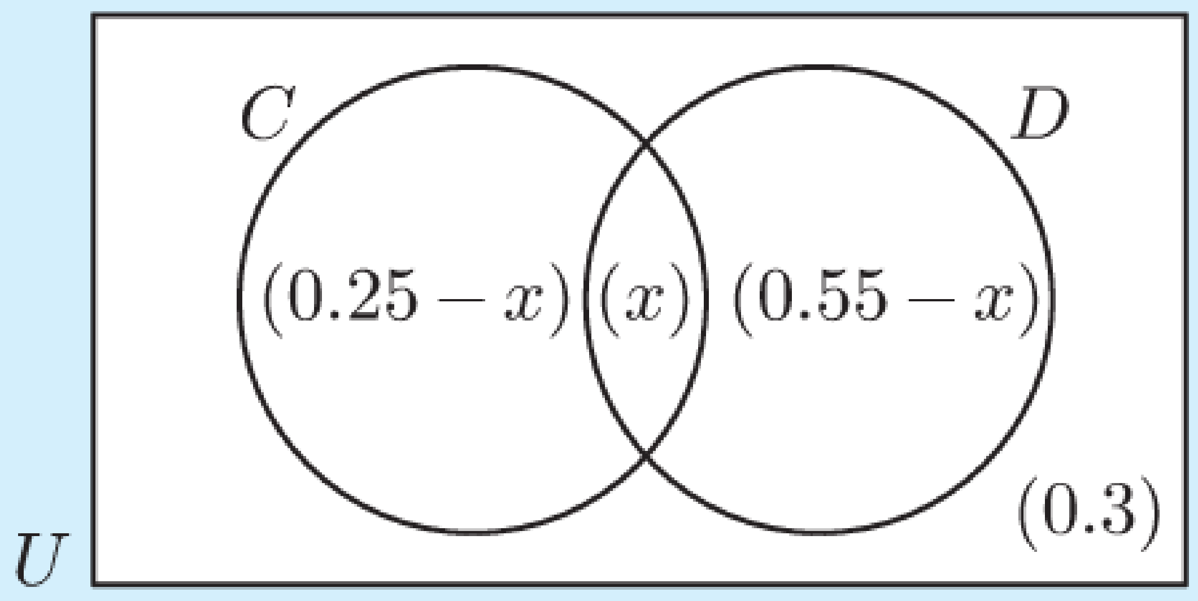
a Let C represent residents who own a cat and D represent residents who own a dog.

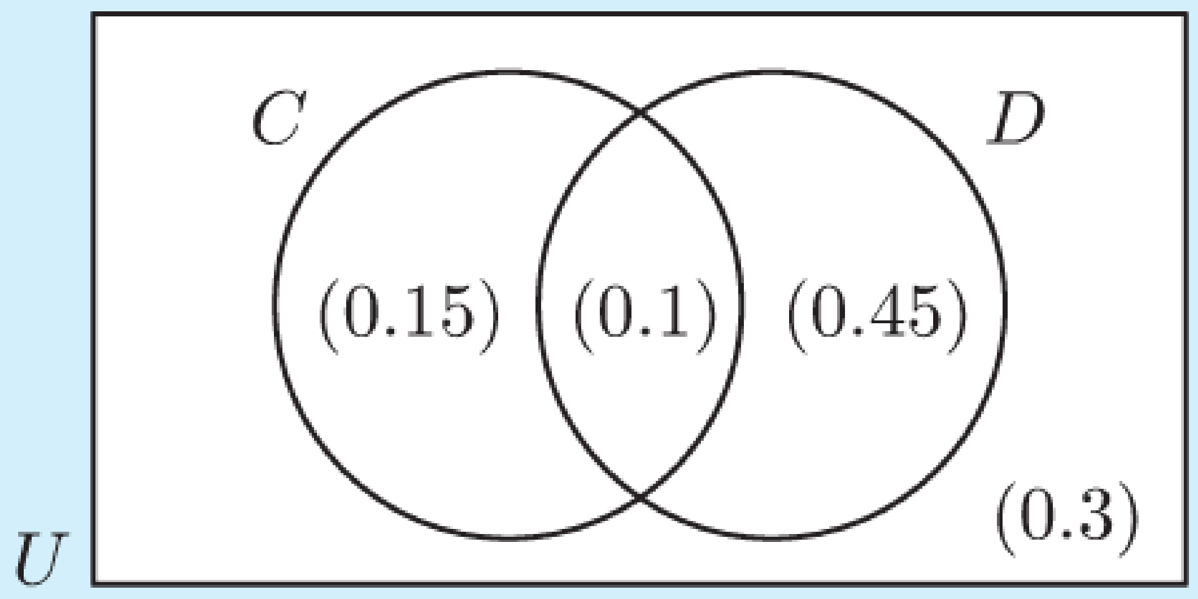
Let the proportion of residents in $C \cap D$ be x.

$\therefore$ the proportion in $C \cap D'$ is $0.25 - x$ and the proportion in $C' \cap D$ is $0.55 - x$.

The proportion in $C' \cap D'$ is 0.3.

$$\therefore \quad (0.25 - x) + x + (0.55 - x) = 0.7$$
$$\therefore \quad 0.8 - x = 0.7$$
$$\therefore \quad x = 0.1$$





b **i** $$\mathrm{P}(C \mid D) = \frac{\mathrm{P}(C \cap D)}{\mathrm{P}(D)} = \frac{0.1}{0.55} \approx 0.182$$

ii $$\mathrm{P}(D' \mid C) = \frac{\mathrm{P}(D' \cap C)}{\mathrm{P}(C)} = \frac{0.15}{0.25} = 0.6$$

6 400 families were surveyed. It was found that 90% had a TV set and 80% had a computer. Every family had at least one of these items. One of the families is randomly selected, and it is found that they have a computer. Find the probability that they also have a TV set.

7 In a certain town three newspapers are published. 20% of the population read A, 16% read B, 14% read C, 8% read A and B, 5% read A and C, 4% read B and C, and 2% read all 3 newspapers. A person is selected at random. Use a Venn diagram to help determine the probability that the person reads:

- **a** none of the papers
- **b** at least one of the papers
- **c** exactly one of the papers
- **d** A or B (or both)
- **e** A, given that the person reads at least one paper
- **f** C, given that the person reads either A or B or both.

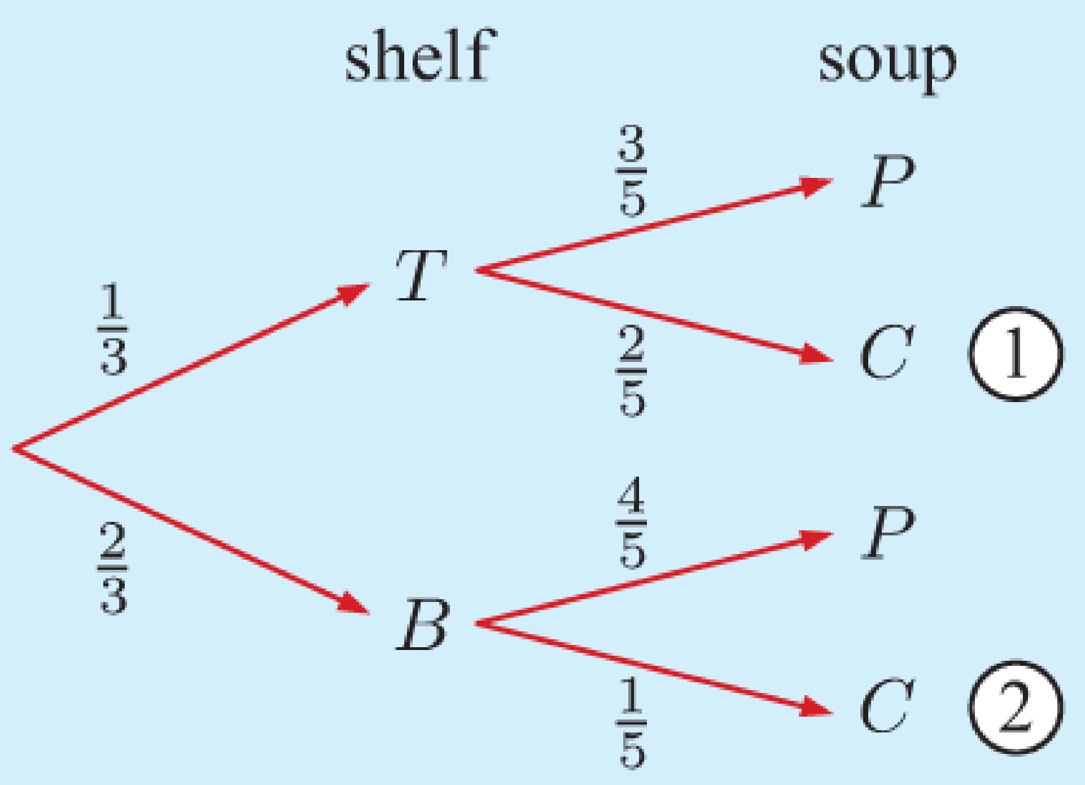
Example 23 **Self Tutor**

The top shelf in a cupboard contains 3 cans of pumpkin soup and 2 cans of chicken soup. The bottom shelf contains 4 cans of pumpkin soup and 1 can of chicken soup. Lukas is twice as likely to take a can from the bottom shelf as he is from the top shelf.

Suppose Lukas takes one can of soup without looking at the label. Find the probability that it:

a is chicken **b** was taken from the top shelf given that it is chicken.

Let T represent the top shelf, B represent the bottom shelf, P represent the pumpkin soup, and C represent the chicken soup.



a $\mathrm{P}(C)$

$$= \underbrace{\mathrm{P}(T \cap C)}_{\text{branch } \textcircled{1}} + \underbrace{\mathrm{P}(B \cap C)}_{\text{branch } \textcircled{2}}$$

$$= \tfrac{1}{3} \times \tfrac{2}{5} + \tfrac{2}{3} \times \tfrac{1}{5}$$

$$= \tfrac{4}{15}$$

b $\mathrm{P}(T \mid C)$

$$= \frac{\mathrm{P}(T \cap C)}{\mathrm{P}(C)}$$

$$= \frac{\frac{1}{3} \times \frac{2}{5}}{\frac{4}{15}} \quad \begin{array}{l} \longleftarrow \text{branch } \textcircled{1} \\ \longleftarrow \text{from } \mathbf{a} \end{array}$$

$$= \tfrac{1}{2}$$

8 Urn A contains 2 red and 3 blue marbles, and urn B contains 4 red and 1 blue marble. Peter selects an urn by tossing a coin, and takes a marble from that urn.

a Determine the probability that the marble is red.

b Given that the marble is red, what is the probability that it came from urn B?

9 When Greta's mother goes shopping, the probability that she takes Greta with her is $\frac{2}{5}$. When Greta goes shopping with her mother she gets an ice cream 70% of the time. When Greta does not go shopping with her mother she gets an ice cream 30% of the time.
Determine the probability that:

a when Greta's mother goes shopping, she buys Greta an ice cream

b Greta went shopping with her mother, given that her mother buys her an ice cream.

10 On a given day, machine X has a 10% chance of malfunctioning and machine Y has a 7% chance of the same.

a Last Thursday *exactly* one of the machines malfunctioned. Find the probability that it was machine X.

b At least one of the machines malfunctioned today. Find the probability that machine Y malfunctioned.

11 Bags A and B each contain red and green balls. In bag B, the ratio of red balls to green balls is 1 : 4.
When a bag is randomly selected and a ball is randomly chosen from it, the probability that the ball is red is $\frac{1}{3}$.
Find the proportion of red balls in bag A.

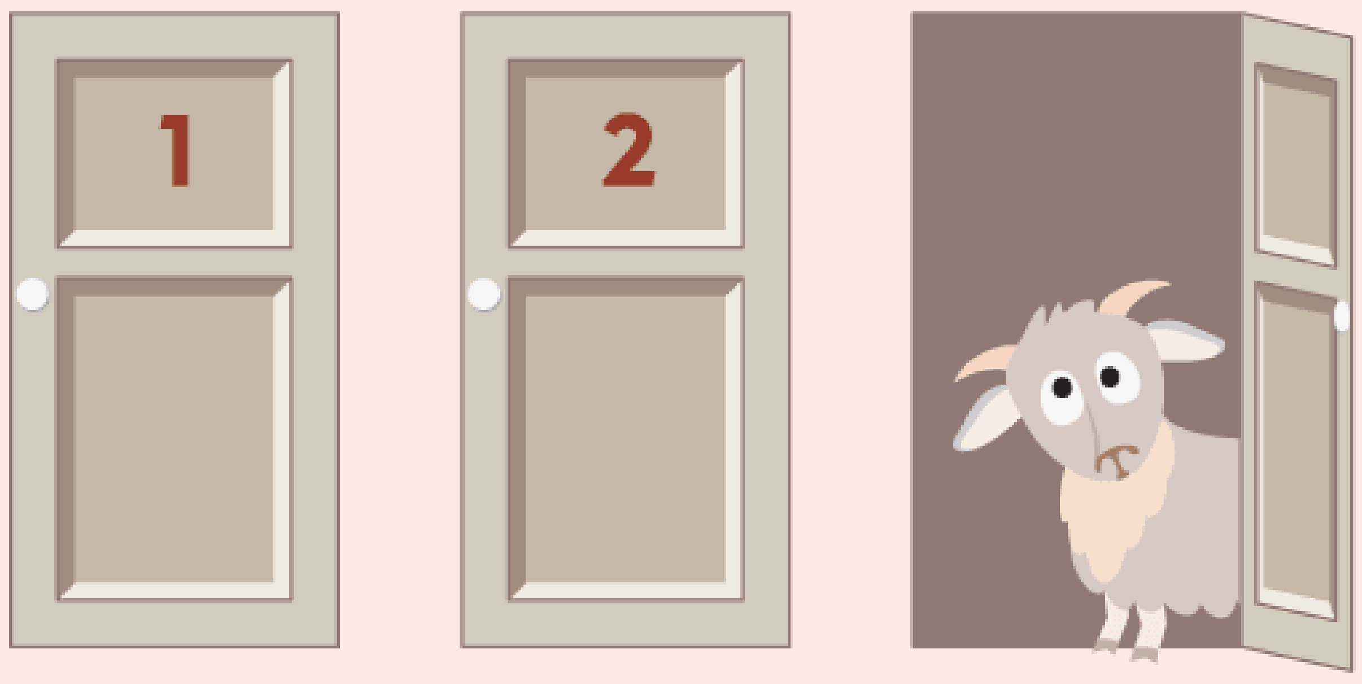
ACTIVITY 2 THE MONTY HALL PROBLEM

The Monty Hall problem is a mathematical paradox first posed by **Steve Selvin** to the *American Statistician* in 1975. It became famous after its publication in *Parade* magazine in 1990. The problem is named after the original host of the American television game show *Let's Make a Deal*, on which the problem is loosely based.

The problem as posed in *Parade* reads:

> *Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to switch your choice to door No. 2?" Is it to your advantage to switch your choice?*

What to do:



1. Draw a tree diagram to represent the problem. Let C represent the event that a contestant's choice is correct, and C' represent the event that the choice is incorrect.
2. Find the probability that:
   - **a** the contestant's first choice has the car
   - **b** the contestant's second choice has the car *given* they decide to change their guess.
3. Suppose this game is being played in front of a live studio audience. One of the audience members arrives late, so when they enter the room, they see two closed doors and the third (incorrect) door open. They do not know the contestant's original choice.
   - **a** If this audience member is asked to choose a door, what is the probability they will choose the one with the car?
   - **b** Explain why the contestant has an advantage over this audience member.

ACTIVITY 3 PENNEY'S GAME

Invented by **Walter Penney** in 1969, **Penney's Game** is a 2-player coin tossing game.

PENNEY'S GAME

Click on the icon to obtain this Activity.

