Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2015

Mathematics

Higher level

Paper 1

22 pages

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

Section A

1. (a) **METHOD 1**

area $= \pi 2^2 - \frac{1}{2}2^2\theta \left(= 3\pi\right)$ ***M1A1***

Note: Award ***M1*** for using area formula.

$\Rightarrow 2\theta = \pi \Rightarrow \theta = \frac{\pi}{2}$ ***A1***

Note: Degrees loses final A1

METHOD 2

let $x = 2\pi - \theta$

area $= \frac{1}{2}2^2 x\,(= 3\pi)$ ***M1***

$\Rightarrow x = \frac{3}{2}\pi$ ***A1***

$\Rightarrow \theta = \frac{\pi}{2}$ ***A1***

METHOD 3

Area of circle is 4π ***A1***

Shaded area is $\frac{3}{4}$ of the circle ***(R1)***

$\Rightarrow \theta = \frac{\pi}{2}$ ***A1***

[3 marks]

(b) arc length $= 2\frac{3\pi}{2}$ ***A1***

perimeter $= 2\frac{3\pi}{2} + 2\times 2$

$= 3\pi + 4$ ***A1***

[2 marks]

Total [5 marks]

2. (a) $\overline{x} = \dfrac{1\times 0 + 19\times 10}{20} = 9.5$ ***(M1)A1***

[2 marks]

(b) median is 10 ***A1***

[1 mark]

(c) (i) 19 ***A1***

(ii) 1 ***A1***

[2 marks]

Total [5 marks]

3. (a) $\int\left(1+\tan^2 x\right)\mathrm{d}x = \int \sec^2 x\,\mathrm{d}x = \tan x\,(+c)$ ***M1A1***

[2 marks]

(b) $\int \sin^2 x\,\mathrm{d}x = \int \dfrac{1-\cos 2x}{2}\mathrm{d}x$ ***M1A1***

$= \dfrac{x}{2} - \dfrac{\sin 2x}{4}(+c)$ ***A1***

Note:	Allow integration by parts followed by trig identity. Award ***M1*** for parts, ***A1*** for trig identity, ***A1*** final answer.

[3 marks]

Total [5 marks]

4. (a) $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$ ***(M1)A1***

[2 marks]

(b) $f'(x) = \lim\limits_{h\to 0}\dfrac{(x+h)^3 - x^3}{h}$ ***(M1)***

$= \lim\limits_{h\to 0}\dfrac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$

$= \lim\limits_{h\to 0}\left(3x^2 + 3xh + h^2\right)$ ***A1***

$= 3x^2$ ***A1***

Note: Do not award final A1 on FT if $= 3x^2$ is not obtained

Note: Final A1 can only be obtained if previous A1 is given

[3 marks]

Total [5 marks]

5. (a) **EITHER**

$f(-x) = f(x)$ ***M1***

$\Rightarrow ax^2 - bx + c = ax^2 + bx + c \Rightarrow 2bx = 0$, $(\forall x \in \mathbb{R})$ ***A1***

OR

y-axis is eqn of symmetry ***M1***

so $\frac{-b}{2a} = 0$ ***A1***

THEN

$\Rightarrow b = 0$ ***AG***

[2 marks]

(b) $g(-x) = -g(x) \Rightarrow p\sin(-x) - qx + r = -p\sin x - qx - r$

$\Rightarrow -p\sin x - qx + r = -p\sin x - qx - r$ ***M1***

Note: ***M1*** is for knowing properties of sin.

$\Rightarrow 2r = 0 \Rightarrow r = 0$ ***A1***

Note: In (a) and (b) allow substitution of a particular value of x

[2 marks]

(c) $h(-x) = h(x) = -h(x) \Rightarrow 2h(x) = 0 \Rightarrow h(x) = 0$, $(\forall x)$ ***M1A1***

Note: Accept geometrical explanations.

[2 marks]

Total [6 marks]

6. (a) $f : x \to y = \dfrac{3x-2}{2x-1}$ $f^{-1} : y \to x$

$y = \dfrac{3x-2}{2x-1} \Rightarrow 3x-2 = 2xy - y$ ***M1***

$\Rightarrow 3x - 2xy = -y + 2$ ***M1***

$x(3-2y) = 2 - y$

$x = \dfrac{2-y}{3-2y}$ ***A1***

$\left(f^{-1}(y) = \dfrac{2-y}{3-2y} \right)$

$f^{-1}(x) = \dfrac{2-x}{3-2x}$ $\left(x \neq \dfrac{3}{2} \right)$ ***A1***

Note: x and y might be interchanged earlier.