J FORMAL DEFINITION OF INDEPENDENCE

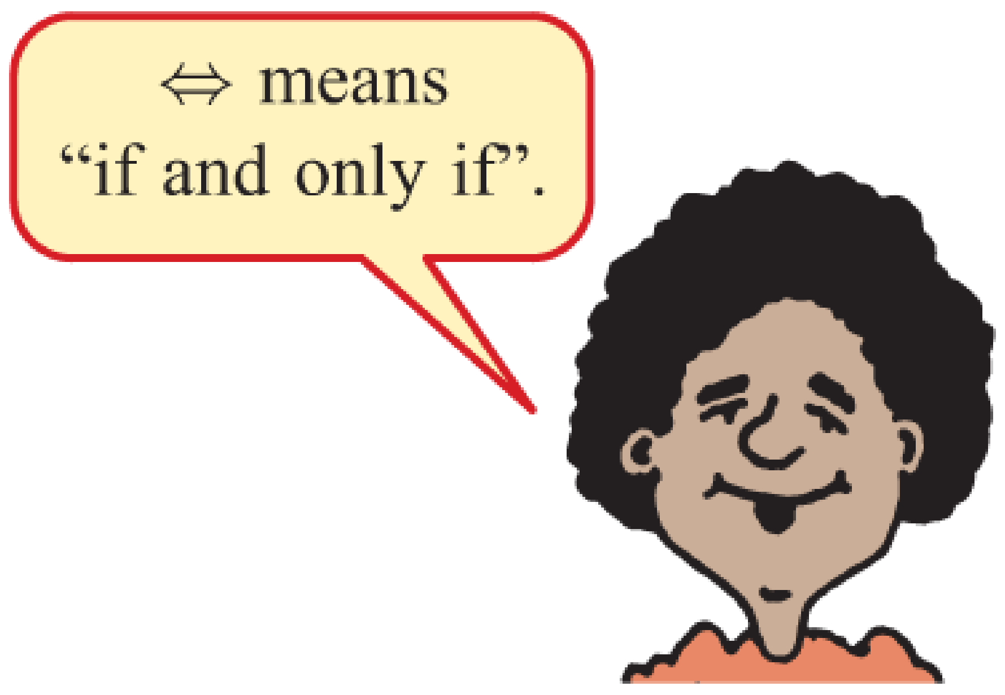
In **Section G** we saw that two events are **independent** if the occurrence of each event does not affect the probability that the other occurs. We can write this definition more formally using conditional probability notation:

> A and B are **independent events** if the occurrence of each one of them does not affect the probability that the other occurs.
>
> This means that $\text{P}(A \mid B) = \text{P}(A \mid B') = \text{P}(A)$
>
> and that $\text{P}(B \mid A) = \text{P}(B \mid A') = \text{P}(B)$.

Using $\text{P}(A \cap B) = \text{P}(A \mid B)\,\text{P}(B)$ we see that

A and B are **independent events** $\Leftrightarrow$ $\text{P}(A \cap B) = \text{P}(A)\,\text{P}(B)$

which is the result we saw earlier.



Example 24 **Self Tutor**

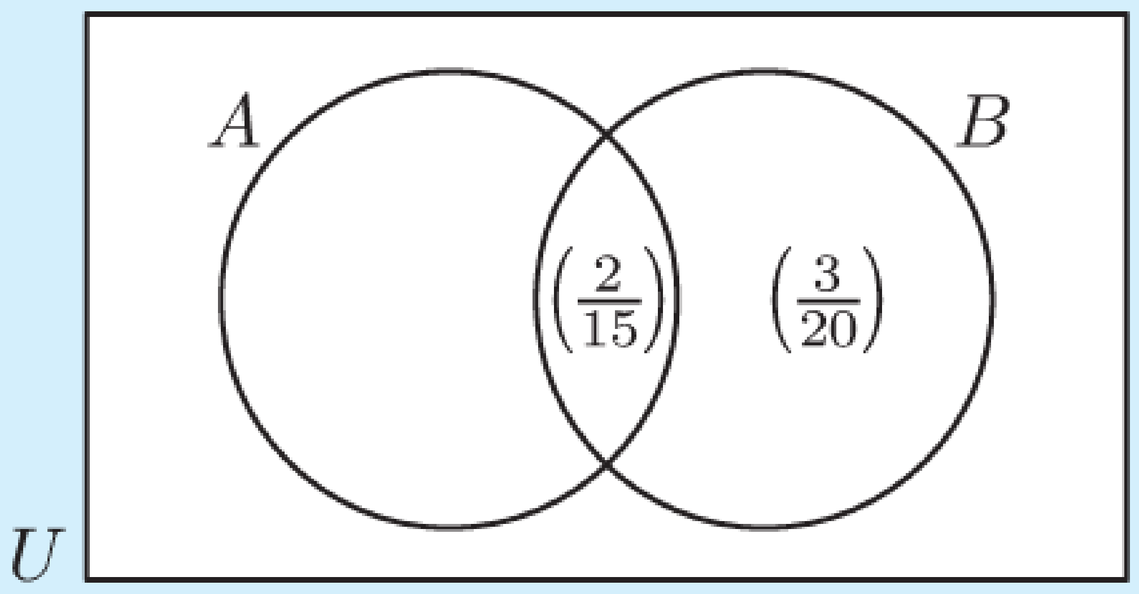
Suppose $\text{P}(A) = \frac{2}{5}$, $\text{P}(B \mid A) = \frac{1}{3}$, and $\text{P}(B \mid A') = \frac{1}{4}$.

a Find $\text{P}(B)$.

b Are A and B independent events? Justify your answer.

$$\begin{aligned}\text{P}(B \cap A) &= \text{P}(B \mid A)\,\text{P}(A)\\ &= \tfrac{1}{3} \times \tfrac{2}{5}\\ &= \tfrac{2}{15}\end{aligned} \qquad \begin{aligned}\text{P}(B \cap A') &= \text{P}(B \mid A')\,\text{P}(A')\\ &= \tfrac{1}{4} \times \tfrac{3}{5}\\ &= \tfrac{3}{20}\end{aligned}$$

$\therefore$ the Venn diagram is:



a $\text{P}(B) = \frac{2}{15} + \frac{3}{20} = \frac{17}{60}$

b $\text{P}(B) \neq \text{P}(B \mid A)$, so A and B are not independent events.

EXERCISE 11J

1 Suppose $\text{P}(R) = 0.4$, $\text{P}(S) = 0.5$, and $\text{P}(R \cup S) = 0.7$. Are R and S independent events? Justify your answer.

2 Suppose $\text{P}(A) = \frac{2}{5}$, $\text{P}(B) = \frac{1}{3}$, and $\text{P}(A \cup B) = \frac{1}{2}$.

- **a** Find:
  - **i** $\text{P}(A \cap B)$
  - **ii** $\text{P}(B \mid A)$
  - **iii** $\text{P}(A \mid B)$
- **b** Are A and B independent events? Justify your answer.

3 Suppose $\text{P}(X) = 0.5$, $\text{P}(Y) = 0.7$, and that X and Y are independent events. Determine the probability of the occurrence of:

- **a** both X and Y
- **b** X or Y or both
- **c** neither X nor Y
- **d** X but not Y
- **e** X given that Y occurs.

4 A and B are independent events. Prove that A' and B' are also independent events.

5 Suppose A and B are independent, mutually exclusive events, and that $\text{P}(A) = \frac{5}{7}$. Find $\text{P}(B)$.

6 Suppose $P(A \cap B) = 0.1$ and $P(A \cap B') = 0.4$. Given that A and B are independent, find $P(A \cup B')$.

7 Suppose $P(C) = \frac{9}{20}$, $P(D \mid C) = \frac{1}{4}$, and $P(D \mid C') = \frac{1}{5}$.

  a Find $P(D)$. **b** Are C and D independent events? Justify your answer.

8 What can be deduced if $A \cap B$ and $A \cup B$ are independent events?

K BAYES' THEOREM

HISTORICAL NOTE

Bayes' theorem is named after **Reverend Thomas Bayes** (1701 - 1761) who was an English statistician, philosopher, and Presbyterian minister. Bayes first described a special case of the theorem in "*An Essay towards solving a Problem in the Doctrine of Chances*" which was posthumously published in 1764.

Despite being attributed to Thomas Bayes, most of the work on the theorem and its interpretation was actually done by French mathematician **Pierre-Simon Laplace**.

Today, Bayes' theorem lies at the heart of a field of statistics called **Bayesian statistics**.

Reverend Thomas Bayes

In **Section I**, we saw the conditional probability formula: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$ .... (1)

Notice that we can also write: $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$

$$\therefore \quad P(A \cap B) = P(B \mid A)\,P(A) \quad .... (2)$$

Now substituting (2) into (1) gives: $$\boxed{P(A \mid B) = \frac{P(B \mid A)\,P(A)}{P(B)}}$$

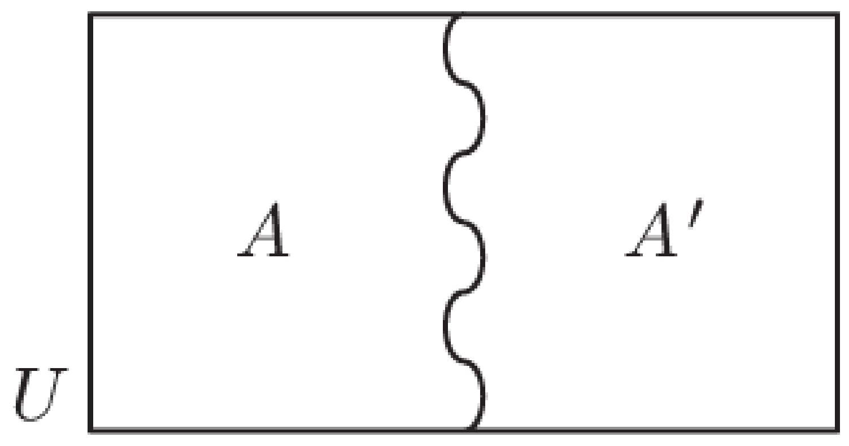
This result is known as **Bayes' theorem**.

PARTITIONS OF THE SAMPLE SPACE

For any event A and its complement A':

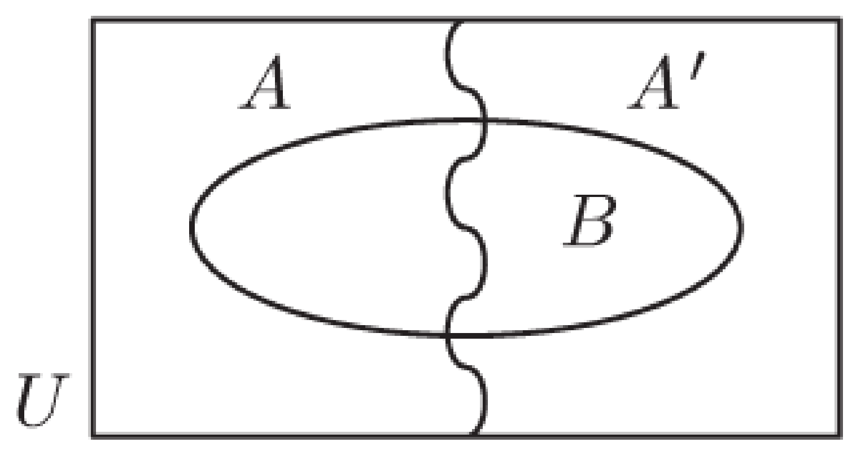
- A and A' are mutually exclusive, so $A \cap A' = \varnothing$
- $A \cup A' = U$, the sample space.

We say that A and A' **partition** the sample space, and we can represent this on a Venn diagram as shown.



For any other event B in the sample space U,

$$\begin{aligned} \mathrm{P}(B) &= \mathrm{P}(B \cap A) + \mathrm{P}(B \cap A') \\ &= \mathrm{P}(B \mid A)\,\mathrm{P}(A) + \mathrm{P}(B \mid A')\,\mathrm{P}(A') \end{aligned}$$



So, Bayes' theorem can alternatively be written as:

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(B \mid A)\,\mathbf{P}(A)}{\mathbf{P}(B \mid A)\,\mathbf{P}(A) + \mathbf{P}(B \mid A')\,\mathbf{P}(A')}$$

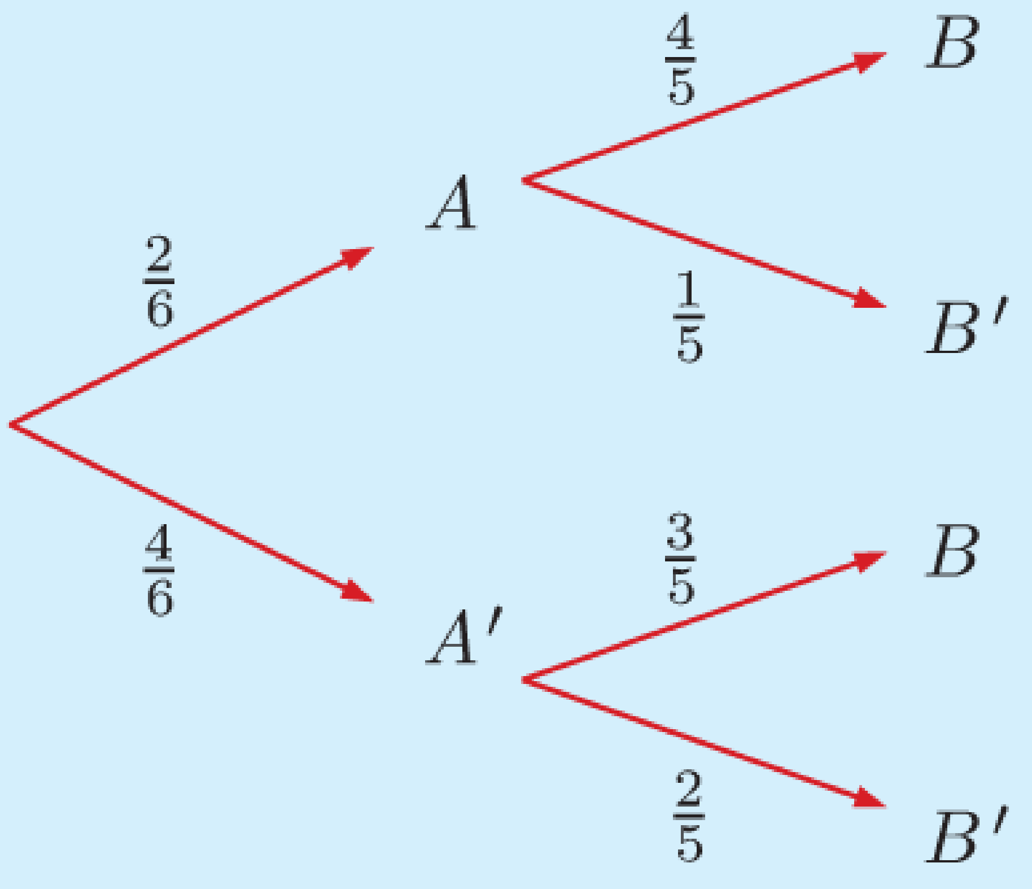
Example 25 **Self Tutor**

A can contains 4 blue and 2 green marbles. One marble is randomly drawn from the can without replacement, and its colour is noted. A second marble is then drawn. Find the probability that:

a the second marble is blue

b the first marble was green, given that the second marble is blue.

Let A be the event that the first marble is green and B be the event that the second marble is blue.



a P(second marble is blue)

$$\begin{aligned} &= \mathrm{P}(B) \\ &= \mathrm{P}(B \mid A)\,\mathrm{P}(A) + \mathrm{P}(B \mid A')\,\mathrm{P}(A') \\ &= \tfrac{4}{5} \times \tfrac{2}{6} + \tfrac{3}{5} \times \tfrac{4}{6} \\ &= \tfrac{2}{3} \end{aligned}$$

b P(first was green | second is blue)

$$\begin{aligned} &= \mathrm{P}(A \mid B) \\ &= \frac{\mathrm{P}(B \mid A)\,\mathrm{P}(A)}{\mathrm{P}(B)} \quad \{\text{Bayes' theorem}\} \\ &= \frac{\frac{4}{5} \times \frac{2}{6}}{\frac{2}{3}} \quad \{\text{using } \mathbf{a}\} \\ &= \tfrac{2}{5} \end{aligned}$$

EXERCISE 11K

1 Coffee machines A and B produce coffee in identically shaped plastic cups. Machine A produces 65% of the coffee sold each day, and machine B produces the remainder. Machine A underfills a cup 4% of the time, while machine B underfills a cup 5% of the time.

a A cup of coffee is chosen at random. Find the probability that it is underfilled.

b A cup of coffee is randomly chosen and is found to be underfilled. Find the probability that it came from machine A.

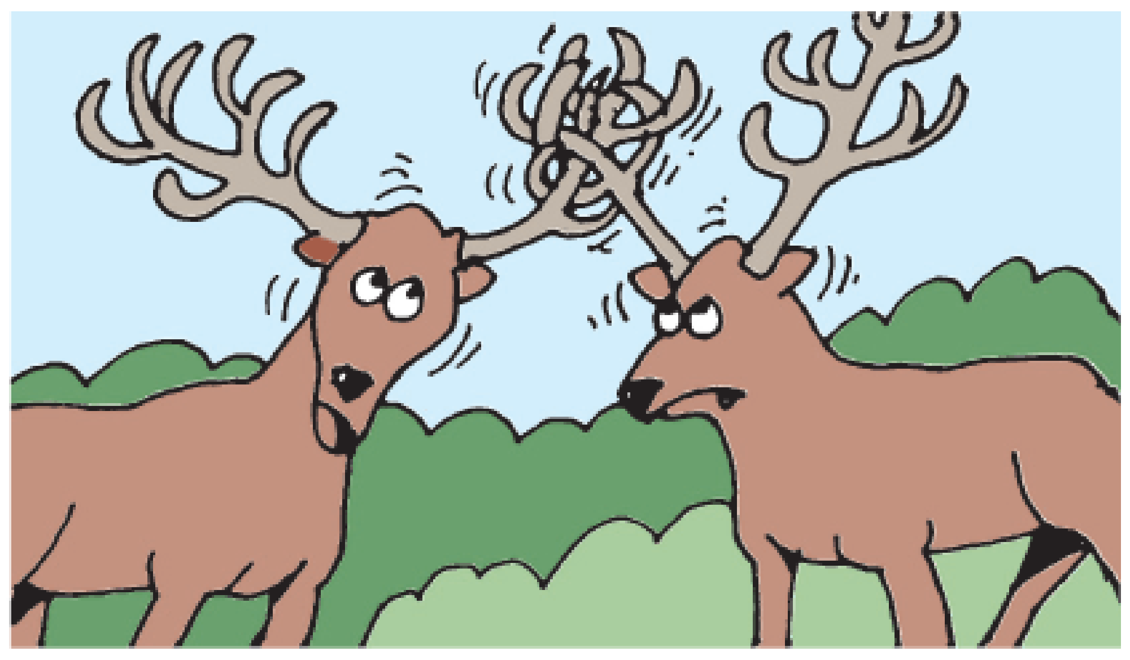
2 54% of the students at a university are female. 8% of the male students are colour-blind, and 2% of the female students are colour-blind.

a A randomly chosen student is colour-blind. Find the probability that the student is male.

b A randomly chosen student is not colour-blind. Find the probability that the student is female.

3 A marble is randomly chosen from a can containing 3 red and 5 blue marbles. It is replaced by two marbles of the other colour. Another marble is then randomly chosen from the can. Given that the marbles chosen have the same colour, what is the probability that they are both blue?

4 35% of the animals in a deer herd carry the TPC gene. 58% of these deer also carry the SD gene, while 23% of the deer without the TPC gene carry the SD gene. A deer is randomly chosen and is found to carry the SD gene. Find the probability that it does not carry the TPC gene.



5 A blood test has been designed to detect a particular form of cancer. The probability that the test correctly identifies someone with the cancer is 0.97, and the probability that the test correctly identifies someone without the cancer is 0.93. Approximately 0.1% of the general population are known to contract this cancer.

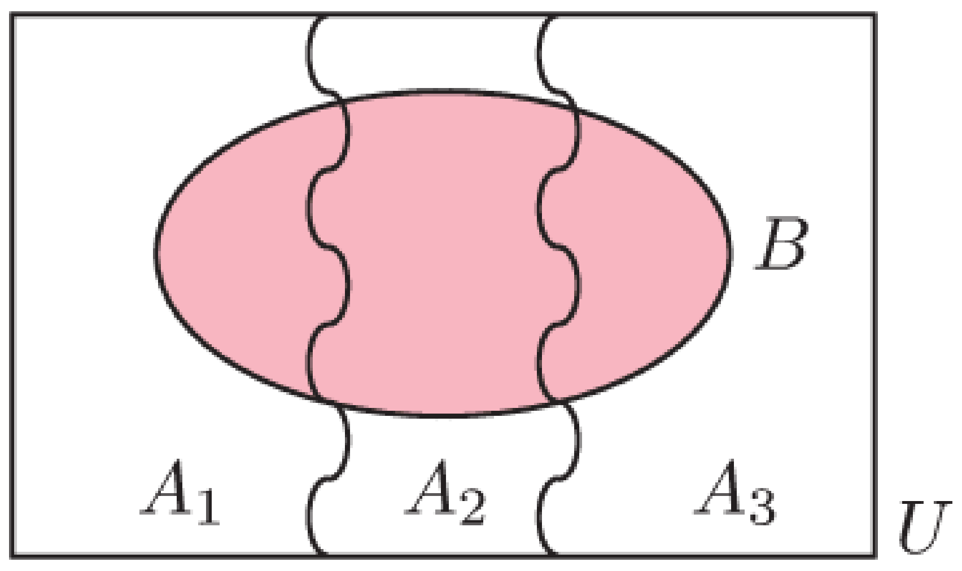
When a patient has a blood test, the test results are positive for the cancer. Find the probability that the patient actually has the cancer.

6 A man drives his car to work 80% of the time. The remainder of the time he rides his bicycle. When he rides his bicycle to work he is late 25% of the time. When he drives his car to work he is late 15% of the time. On a particular day, the man arrives early. Find the probability that he rode his bicycle to work that day.

7 The probabilities that Hiran's mother and father will be alive after ten years are 0.99 and 0.98 respectively. If only one of them is alive after ten years, find the probability that it will be his mother.

8 A manufacturer produces drink bottles using two machines. Machine A produces 60% of the bottles, and 3% of those it makes are defective. Machine B produces 40% of the bottles, and 5% of those it makes are defective. Find the probability that a defective bottle comes from:

a machine A **b** machine B.

9 

A sample space U is partitioned into three by the mutually exclusive events A_1, A_2, and A_3.
The sample space also contains another event B.

a Show that $\mathrm{P}(B) = \mathrm{P}(B \mid A_1)\,\mathrm{P}(A_1) + \mathrm{P}(B \mid A_2)\,\mathrm{P}(A_2) + \mathrm{P}(B \mid A_3)\,\mathrm{P}(A_3)$.

b Hence show that Bayes' theorem for the case of three partitions is

$$\mathrm{P}(A_i \mid B) = \frac{\mathrm{P}(B \mid A_i)\,\mathrm{P}(A_i)}{\mathrm{P}(B)}, \quad i \in \{1, 2, 3\} \quad \text{where} \quad \mathrm{P}(B) = \sum_{j=1}^{3} \mathrm{P}(B \mid A_j)\,\mathrm{P}(A_j).$$

10 A newspaper printer has three presses A, B, and C which print 30%, 40%, and 30% of daily production respectively. Due to the age of the machines, the presses will produce streaks on their output 3%, 5%, and 7% of the time, respectively.

a Find the probability that a randomly chosen newspaper does not have streaks.

b If a randomly chosen newspaper does not have streaks, find the probability that it was printed by press A.

c If a randomly chosen newspaper has streaks, find the probability that it was printed by either press A or C.

11 12% of the over-60 population of Agento have lung cancer. Of those with lung cancer, 50% were heavy smokers, 40% were moderate smokers, and 10% were non-smokers. Of those without lung cancer, 5% were heavy smokers, 15% were moderate smokers, and 80% were non-smokers.

A member of the over-60 population of Agento is chosen at random.

- **a** Find the probability that the person was a heavy smoker.
- **b** Given the person was a moderate smoker, find the probability that the person has lung cancer.
- **c** Given the person was a non-smoker, find the probability that the person has lung cancer.

12 205 teachers and 52 headmasters attended an educational conference. Attendees were given t-shirts at the start of the conference.

Of the attending teachers, 39 received a small t-shirt, 120 received a medium t-shirt, and 46 received a large t-shirt.

Of the attending headmasters, 16 received a small t-shirt, 25 received a medium t-shirt, and 11 received a large t-shirt.

- **a** A teacher lost their conference t-shirt. Find the probability that it is a large t-shirt.
- **b** A large t-shirt was left behind at a plenary talk. Find the probability that it belongs to a teacher.
- **c** Use Bayes' theorem to explain why your answers to **a** and **b** are different.

THEORY OF KNOWLEDGE

Modern probability theory began in 1653 when gambler Chevalier de Mere contacted mathematician **Blaise Pascal** with a problem on how to divide the stakes when a gambling game is interrupted during play. Pascal involved **Pierre de Fermat**, a lawyer and amateur mathematician, and together they solved the problem. In the process they laid the foundations upon which the laws of probability were formed.

Applications of probability are now found from quantum physics to medicine and industry.

Agner Krarup Erlang

The first research paper on **queueing theory** was published in 1909 by the Danish engineer **Agner Krarup Erlang** who worked for the Copenhagen Telephone Exchange. In the last hundred years this theory has become an integral part of the huge global telecommunications industry, but it is equally applicable to modelling car traffic or queues at your local supermarket.

Statistics and probability are used extensively to predict the behaviour of the global stock market. For example, American mathematician **Edward Oakley Thorp** developed and applied hedge fund techniques for the financial markets in the 1960s.

On the level of an individual investor, money is put into the stock market if there is a good probability that the value of the shares will increase. This investment has risk, however, as witnessed by historic stock market crashes like that of Wall Street in 1929 which triggered the Great Depression, the Black Monday crash of 1987, and the Global Financial Crisis of 2008 - 2009.

However, the question "What is probability?" is as much a philosophical question as it is mathematical.

In statistics, there are two main interpretations of probability:

- The **frequentist** interpretation considers probability as a measure of how *frequently* an event occurs if an experiment is repeated many times.
- The **Bayesian** interpretation considers probability to be a measure of the strength of one's prior beliefs of an event occurring. Such beliefs are often *subjective*.

1 Discuss the frequentist and Bayesian interpretations. Which interpretation makes the most sense to you?

2 How does a knowledge of probability theory affect decisions we make?

3 What roles should ethics play in the use of mathematics? You may wish to consider:

- What responsibility does a casino have to operate as a functioning business? What responsibility does it have to the welfare of habitual gamblers? How do these responsibilities affect the way casinos operate?
- By clever mathematical modelling of the global stock markets, you may be able to gain a market advantage. In your gain, does somebody else lose? What rules are in place to protect against financial corruption?
- Do rich countries adopt foreign policies and control trade in order that poor countries remain poor? How much power is associated with financial wealth?

REVIEW SET 11A

1 Kate recorded the number of emails she sent each day for 30 days. Find, to 2 decimal places, the experimental probability that tomorrow she will send:

a 5 emails **b** less than 3 emails.

Number of emails	*Frequency*
0	2
1	5
2	9
3	5
4	4
5	4
6	1

2 A coin is tossed and a square spinner labelled A, B, C, D is twirled.

a Draw a 2-dimensional grid to illustrate the sample space.

b Determine the probability of obtaining:

i a head and consonant **ii** a tail and C

iii a tail or a vowel (or both).

3 Explain what is meant by:

a independent events **b** mutually exclusive events.

4 The students A, B, and C have 10%, 20%, and 30% chance of independently solving a certain maths problem. If they all try independently of one another, what is the probability that at least one of them will solve the problem?

5 A and B are mutually exclusive events where $\mathrm{P}(A) = x$ and $\mathrm{P}(B') = 0.43$.

a Write $\mathrm{P}(A \cup B)$ in terms of x. **b** Find x given that $\mathrm{P}(A \cup B) = 0.73$.

6 On any one day, there is a 25% chance of rain and 36% chance that it will be windy.

- **a** Draw a tree diagram showing the probabilities of wind or rain on a particular day.
- **b** Hence determine the probability that on a particular day there will be:
  - **i** rain and wind
  - **ii** rain or wind (or both).
- **c** What assumption have you made in your answers?

7 Given $P(Y) = 0.35$ and $P(X \cup Y) = 0.8$, and that X and Y are mutually exclusive events, find:

- **a** $P(X \cap Y)$
- **b** $P(X)$
- **c** $P(X$ or Y but not both).

8
- **a** Graph the sample space of all possible outcomes when a pair of dice is rolled.
- **b** Hence determine the probability of getting:
  - **i** a sum of 7 or 11
  - **ii** a sum of at least 8.

9 In a group of 40 students, 22 study Economics, 25 study Law, and 3 study neither of these subjects.

- **a** Draw a Venn diagram to display this information.
- **b** Determine the probability that a randomly chosen student studies:
  - **i** both Economics and Law
  - **ii** at least one of these subjects
  - **iii** Economics given that they study Law.

10 The probability that a tomato seed will germinate is 0.87. If a market gardener plants 5000 seeds, how many are expected to germinate?

11 A bag contains 3 red, 4 yellow, and 5 blue marbles. Two marbles are randomly selected from the bag with replacement. Find the probability that:

- **a** both are blue
- **b** they are the same colour
- **c** at least one is red
- **d** exactly one is yellow.

12 Of the people who apply for executive positions, those with a university degree have 0.33 chance of eventually being successful, while those without have only 0.17 chance. If 78% of all applicants for a particular executive position have a university degree, find the probability that the successful applicant does not have one.

13 A survey of 200 people included 90 females. It found that 60 people smoked, 40 of whom were male.

- **a** Use the given information to complete the two-way table.
- **b** A person is selected at random. Find the probability that this person is:
  - **i** a female non-smoker
  - **ii** a male given the person was a non-smoker.

	Female	*Male*	*Total*
Smoker			
Non-smoker			
Total			

- **c** If two people from the survey are selected at random, calculate the probability that:
  - **i** both of them are non-smoking females
  - **ii** one is a smoker and the other is a non-smoker.

14 The government of Australia is divided up into the Senate and the House of Representatives. As of 2018:

- Of the 76 Senators, 31 are part of the Coalition, 26 are part of the Opposition, and 19 are crossbenchers.
- Of the 150 members of parliament (MPs) in the House of Representatives, 73 are part of the Coalition, 69 are part of the Opposition, and 8 are crossbenchers.

Given that a randomly selected politician is part of the Opposition, find the probability that the politician is an MP.

15 For two events A and B, it is known that $\text{P}(A) = \frac{2}{5}$, $\text{P}(B) = \frac{3}{10}$, and $\text{P}(B \mid A) = \frac{1}{2}$.