Note: First ***M1*** is for interchange of variables second ***M1*** for manipulation

Note: Final answer must be a function of x

[4 marks]

(b) $\dfrac{3x-2}{2x-1} = A + \dfrac{B}{2x-1} \Rightarrow 3x-2 = A(2x-1) + B$

equating coefficients $3 = 2A$ and $-2 = -A + B$ ***(M1)***

$A = \dfrac{3}{2}$ and $B = -\dfrac{1}{2}$ ***A1***

Note: Could also be done by division or substitution of values.

[2 marks]

(c) $\int f(x)\,\mathrm{d}x = \dfrac{3}{2}x - \dfrac{1}{4}\ln|2x-1| + c$ ***A1***

Note: accept equivalent e.g. ln $|4x-2|$

[1 mark]

Total [7 marks]

7. (a) (i) $\left(-\frac{a_{n-1}}{a_n}=\right)-\frac{1}{2}$ ***A1***

(ii) $\left((-1)^n\frac{a_0}{a_n}=\right)-\frac{36}{2}=(-18)$ ***A1A1***

Note: First ***A1*** is for the negative sign.

[3 marks]

(b) **METHOD 1**

if λ satisfies $p(\lambda)=0$ then $q(\lambda-4)=0$

so the roots of $q(x)$ are each 4 less than the roots of $p(x)$ ***(R1)***

so sum of roots is $-\frac{1}{2}-4\times 5=-20.5$ ***A1***

METHOD 2

$p(x+4)=2x^5+2\times 5\times 4x^4\ldots+x^4\ldots=2x^5+41x^4\ldots$ ***(M1)***

so sum of roots is $-\frac{41}{2}=-20.5$ ***A1***

[2 marks]

Total [5 marks]

8. $\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x$ ***(A1)***

EITHER

integral is $\int \frac{\mathrm{e}^x}{\left(\mathrm{e}^x + 3\right)^2 + 2^2} \mathrm{d}x$ ***M1A1***

$= \int \frac{1}{u^2 + 2^2} \mathrm{d}u$ ***M1A1***

Note: Award ***M1*** only if the integral has completely changed to one in u.

Note: $\mathrm{d}u$ needed for final ***A1***

OR

$\mathrm{e}^x = u - 3$

integral is $\int \frac{1}{(u-3)^2 + 6(u-3) + 13} \mathrm{d}u$ ***M1A1***

Note: Award ***M1*** only if the integral has completely changed to one in u.

$= \int \frac{1}{u^2 + 2^2} \mathrm{d}u$ ***M1A1***

Note: In both solutions the two method marks are independent.

THEN

$= \frac{1}{2} \arctan\left(\frac{u}{2}\right) (+c)$ ***(A1)***

$= \frac{1}{2} \arctan\left(\frac{\mathrm{e}^x + 3}{2}\right) (+c)$ ***A1***

Total [7 marks]

9. (a) $g \circ f(x) = g(f(x))$ ***M1***

$= g\left(2x + \frac{\pi}{5}\right)$

$= 3\sin\left(2x + \frac{\pi}{5}\right) + 4$ ***AG***

[1 mark]

(b) since $-1 \le \sin\theta \le +1$, range is $[1, 7]$ ***(R1)A1***

[2 marks]

(c) $3\sin\left(2x + \frac{\pi}{5}\right) + 4 = 7 \Rightarrow 2x + \frac{\pi}{5} = \frac{\pi}{2} + 2n\pi \Rightarrow x = \frac{3\pi}{20} + n\pi$ ***(M1)***

so next biggest value is $\frac{23\pi}{20}$ ***A1***

Note: Allow use of period.

[2 marks]

(d)

Note: Transformations can be in any order but see notes below.

stretch scale factor 3 parallel to y axis (vertically) ***A1***
vertical translation of 4 up ***A1***

Note: Vertical translation is $\frac{4}{3}$ up if it occurs before stretch parallel to y axis.

stretch scale factor $\frac{1}{2}$ parallel to x axis (horizontally) ***A1***

horizontal translation of $\frac{\pi}{10}$ to the left ***A1***

Note: Horizontal translation is $\frac{\pi}{5}$ to the left if it occurs before stretch parallel to x axis.

Note: Award ***A1*** for magnitude and direction in each case. Accept any correct terminology provided that the meaning is clear *eg* shift for translation.

[4 marks]

Total [9 marks]

10. **METHOD 1**

to have 3 consecutive losses there must be exactly 5, 4 or 3 losses

the probability of exactly 5 losses (must be 3 consecutive) is $\left(\frac{1}{3}\right)^5$ ***A1***

the probability of exactly 4 losses (with 3 consecutive) is $4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)$ ***A1A1***

Note: First ***A1*** is for the factor 4 and second ***A1*** for the other 2 factors.

the probability of exactly 3 losses (with 3 consecutive) is $3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2$ ***A1A1***

Note: First ***A1*** is for the factor 3 and second ***A1*** for the other 2 factors.