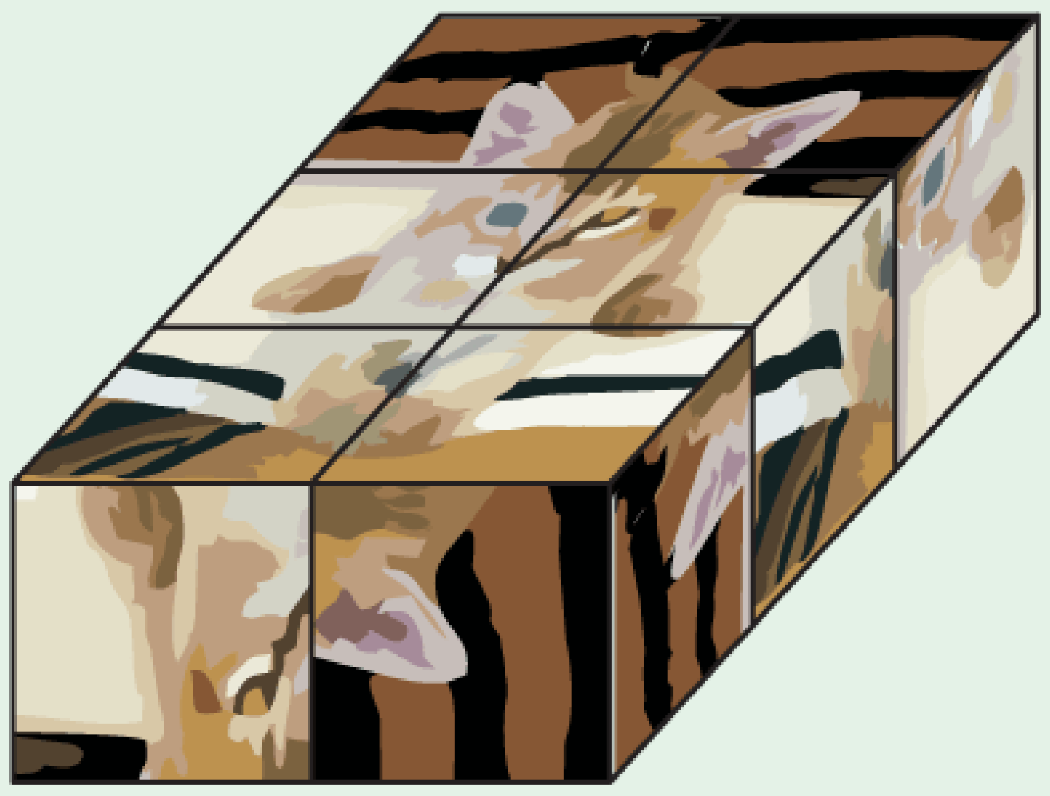
a Calculate $\text{P}(A \cap B)$.

b Show that A and B are not independent.

c Calculate $\text{P}(A \mid B)$.

16 A picture is divided into 6 squares of equal size. One of each of the squares is printed on each of the 6 faces of a cube. The cube is then copied until there are 6 identical cubes.

If the 6 cubes are "rolled", find the probability that the upmost faces can be assembled into the original image.



REVIEW SET 11B

1 T and M are events such that $n(U) = 30$, $n(T) = 10$, $n(M) = 17$, and $n((T \cup M)') = 5$.

a Draw a Venn diagram to display this information.

b Hence find:

  i $\text{P}(T \cap M)$

  ii $\text{P}((T \cap M) \mid M)$

2 A school photocopier has a 95% chance of working on any particular day. Find the probability that it will be working on at least one of the next two days.

3 Suppose A and B are independent events, $\text{P}(A) = 0.4$, and $\text{P}(B) = 0.7$.

a Calculate $\text{P}(A \cap B)$. Hence explain why A and B cannot be mutually exclusive.

b Calculate $\text{P}(A \cup B)$.

4 The probability that a particular salesman will leave his sunglasses behind in any store is $\frac{1}{5}$. Suppose the salesman visits two stores in succession and leaves his sunglasses behind in one of them. What is the probability that the salesman left his sunglasses in the first store?

5 A survey of 50 men and 50 women was conducted to see how many people prefer coffee or tea. It was found that 15 men and 24 women prefer tea.

Let C represent the people who prefer coffee and M represent the men.

a Represent C and M on a Venn diagram.

b Calculate $\text{P}(M \mid C)$.

6 Niklas and Rolf play tennis with the winner being the first to win two sets. Niklas has a 40% chance of beating Rolf in any set.

- **a** Draw a tree diagram showing the possible outcomes.
- **b** Hence determine the probability that Niklas will win the match.

7 A and B are independent events where $\mathrm{P}(A) = 0.8$ and $\mathrm{P}(B) = 0.65$. Determine:

- **a** $\mathrm{P}(A \cup B)$
- **b** $\mathrm{P}(A \mid B)$
- **c** $\mathrm{P}(A' \mid B')$
- **d** $\mathrm{P}(B \mid A)$.

8 If I buy 4 tickets in a 500 ticket lottery and the prizes are drawn without replacement, determine the probability that I will win:

- **a** the first 3 prizes
- **b** at least one of the first 3 prizes.

9 The students in a school are all invited to participate in a survey. 48% of the students at the school are males, of whom 16% will participate in the survey. 35% of the females will also participate in the survey. A student is randomly chosen from the school. Find the probability that the student:

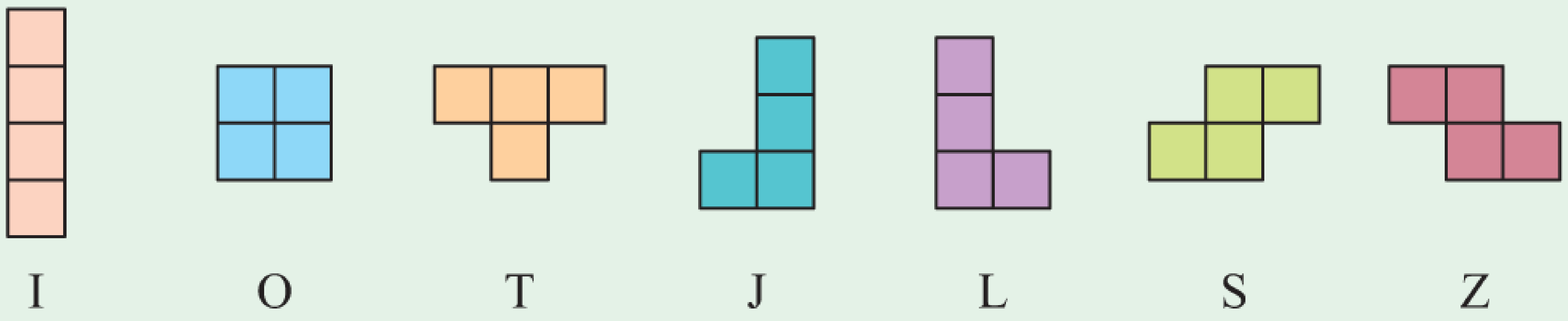
- **a** will participate in the survey
- **b** is female given that he or she will participate in the survey.

10 For the two events A and B, $\mathrm{P}(A) = \frac{3}{7}$ and $\mathrm{P}(B') = \frac{2}{3}$.

- **a** Determine $\mathrm{P}(B)$.
- **b** Calculate $\mathrm{P}(A \cup B)$ if A and B are:
  - **i** mutually exclusive
  - **ii** independent.

11 Suppose $\mathrm{P}(X' \mid Y) = \frac{2}{3}$, $\mathrm{P}(Y) = \frac{5}{6}$, and $X' \cap Y' = \varnothing$. Find $\mathrm{P}(X)$.

12 In a particular video game there are 7 distinct pieces called *tetrominos*, each of which represents a letter:



The pieces are given to the player in a random sequence.

- **a** Find the probability of getting at least one "I" in a sequence of 7 pieces.
- **b** Find the expected number of "O"s in a sequence of 100 pieces.

13 Jon goes cycling on three random mornings of each week. When he goes cycling he has eggs for breakfast 70% of the time. When he does not go cycling he has eggs for breakfast 25% of the time. Determine the probability that Jon:

- **a** has eggs for breakfast
- **b** goes cycling given that he has eggs for breakfast.

14 With each pregnancy, a particular woman will give birth to either a single baby or twins. There is a 15% chance of having twins during each pregnancy. Suppose that after 2 pregnancies she has given birth to 3 children. Find the probability that she had twins first.

15 The table alongside shows the number of balloons in a giant party pack.

	Red	*Yellow*	*Blue*
Large	12	5	9
Medium	15	8	10
Small	24	11	6

a State the:

- **i** total number of balloons in the pack
- **ii** number of medium balloons in the pack.

b One balloon is chosen at random from the pack. Find the probability that:

- **i** the balloon is not yellow
- **ii** the balloon is either medium or small.

c Two balloons are selected at random from the pack. Find the probability that:

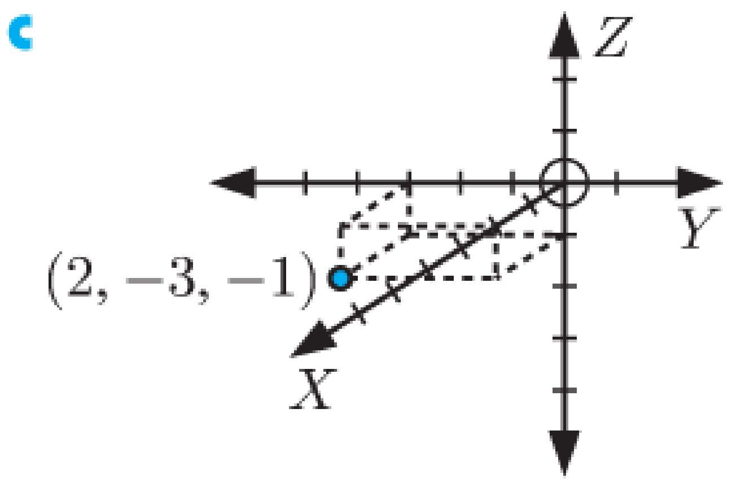
- **i** both balloons are red
- **ii** neither of the balloons are large
- **iii** exactly one of the balloons is blue
- **iv** at least one of the balloons is blue.

d Three balloons are selected at random from the pack. Find the probability that:

- **i** all three balloons are small and yellow
- **ii** exactly two balloons are medium and red.

16 Answer the questions in the **Opening Problem** on page **248**.

c



2 **a** **i** 6 units **ii** $(-1, -4, 1)$

b **i** $3\sqrt{10}$ units **ii** $(-\frac{5}{2}, -3, \frac{7}{2})$

3 isosceles with $AB = AC = \sqrt{41}$ units

4 **a** 128 units^3 **b** $M(8, 4, 0)$ **c** $2\sqrt{13}$ units

d $32(2 + \sqrt{13})$ units^2 ≈ 179 units^2 **e** $\approx 29.0°$

5 **a** $\sqrt{29}$ units

b volume ≈ 327 units^3, surface area ≈ 273 units^2

6 **a** $M(5, 4, 3)$ **b** $\approx 64.9°$ **c** **i** $\approx 43.1°$ **ii** $\approx 25.1°$

7 $\approx 61.4°$ **8** **a** $k = 2$ **b** ≈ 68.9 units^2

9 **a** $P(2, 7, -2.5)$, $Q(8, 3, -2.9)$ **b** ≈ 7.22 m

c $\approx 3.17°$

REVIEW SET 10B

1 **a** **i** $\sqrt{41}$ units **ii** $(-2, 3, \frac{9}{2})$

b **i** $\sqrt{83}$ units **ii** $(-\frac{9}{2}, \frac{5}{2}, \frac{5}{2})$

2 **a** $PQ = \sqrt{14}$ units, $PR = \sqrt{45}$ units, $QR = \sqrt{59}$ units

$PQ^2 + PR^2 = (\sqrt{14})^2 + (\sqrt{45})^2 = 59 = QR^2$

$\therefore$ triangle PQR is right angled.

b $\approx 60.8°$

3 $k = 1 \pm \sqrt{30}$

4 **a** 96 units^3 **b** $2\sqrt{13}$ units

c $(104 + 16\sqrt{13}) \approx 162$ units^2

5 **a** $(-1, 0, -1)$ **b** $3\sqrt{5}$ units

c volume ≈ 1260 units^3, surface area ≈ 565 units^2

6 **a** $\approx 21.4°$ **b** ≈ 3.53 units^2

7 **a** $M(6, 9, 5)$ **b** $\approx 71.6°$ **c** **i** $\approx 54.2°$ **ii** $\approx 36.7°$

8 **a** $H(2, -4, \frac{1}{5})$ **b** ≈ 4.48 km

c **i** $M(-4, 1, \frac{1}{2})$ **ii** ≈ 7.82 km **iii** $\approx 2.20°$

9 **a** $R(0, 0, 3)$ **b** $\widehat{PRO} \approx 46.5°$, $\widehat{QRO} \approx 36.7°$

c $\widehat{PRQ} \approx 60.6°$

EXERCISE 11A

1 **a** ≈ 0.78 **b** ≈ 0.22

2 **a** ≈ 0.487 **b** ≈ 0.051 **c** ≈ 0.731

3 **a** 43 days **b** **i** ≈ 0.0465 **ii** ≈ 0.186 **iii** ≈ 0.465

4 **a** ≈ 0.0895 **b** ≈ 0.126

5 **a** ≈ 0.265 **b** ≈ 0.861 **c** ≈ 0.222

6 **a** ≈ 0.146 **b** ≈ 0.435 **c** ≈ 0.565

7 **a** **i** ≈ 0.171 **ii** ≈ 0.613 **b** ≈ 0.366 **c** ≈ 0.545

EXERCISE 11B

1 **a** 7510 **b** **i** ≈ 0.325 **ii** ≈ 0.653 **iii** ≈ 0.243

2 **a**

	Junior	*Middle*	*Senior*	*Total*
Sport	131	164	141	436
No sport	28	81	176	285
Total	159	245	317	721

b **i** $\frac{436}{721} \approx 0.605$ **ii** $\frac{131}{721} \approx 0.182$ **iii** $\frac{257}{721} \approx 0.356$

3 **a** **i** $\frac{743}{1235} \approx 0.602$ **ii** $\frac{148}{1235} \approx 0.120$ **iii** $\frac{1085}{1235} \approx 0.879$