(Since the events are mutually exclusive)

the total probability is $\frac{1+8+12}{3^5}=\frac{21}{243}\left(=\frac{7}{81}\right)$ ***A1***

[6 marks]

METHOD 2

Roy loses his job if

A – first 3 games are all lost (so the last 2 games can be any result)

B – first 3 games are not all lost, but middle 3 games are all lost (so the first game is not a loss and the last game can be any result)

or C – first 3 games are not all lost, middle 3 games are not all lost but last 3 games are all lost, (so the first game can be any result but the second game is not a loss)

for A 4th & 5th games can be anything

$\mathrm{P}(A)=\left(\frac{1}{3}\right)^3=\frac{1}{27}$ ***A1***

for B 1st game not a loss & 5th game can be anything ***(R1)***

$\mathrm{P}(B)=\frac{2}{3}\times\left(\frac{1}{3}\right)^3=\frac{2}{81}$ ***A1***

for C 1st game anything, 2nd game not a loss ***(R1)***

$\mathrm{P}(C)=1\times\frac{2}{3}\times\left(\frac{1}{3}\right)^3=\frac{2}{81}$ ***A1***

(Since the events are mutually exclusive)

total probability is $\frac{1}{27}+\frac{2}{81}+\frac{2}{81}=\frac{7}{81}$ ***A1***

continued…

Question 10 continued.

Note:	In both methods all the ***A*** marks are independent.

Note:	If the candidate misunderstands the question and thinks that it is asking for exactly 3 losses award ***A1 A1*** and ***A1*** for an answer of $\frac{12}{243}$ as in the last lines of Method 1.

[6 marks]

Total [6 marks]

Section B

11. (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \times \mathrm{e}^{3x} + x \times 3\mathrm{e}^{3x} = \left(\mathrm{e}^{3x} + 3x\mathrm{e}^{3x}\right)$ ***M1A1***

[2 marks]

(b) let $P(n)$ be the statement $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = n3^{n-1}\mathrm{e}^{3x} + x3^n\mathrm{e}^{3x}$

prove for $n = 1$ ***M1***

LHS of $P(1)$ is $\frac{\mathrm{d}y}{\mathrm{d}x}$ which is $1 \times \mathrm{e}^{3x} + x \times 3\mathrm{e}^{3x}$ and RHS is $3^0\mathrm{e}^{3x} + x3^1\mathrm{e}^{3x}$ ***R1***

as LHS = RHS, $P(1)$ is true

assume $P(k)$ is true and attempt to prove $P(k+1)$ is true ***M1***

assuming $\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = k3^{k-1}\mathrm{e}^{3x} + x3^k\mathrm{e}^{3x}$

$\frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^k y}{\mathrm{d}x^k}\right)$ (***M1***)

$= k3^{k-1} \times 3\mathrm{e}^{3x} + 1 \times 3^k\mathrm{e}^{3x} + x3^k \times 3\mathrm{e}^{3x}$ ***A1***

$= (k+1)3^k\mathrm{e}^{3x} + x3^{k+1}\mathrm{e}^{3x}$ (as required) ***A1***

Note: Can award the ***A*** marks independent of the ***M*** marks

since $P(1)$ is true and $P(k)$ is true $\Rightarrow P(k+1)$ is true

then (by PMI), $P(n)$ is true ($\forall n \in \mathbb{Z}^+$) ***R1***

Note: To gain last ***R1*** at least four of the above marks must have been gained.

[7 marks]

continued…

Question 11 continued

(c) $e^{3x} + x \times 3e^{3x} = 0 \Rightarrow 1 + 3x = 0 \Rightarrow x = -\frac{1}{3}$ ***M1A1***

point is $\left(-\frac{1}{3}, -\frac{1}{3e}\right)$ ***A1***

EITHER

$\frac{d^2y}{dx^2} = 2 \times 3e^{3x} + x \times 3^2 e^{3x}$

when $x = -\frac{1}{3}$, $\frac{d^2y}{dx^2} > 0$ therefore the point is a minimum ***M1A1***

OR

x	$-\frac{1}{3}$
$\frac{dy}{dx}$	$-ve$ 0 $+ve$

nature table shows point is a minimum ***M1A1***

[5 marks]

(d) $\frac{d^2y}{dx^2} = 2 \times 3e^{3x} + x \times 3^2 e^{3x}$ ***A1***

$2 \times 3e^{3x} + x \times 3^2 e^{3x} = 0 \Rightarrow 2 + 3x = 0 \Rightarrow x = -\frac{2}{3}$ ***M1A1***

point is $\left(-\frac{2}{3}, -\frac{2}{3e^2}\right)$ ***A1***

x	$-\frac{2}{3}$
$\frac{d^2y}{dx^2}$	$-ve$ 0 $+ve$

since the curvature does change (concave down to concave up) it is a point of inflection ***R1***