iv $\frac{795}{1235} \approx 0.644$

b $\frac{52}{492} \approx 0.106$ **c** $\frac{518}{862} \approx 0.601$

EXERCISE 11C

1 **a** {A, B, C, D} **b** {1, 2, 3, 4, 5, 6, 7, 8}

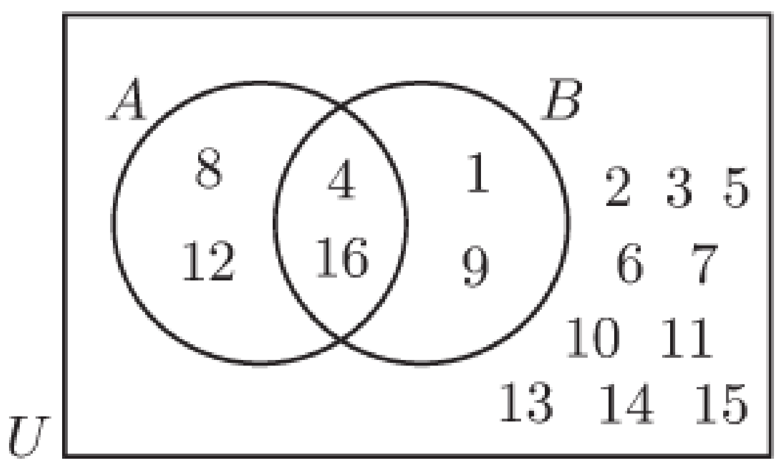
c {MM, MF, FM, FF}

2 **a** $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$

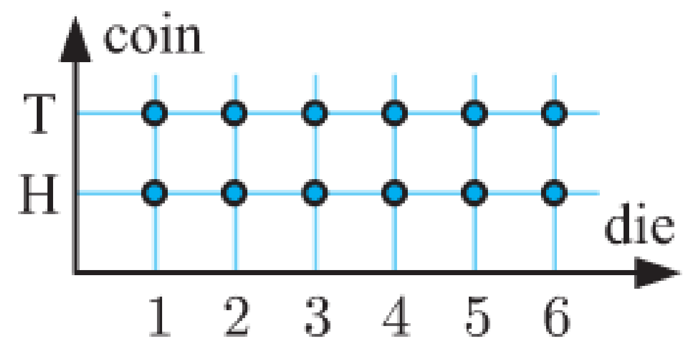
b **i** $A = \{4, 8, 12, 16\}$

ii $B = \{1, 4, 9, 16\}$

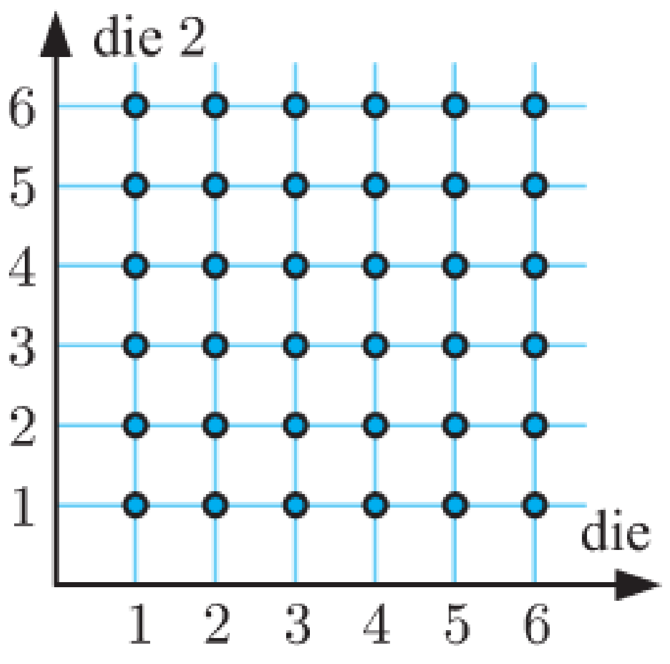
c



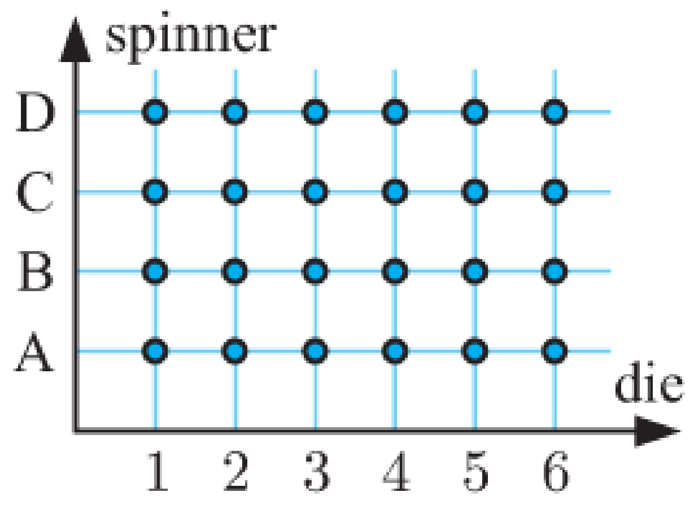
3 **a**



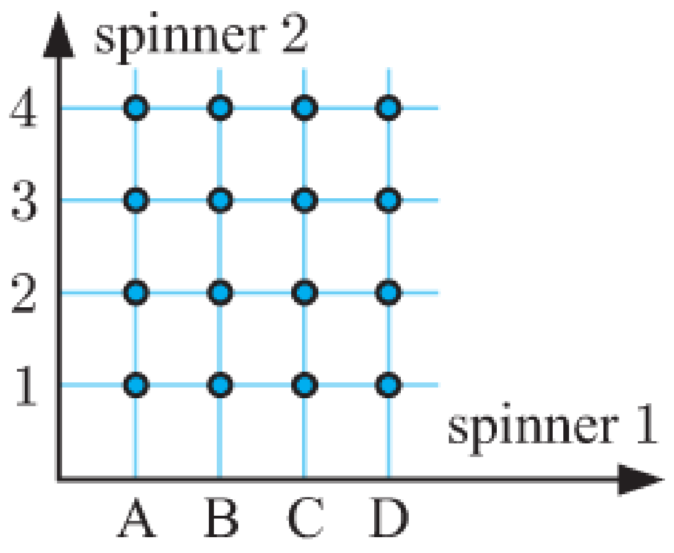
b



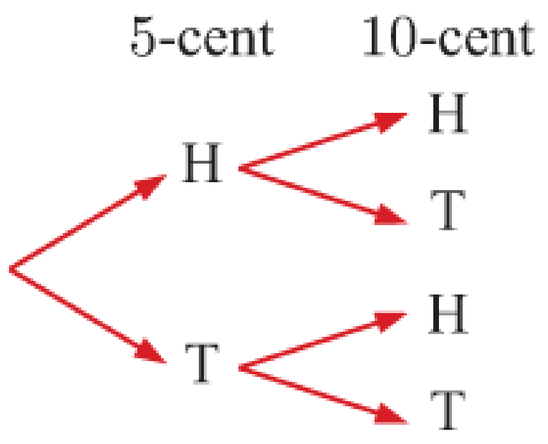
c



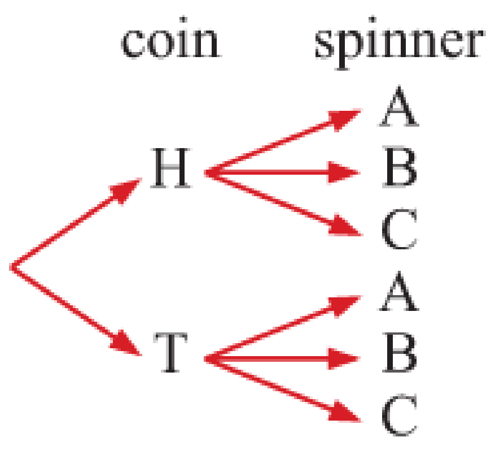
d



4 **a**



b



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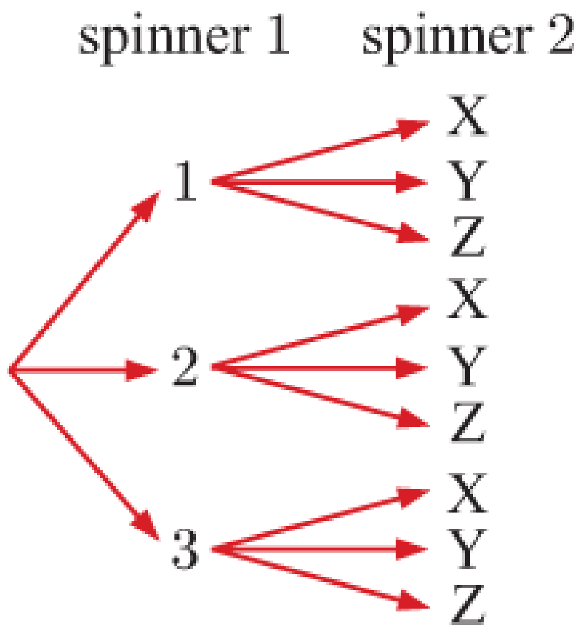

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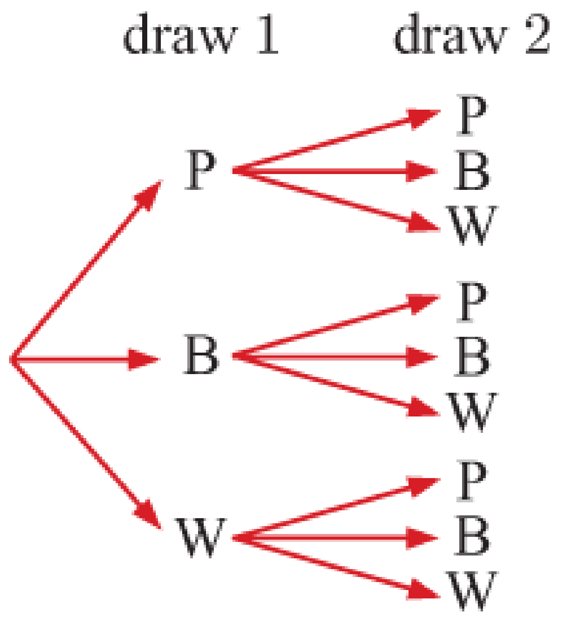
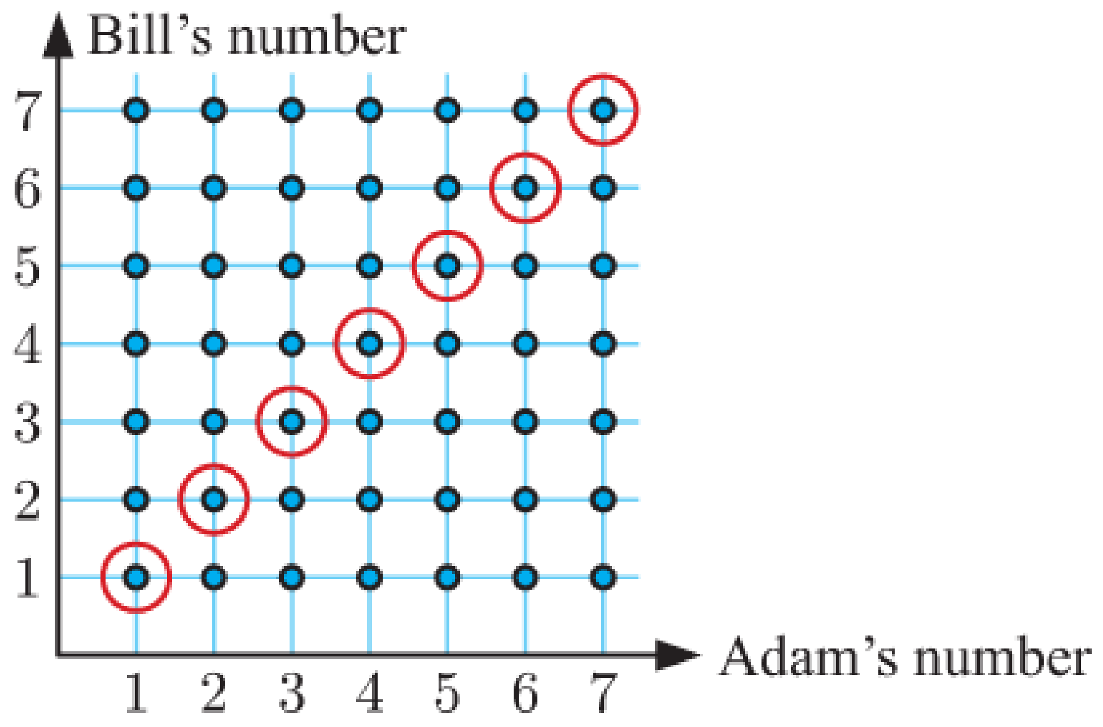
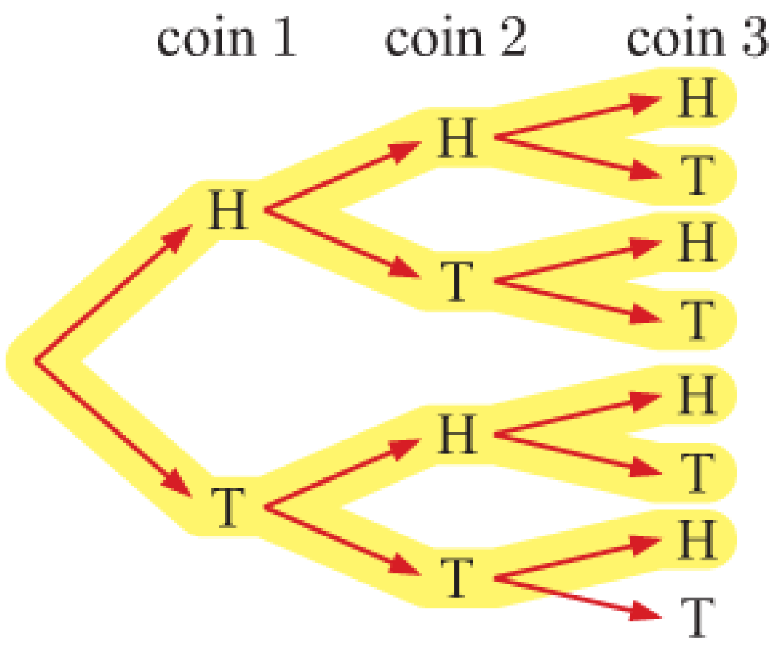


5



6



EXERCISE 11D

1 **a** $\frac{1}{5}$ **b** $\frac{1}{3}$ **c** $\frac{7}{15}$ **d** $\frac{4}{5}$ **e** $\frac{1}{5}$ **f** $\frac{8}{15}$

2 **a** $\frac{1}{2}$ **b** $\frac{2}{25}$ **c** $\frac{39}{200}$

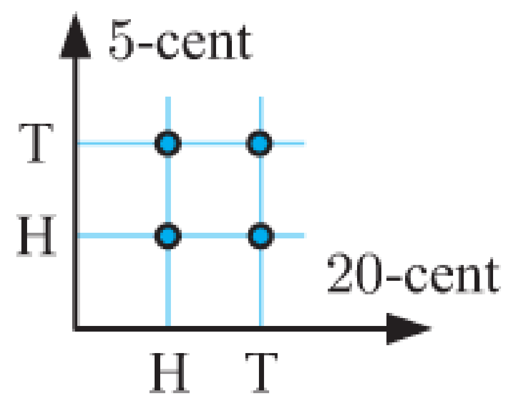
3 **a** $\frac{1}{4}$ **b** $\frac{1}{9}$ **c** $\frac{4}{9}$ **d** $\frac{1}{18}$ **e** $\frac{1}{6}$ **f** $\frac{13}{36}$
g $\frac{1}{12}$ **h** $\frac{1}{3}$

4 **a** $\frac{1}{7}$ **b** $\frac{2}{7}$ **c** $\frac{124}{1461}$ **d** $\frac{237}{1461}$
e $\frac{729}{1461}$ {remember leap years}

5 **a** {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}
b **i** $\frac{1}{8}$ **ii** $\frac{1}{8}$ **iii** $\frac{1}{8}$ **iv** $\frac{3}{8}$ **v** $\frac{1}{2}$ **vi** $\frac{7}{8}$

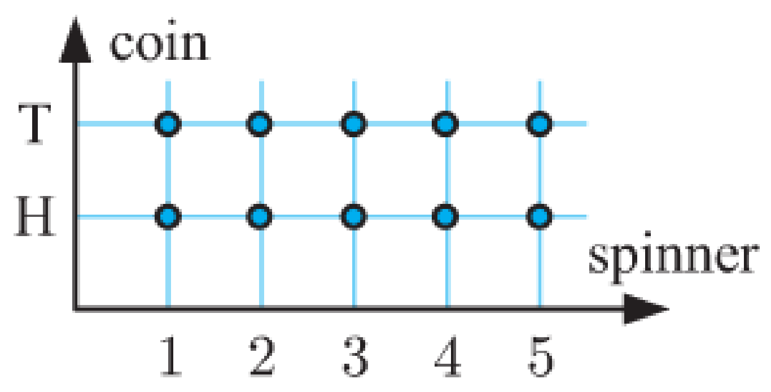
6 **a** {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}
b **i** $\frac{1}{2}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{2}$ **iv** $\frac{1}{2}$

7 **a**



b **i** $\frac{1}{4}$ **ii** $\frac{1}{4}$
iii $\frac{1}{2}$ **iv** $\frac{3}{4}$

8 **a**

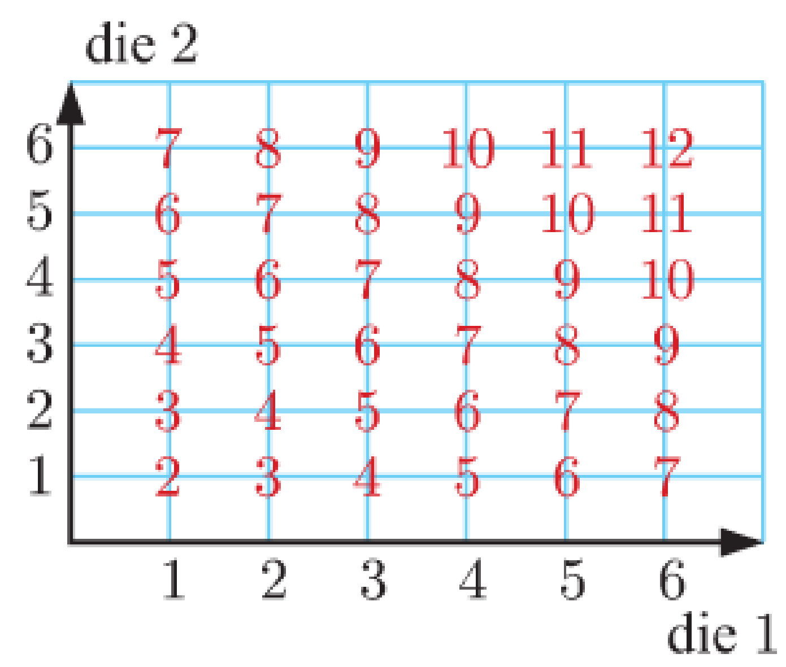


b **i** $\frac{1}{10}$ **ii** $\frac{3}{10}$
iii $\frac{2}{5}$ **iv** $\frac{3}{5}$

9 **a** $\frac{1}{36}$ **b** $\frac{1}{18}$ **c** $\frac{5}{9}$ **d** $\frac{11}{36}$ **e** $\frac{5}{18}$ **f** $\frac{25}{36}$

10 **a** Both grids show the sample space correctly, although **B** is more useful for calculating probabilities.
b $\frac{1}{6}$

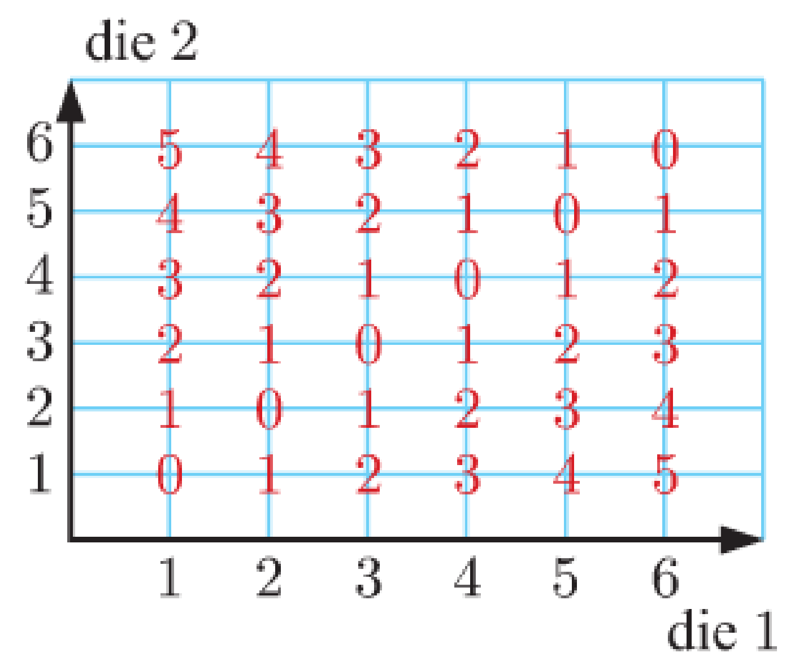
11 **a**



6	7	8	9	10	11	12
5	6	7	8	9	10	11
4	5	6	7	8	9	10
3	4	5	6	7	8	9
2	3	4	5	6	7	8
1	2	3	4	5	6	7
	1	2	3	4	5	6

b **i** $\frac{2}{36} = \frac{1}{18}$
ii $\frac{5}{36}$
iii $\frac{9}{36} = \frac{1}{4}$
iv $\frac{10}{36} = \frac{5}{18}$
v $\frac{10}{36} = \frac{5}{18}$
vi $\frac{26}{36} = \frac{13}{18}$

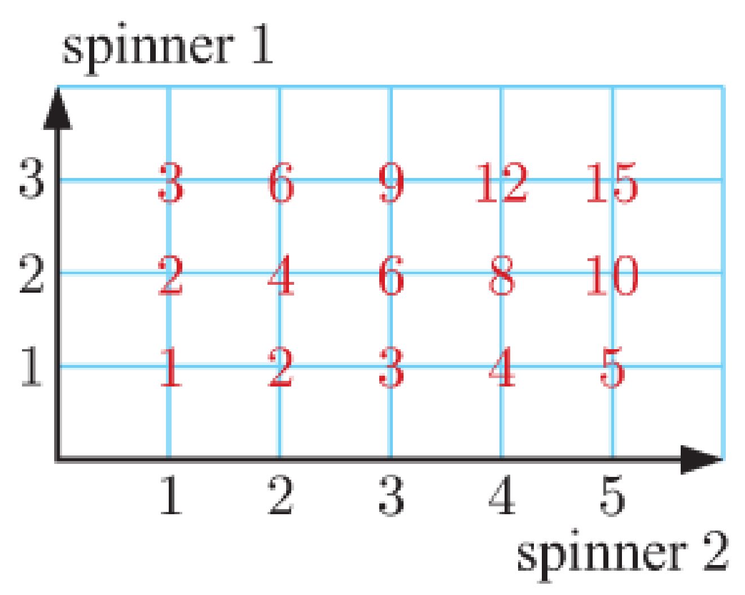
12 **a**



6	5	4	3	2	1	0
5	4	3	2	1	0	1
4	3	2	1	0	1	2
3	2	1	0	1	2	3
2	1	0	1	2	3	4
1	0	1	2	3	4	5
	1	2	3	4	5	6

b **i** $\frac{6}{36} = \frac{1}{6}$
ii $\frac{8}{36} = \frac{2}{9}$
iii $\frac{18}{36} = \frac{1}{2}$
iv $\frac{6}{36} = \frac{1}{6}$
v $\frac{24}{36} = \frac{2}{3}$

13 **a**



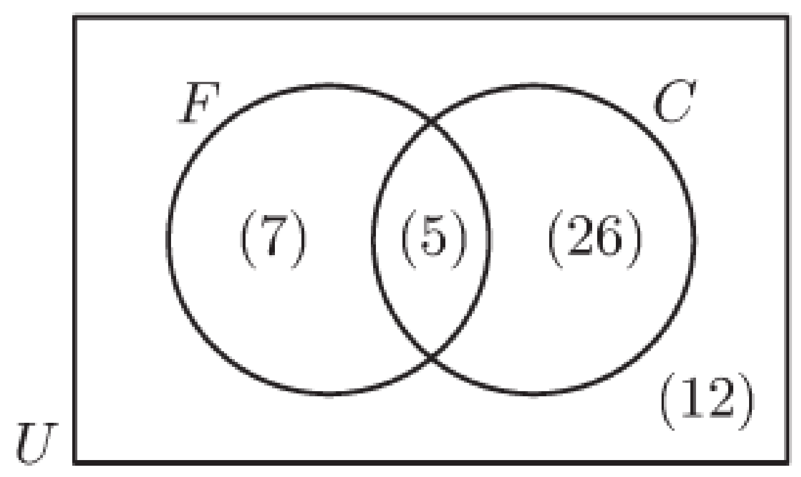
3	3	6	9	12	15
2	2	4	6	8	10
1	1	2	3	4	5
	1	2	3	4	5

b **i** $\frac{2}{15}$
ii $\frac{7}{15}$
iii $\frac{6}{15} = \frac{2}{5}$

14 **a** $\frac{3}{17}$ **b** $\frac{14}{17}$ **15** **a** $\frac{9}{65}$ **b** $\frac{4}{65}$ **c** $\frac{4}{5}$

16 **a** $\frac{17}{29}$ **b** $\frac{26}{29}$ **c** $\frac{5}{29}$ **17** **a** $\frac{37}{50}$ **b** $\frac{2}{5}$ **c** $\frac{17}{50}$

18 **a**



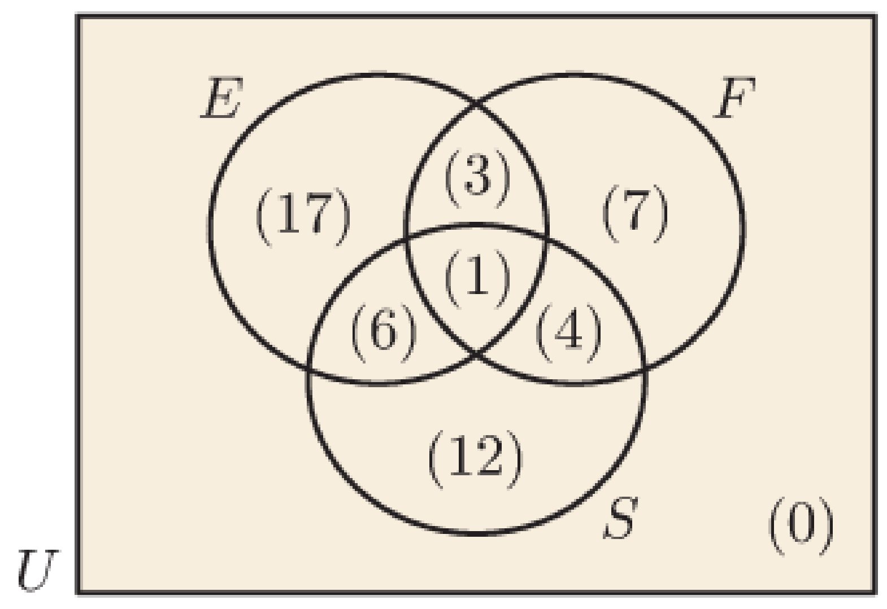
b **i** $\frac{19}{25}$
ii $\frac{13}{25}$
iii $\frac{6}{25}$

19 **a** $\frac{19}{40}$ **b** $\frac{1}{2}$ **c** $\frac{4}{5}$ **d** $\frac{5}{8}$ **e** $\frac{13}{40}$

20 **a** $\frac{7}{15}$ **b** $\frac{1}{15}$ **c** $\frac{2}{15}$

21 **a** $k = 5$
b **i** $\frac{7}{30}$ **ii** $\frac{11}{60}$ **iii** $\frac{7}{60}$ **iv** $\frac{53}{60}$ **v** $\frac{7}{60}$
vi $\frac{2}{15}$ **vii** $\frac{41}{60}$ **viii** $\frac{31}{60}$

22 **a**



b **i** $\frac{27}{50}$
ii $\frac{3}{10}$
iii $\frac{8}{25}$
iv $\frac{1}{5}$
v $\frac{2}{25}$

23 **a** $a = 3$, $b = 3$
b **i** $\frac{3}{10}$ **ii** $\frac{1}{10}$ **iii** $\frac{7}{40}$ **iv** $\frac{3}{8}$ **v** $\frac{5}{8}$

EXERCISE 11E

1 27 saves **2** $\approx$ 16 times **3** **a** $\frac{1}{4}$ **b** 50 occasions

4 15 days **5** 30 occasions

6 **a** **i** ≈ 0.55 **ii** ≈ 0.29 **iii** ≈ 0.16
b **i** $\approx$ 4125 people **ii** $\approx$ 2175 people
iii $\approx$ 1200 people

EXERCISE 11F

1 $\text{P}(A \cup B) = 0.55$ **2** $\text{P}(B) = 0.6$ **3** $\text{P}(X \cap Y) = 0.2$

4 **a** $\text{P}(A \cap B) = 0$ **b** A and B are mutually exclusive.

5 $\text{P}(A) = 0.35$

6 **a** yes
b **i** $\text{P}(A) = \frac{4}{15}$ **ii** $\text{P}(B) = \frac{7}{15}$ **iii** $\text{P}(A \cup B) = \frac{11}{15}$

7 **a** $\frac{11}{25}$ **b** $\frac{12}{25}$ **c** $\frac{8}{25}$ **d** $\frac{7}{25}$ **e** $\frac{4}{25}$ **f** $\frac{23}{25}$
g not possible **h** $\frac{11}{25}$ **i** not possible **j** $\frac{12}{25}$

8 $\text{P}(A \cup B) = 1$
Hint: Show $\text{P}(A' \cup B') = 2 - \text{P}(A \cup B)$

EXERCISE 11G

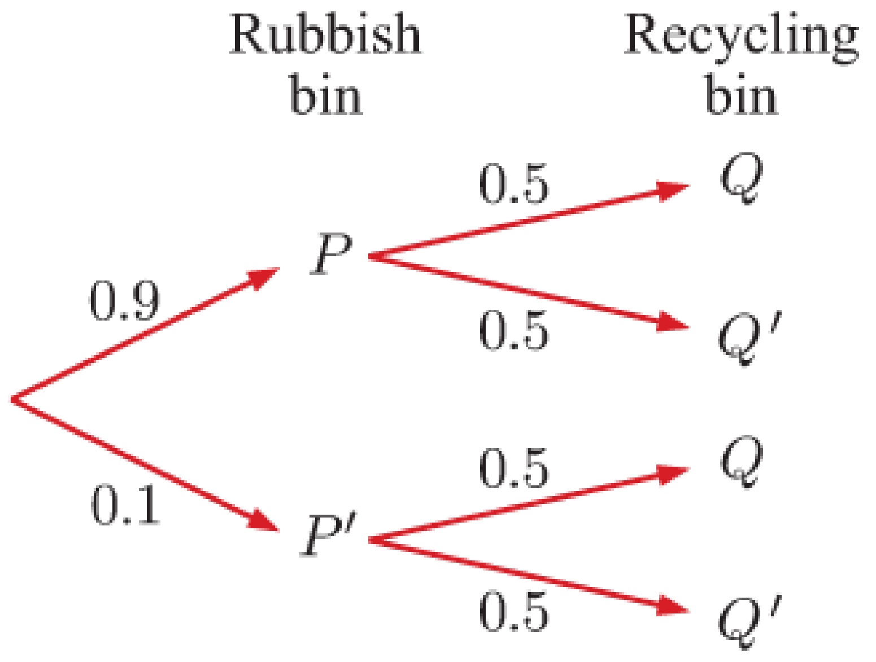
1 **a** $\frac{1}{24}$ **b** $\frac{1}{6}$ **2** **a** $\frac{1}{8}$ **b** $\frac{1}{8}$

3 **a** 0.0096 **b** 0.8096

4 **a** 0.56 **b** 0.06 **c** 0.14 **d** 0.24

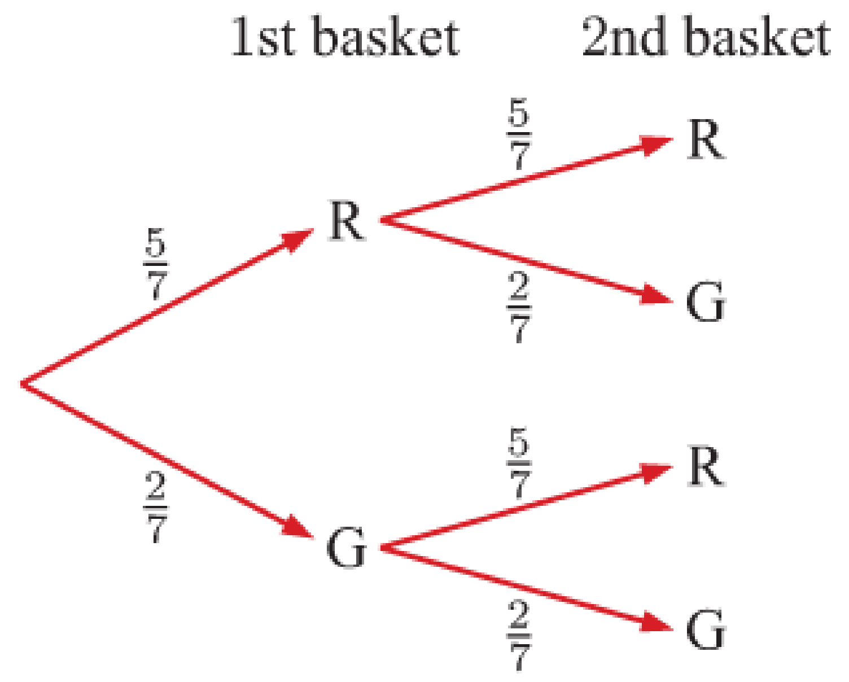
5 **a** $\frac{8}{125}$ **b** $\frac{12}{125}$ **c** $\frac{27}{125}$

6 **a**



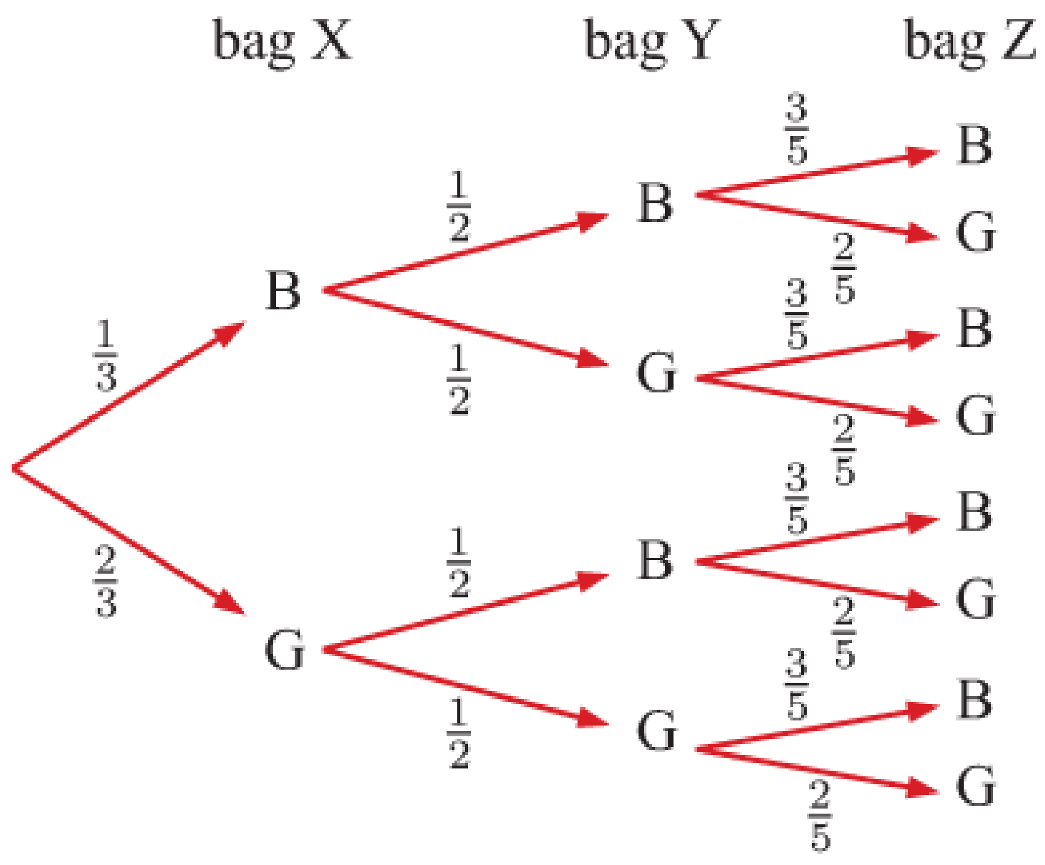
b **i** 0.45
ii 0.05

7 **a**



b **i** $\frac{25}{49}$ **ii** $\frac{20}{49}$

8 **a**



b **i** $\frac{1}{10}$ **ii** $\frac{1}{5}$ **iii** $\frac{13}{15}$

9 **a** $2p^2 - p^4$ **b** $p \approx 0.541$

10 Penny - Quentin - Penny
To win 2 matches in a row, Kane must win the middle match, so he should play against the weaker player in this match.

EXERCISE 11H

1 **a** $\frac{7}{15}$ **b** $\frac{7}{30}$ **2** **a** $\frac{2}{15}$ **b** $\frac{4}{15}$ **3** **a** $\frac{14}{55}$ **b** $\frac{1}{55}$

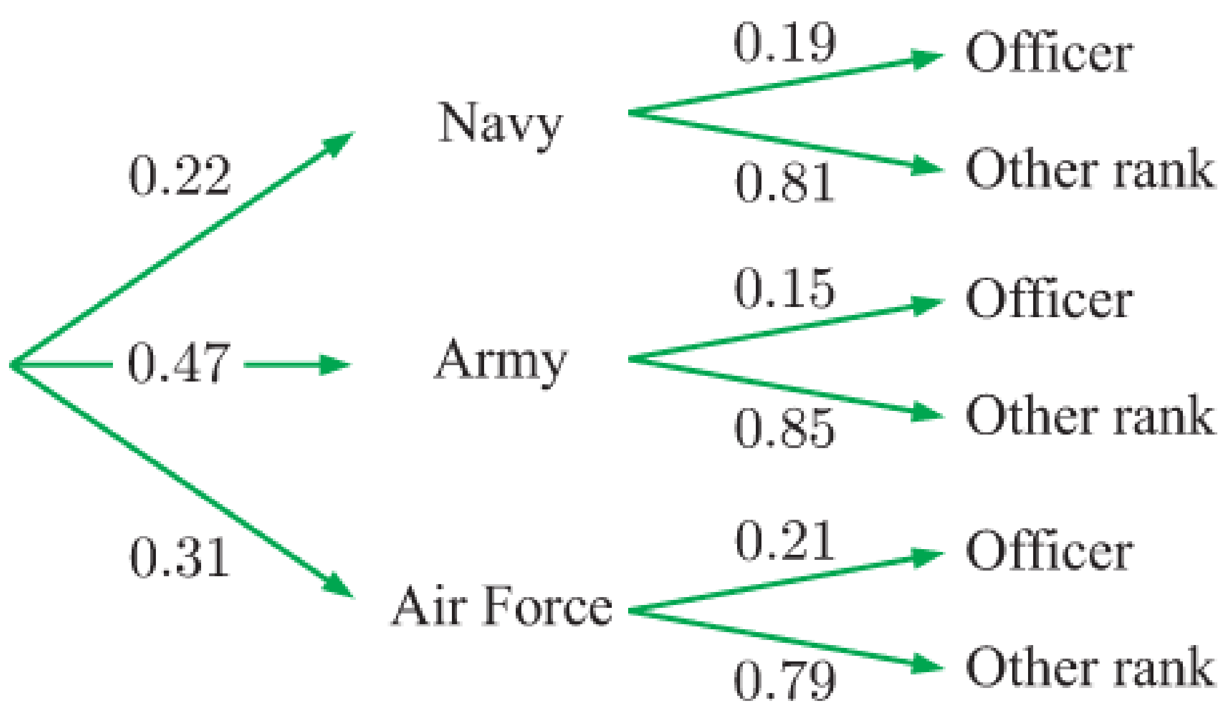
4 **a** $\frac{3}{100}$ **b** $\frac{3}{100} \times \frac{2}{99} \approx 0.000\,606$

c $\frac{3}{100} \times \frac{2}{99} \times \frac{1}{98} \approx 0.000\,006\,18$

d $\frac{97}{100} \times \frac{96}{99} \times \frac{95}{98} \approx 0.912$

5 **a** $\frac{4}{7}$ **b** $\frac{2}{7}$ **6** **a** $\frac{10}{21}$ **b** $\frac{1}{21}$

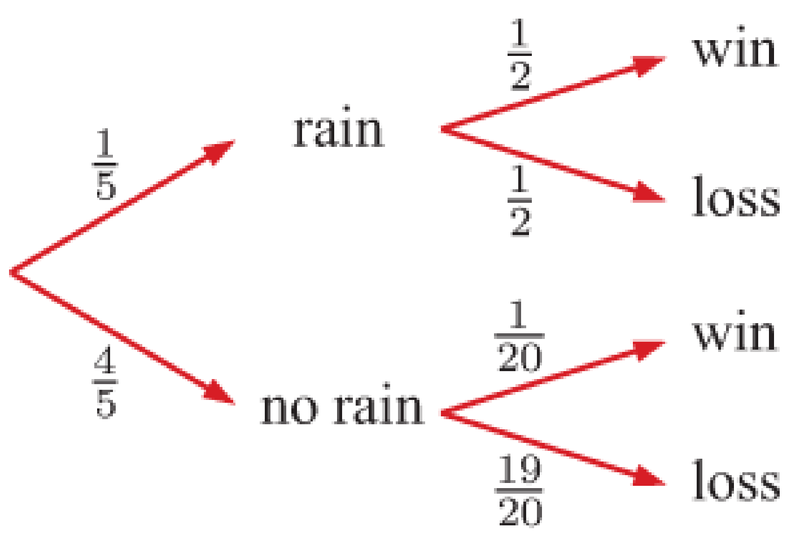
7 **a**



b **i** 0.1774 **ii** 0.9582 **iii** 0.8644

8 **a** 0.28 **b** 0.24

9 **a**



b $\frac{7}{50}$

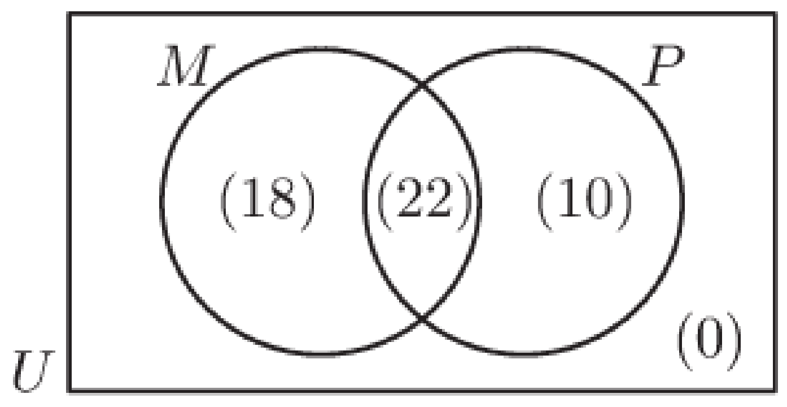
10 0.032 **11** $\frac{9}{38}$ **12** **a** $\frac{11}{30}$ **b** $\frac{19}{30}$

13 $\frac{187}{460} \approx 0.407$ **14** **a** $\frac{325}{833} \approx 0.390$ **b** $\frac{787}{833} \approx 0.945$

EXERCISE 11I

1 **a** $\frac{1}{4}$ **b** $\frac{1}{2}$ **c** 0 **2** $\frac{1}{2}$

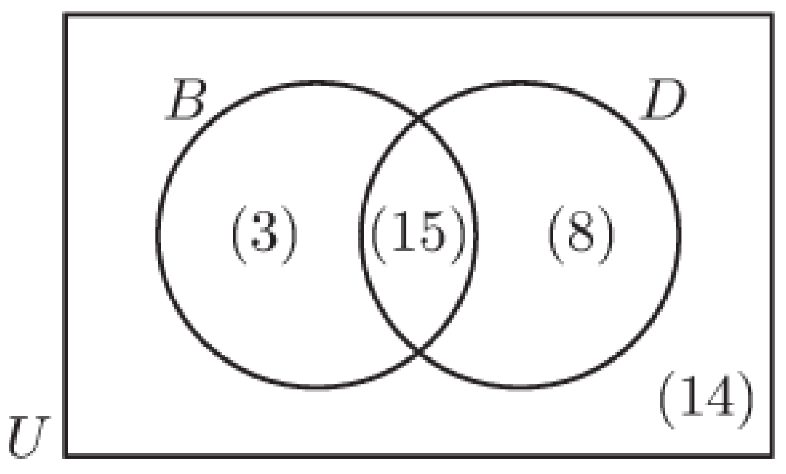
3 **a**



22 study both

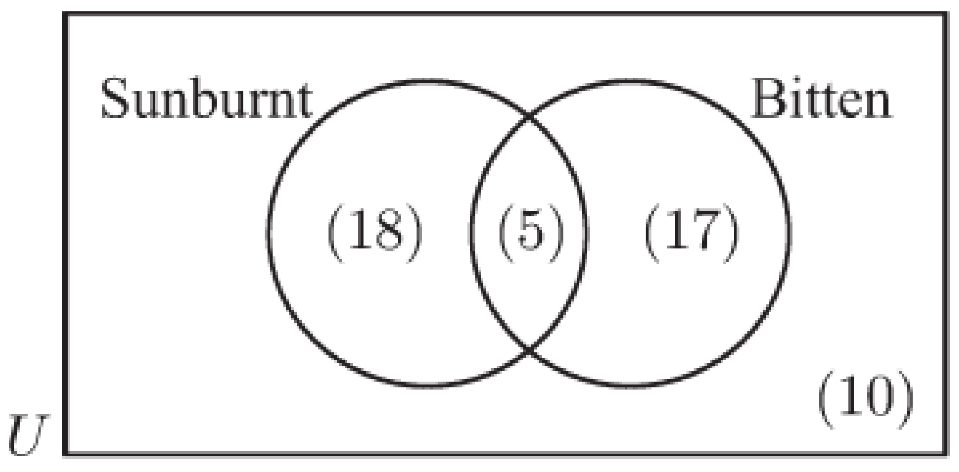
b **i** $\frac{9}{25}$ **ii** $\frac{11}{20}$

4 **a**



b **i** $\frac{3}{8}$ **ii** $\frac{15}{23}$

5 **a**



b **i** $\frac{14}{25}$ **ii** $\frac{4}{5}$ **iii** $\frac{5}{23}$ **iv** $\frac{9}{14}$

6 $\frac{7}{8}$

7 **a** $\frac{13}{20}$ **b** $\frac{7}{20}$ **c** $\frac{11}{50}$ **d** $\frac{7}{25}$ **e** $\frac{4}{7}$ **f** $\frac{1}{4}$

8 **a** $\frac{3}{5}$ **b** $\frac{2}{3}$ **9** **a** $\frac{23}{50}$ **b** $\frac{14}{23}$

10 **a** $\frac{10}{17}$ **b** $\frac{70}{163}$ **11** $\frac{7}{15}$

EXERCISE 11J

1 $\mathrm{P}(R \cap S) = 0.4 + 0.5 - 0.7 = 0.2$ and $\mathrm{P}(R) \times \mathrm{P}(S) = 0.2$
∴ R and S are independent events.

2 **a** **i** $\frac{7}{30}$ **ii** $\frac{7}{12}$ **iii** $\frac{7}{10}$

b No, as $\mathrm{P}(A \mid B) \neq \mathrm{P}(A)$.

3 **a** 0.35 **b** 0.85 **c** 0.15 **d** 0.15 **e** 0.5

4 **Hint:** Show $\mathrm{P}(A' \cap B') = \mathrm{P}(A')\,\mathrm{P}(B')$ using a Venn diagram and $\mathrm{P}(A \cap B)$.

5 $\mathrm{P}(B) = 0$ **6** 0.9

7 **a** $\mathrm{P}(D) = \frac{89}{400}$ **b** No, as $\mathrm{P}(D \mid C) \neq \mathrm{P}(D)$.

8 $\mathrm{P}(A \cup B) = 1$ or $\mathrm{P}(A \cap B) = 0$

EXERCISE 11K

1 **a** 0.0435 **b** ≈ 0.598 **2** **a** ≈ 0.773 **b** ≈ 0.556

3 $\frac{10}{13}$ **4** ≈ 0.424 **5** 0.0137 **6** $\frac{15}{83}$ **7** $\frac{99}{148}$

8 **a** $\frac{9}{19}$ **b** $\frac{10}{19}$ **10** **a** 0.95 **b** ≈ 0.306 **c** 0.6

11 **a** 0.104 **b** ≈ 0.267 **c** ≈ 0.0168

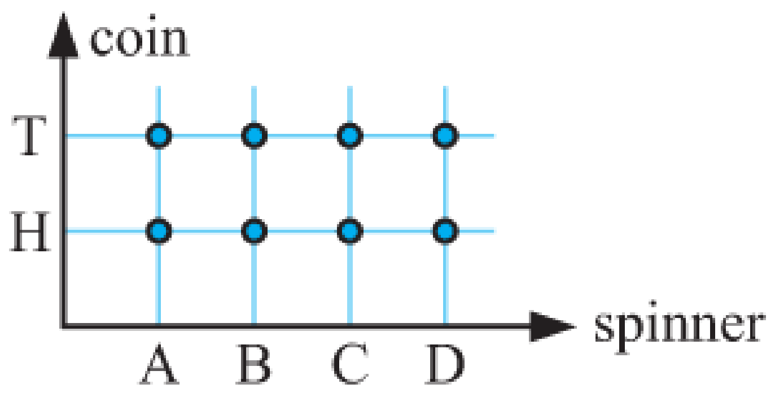
12 **a** $\mathrm{P}(L \mid T) = \frac{46}{205}$ **b** $\mathrm{P}(T \mid L) = \frac{46}{57}$

c Bayes' theorem tells us that $\mathrm{P}(L \mid T) = \mathrm{P}(T \mid L)\,\frac{\mathrm{P}(L)}{\mathrm{P}(T)}$.
Our answers to **a** and **b** differ since $\mathrm{P}(L) \neq \mathrm{P}(T)$.

REVIEW SET 11A

1 **a** ≈ 0.13 **b** ≈ 0.53

2 **a**



b **i** $\frac{3}{8}$ **ii** $\frac{1}{8}$ **iii** $\frac{5}{8}$

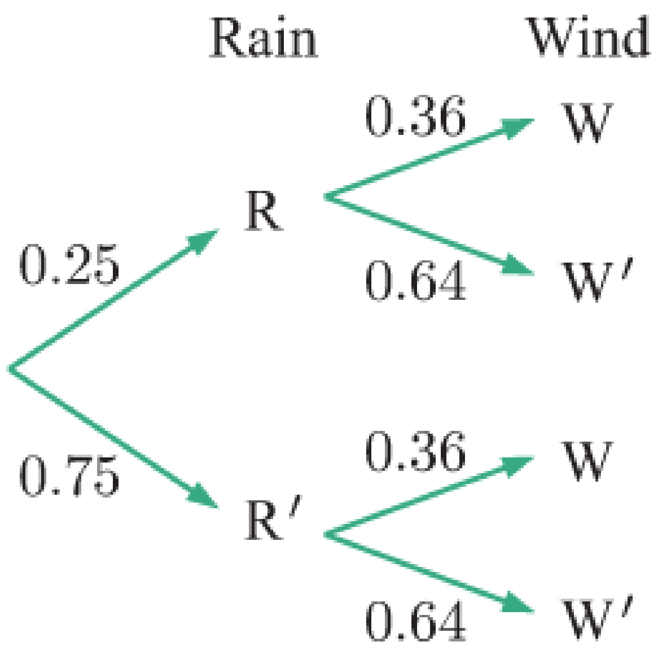
3 **a** Two events are independent if the occurrence of either event does not affect the probability that the other occurs. For A and B independent, $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$.

b Two events A and B are mutually exclusive if they have no common outcomes. $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$

4 0.496

5 **a** $\mathrm{P}(A \cup B) = x + 0.57$ **b** $x = 0.16$

6 **a**

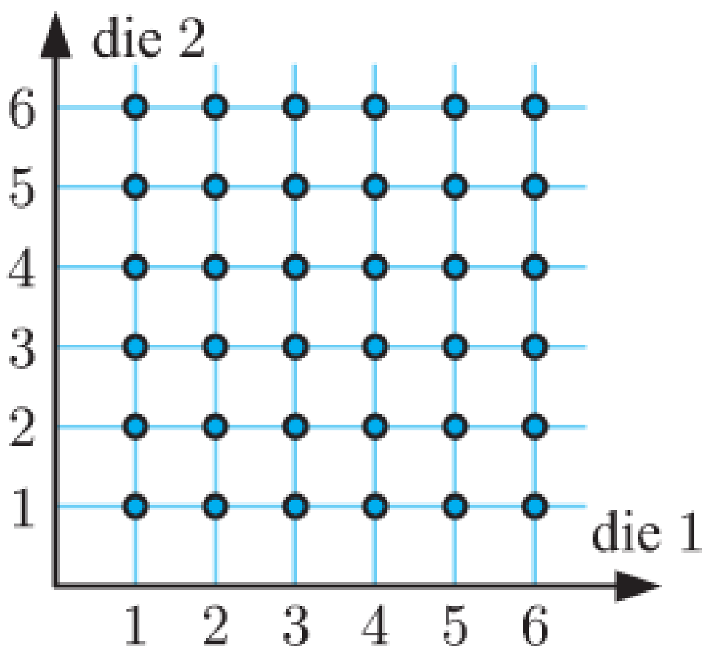


b **i** 0.09 **ii** 0.52

c It is assumed that the events are independent.

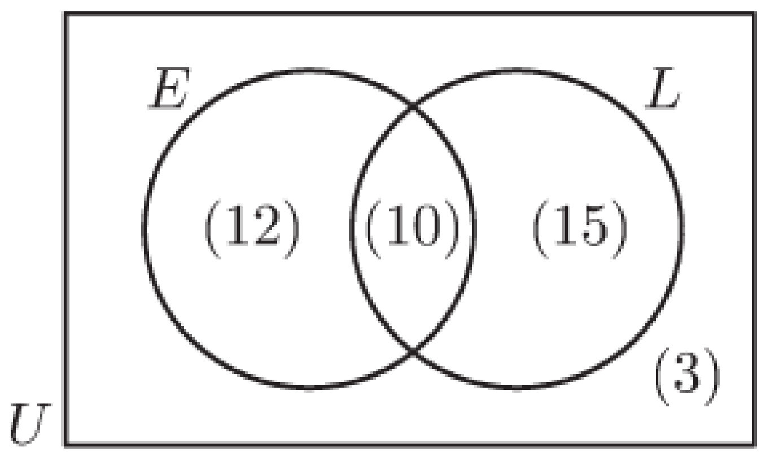
7 **a** 0 **b** 0.45 **c** 0.8

8 **a**



b **i** $\frac{2}{9}$ **ii** $\frac{5}{12}$

9 **a**



b **i** $\frac{1}{4}$ **ii** $\frac{37}{40}$ **iii** $\frac{2}{5}$

10 4350 seeds **11** **a** $\frac{25}{144}$ **b** $\frac{25}{72}$ **c** $\frac{7}{16}$ **d** $\frac{4}{9}$

12 ≈ 0.127

13 **a**

	Female	*Male*	*Total*
Smoker	20	40	60
Non-smoker	70	70	140
Total	90	110	200

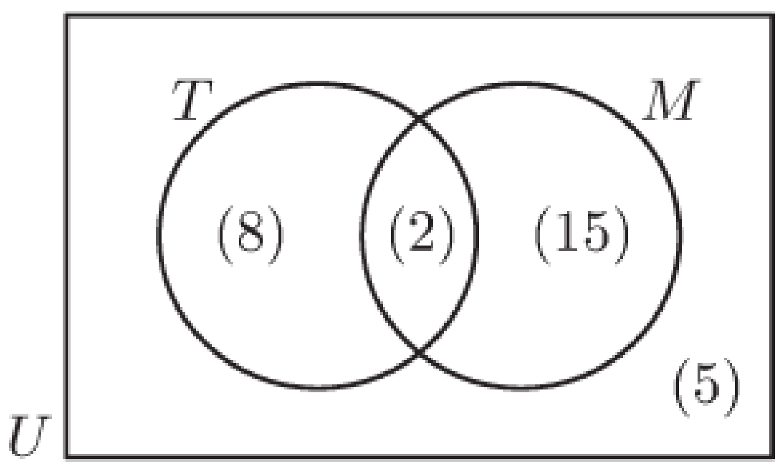
b **i** $\frac{7}{20}$ **ii** $\frac{1}{2}$

c **i** ≈ 0.121 **ii** ≈ 0.422

14 $\frac{69}{95}$ **15** **a** $\frac{1}{5}$ **b** $P(B \mid A) \neq P(B)$ **c** $\frac{2}{3}$ **16** $\frac{5}{324}$

REVIEW SET 11B

1 **a**



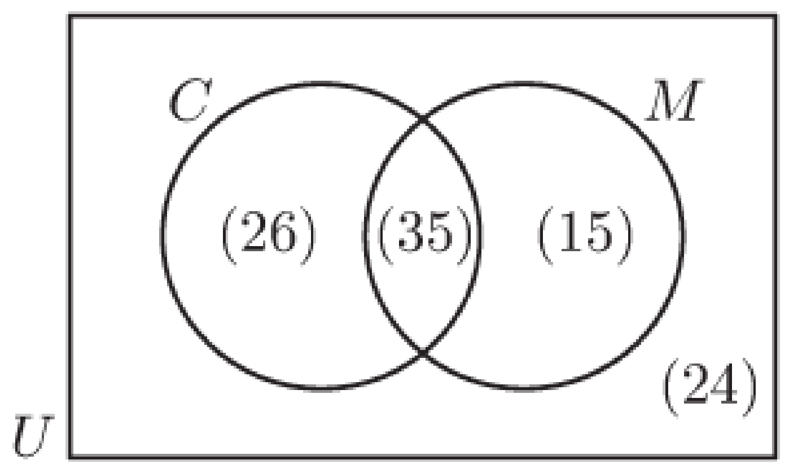
b **i** $\frac{1}{15}$ **ii** $\frac{2}{17}$

2 0.9975

3 **a** $P(A \cap B) = 0.28$ which is not equal to 0.
$\therefore$ A and B are not mutually exclusive.

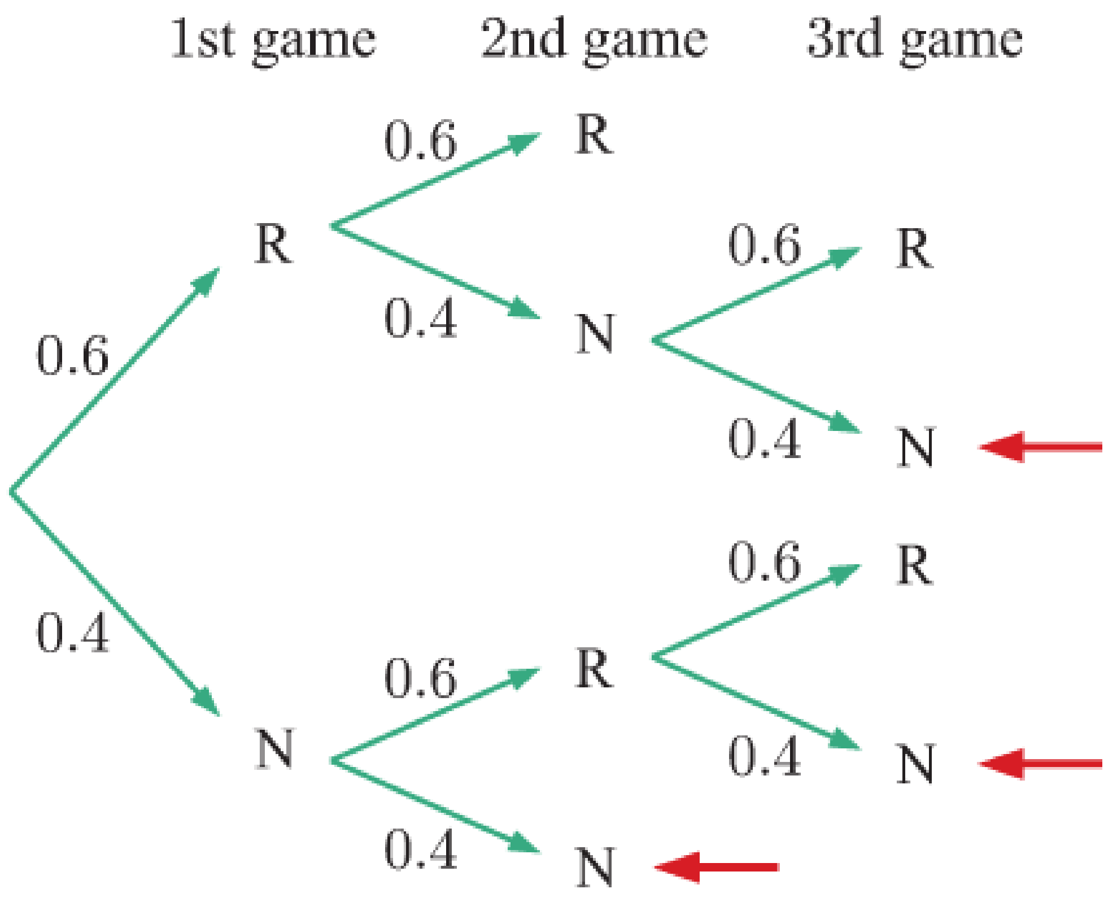
b $P(A \cup B) = 0.82$

4 $\frac{5}{9}$ **5** **a**



b $\frac{35}{61}$

6 **a**



b P(N wins) $= 0.352$

7 **a** 0.93 **b** 0.8 **c** 0.2 **d** 0.65

8 **a** $\frac{4}{500} \times \frac{3}{499} \times \frac{2}{498} \approx 0.000\,000\,193$

b $1 - \frac{496}{500} \times \frac{495}{499} \times \frac{494}{498} \approx 0.0239$

9 **a** 0.2588 **b** ≈ 0.703

10 **a** $P(B) = \frac{1}{3}$ **b** **i** $\frac{16}{21}$ **ii** $\frac{13}{21}$

11 $\frac{4}{9}$ **12** **a** ≈ 0.660 **b** $\frac{100}{7} \approx 14.3$ or 14 pieces

13 **a** $\frac{31}{70}$ **b** $\frac{21}{31}$ **14** $\frac{1}{2}$

15 **a** **i** 100 balloons **ii** 33 balloons

b **i** $\frac{19}{25}$ **ii** $\frac{37}{50}$

c **i** $\frac{17}{66}$ **ii** $\frac{2701}{4950}$ **iii** $\frac{25}{66}$ **iv** $\frac{29}{66}$

d **i** $\frac{1}{980}$ **ii** $\frac{17}{308}$

16 **b** ≈ 0.988 **c** **i** ≈ 0.547 **ii** ≈ 0.266 **d** females

e A 20 year old is expected to live much longer than 30 more years, so it is unlikely the insurance company will have to pay out the policy. A 50 year old however is expected to live for only another 26.45 years (males) or 31.59 years (females), so the insurance company may have to pay out the policy.

g For "third world" countries with poverty, lack of sanitation, and so on, the tables would show a significantly lower life expectancy.

EXERCISE 12A

1 This sample is too small to draw reliable conclusions from.

2
- The sample size is very small and may not be representative of the whole population.
- The sample was taken in a Toronto shopping mall. People living outside of the city are probably not represented.

3 **a** The sample is likely to under-represent full-time weekday working voters.

b The members of the golf club may not be representative of the whole electorate.

c Only people who catch the train in the morning such as full-time workers or students will be sampled.

d The voters in the street may not be representative of those in the whole electorate.

4 **a** The sample size is too small.

b With only 10 sheep being weighed, any errors in the measuring of weights will have more impact on the results.

5 **a** The whole population is being considered, not just a sample. There will be no sampling error as this is a census.

b measurement error

6 **a** Many of the workers may not return or even complete the survey.

b There may be more responses to the survey as many workers would feel that it is easier to complete a survey online rather than on paper and mailing it back. Responses would also be received more quickly however some workers may not have internet access and will therefore be unable to complete the survey.

7 **a** Yes; members with strong negative opinions regarding the management structure of the organisation are more likely to respond.

b No; the feedback from the survey is still valid. Although it might be biased, the feedback might bring certain issues to attention.

EXERCISE 12B

1 **Note:** Sample answers only - many answers are possible.

a 12, 6, 23, 10, 21, 25