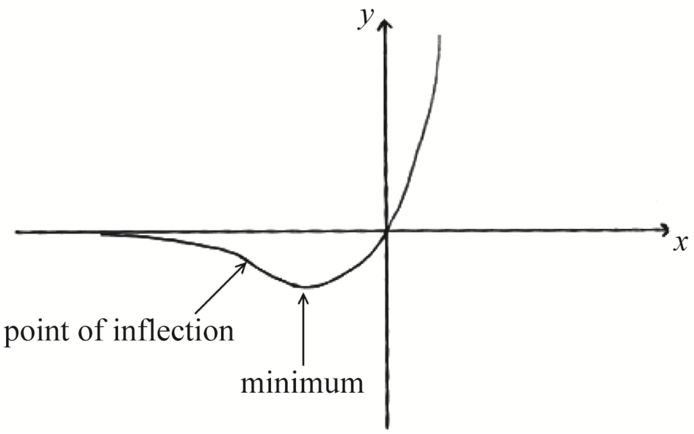
Note: Allow 3rd derivative is not zero at $-\frac{2}{3}$

[5 marks]

continued…

Question 11 continued

(e)



(general shape including asymptote and through origin) ***A1***
showing minimum and point of inflection ***A1***

Note:	Only indication of position of answers to (c) and (d) required, not coordinates.

[2 marks]

Total [21 marks]

12. (a) (i) **METHOD 1**

$\frac{v_{n+1}}{v_n} = \frac{2^{u_{n+1}}}{2^{u_n}}$ ***M1***

$= 2^{u_{n+1}-u_n} = 2^d$ ***A1***

METHOD 2

$\frac{v_{n+1}}{v_n} = \frac{2^{a+nd}}{2^{a+(n-1)d}}$ ***M1***

$= 2^d$ ***A1***

(ii) 2^a ***A1***

Note: Accept 2^{u_1}.

(iii) **EITHER**

v_n is a GP with first term 2^a and common ratio 2^d

$v_n = 2^a(2^d)^{(n-1)}$

OR

$u_n = a+(n-1)d$ as it is an AP

THEN

$v_n = 2^{a+(n-1)d}$ ***A1***

[4 marks]

(b) (i) $S_n = \frac{2^a\left((2^d)^n - 1\right)}{2^d - 1} = \frac{2^a\left(2^{dn} - 1\right)}{2^d - 1}$ ***M1A1***

Note: Accept either expression.

(ii) for sum to infinity to exist need $-1 < 2^d < 1$ ***R1***

$\Rightarrow \log 2^d < 0 \Rightarrow d\log 2 < 0 \Rightarrow d < 0$ ***(M1)A1***

Note: Also allow graph of 2^d.

(iii) $S_\infty = \frac{2^a}{1-2^d}$ ***A1***

continued…

Question 12 continued

(iv) $\dfrac{2^a}{1-2^d} = 2^{a+1} \Rightarrow \dfrac{1}{1-2^d} = 2$ ***M1***

$\Rightarrow 1 = 2 - 2^{d+1} \Rightarrow 2^{d+1} = 1$

$\Rightarrow d = -1$ ***A1***

[8 marks]

(c) **METHOD 1**

$w_n = pq^{n-1}$, $z_n = \ln pq^{n-1}$ ***(A1)***

$z_n = \ln p + (n-1)\ln q$ ***M1A1***

$z_{n+1} - z_n = (\ln p + n\ln q) - (\ln p + (n-1)\ln q) = \ln q$

which is a constant so this is an AP

(with first term $\ln p$ and common difference $\ln q$)

$\sum_{i=1}^{n} z_i = \dfrac{n}{2}\left(2\ln p + (n-1)\ln q\right)$ ***M1***

$= n\left(\ln p + \ln q^{\left(\frac{n-1}{2}\right)}\right) = n\ln\left(pq^{\left(\frac{n-1}{2}\right)}\right)$ ***(M1)***

$= \ln\left(p^n q^{\frac{n(n-1)}{2}}\right)$ ***A1***

METHOD 2

$\sum_{i=1}^{n} z_i = \ln p + \ln pq + \ln pq^2 + \ldots + \ln pq^{n-1}$ ***(M1)A1***

$= \ln\left(p^n q^{(1+2+3+\ldots+(n-1))}\right)$ ***(M1)A1***

$= \ln\left(p^n q^{\frac{n(n-1)}{2}}\right)$ ***(M1)A1***

[6 marks]

Total [18 marks]

13. (a) $\overrightarrow{OP} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k} + \lambda(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

$\overrightarrow{OQ} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + \mu(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$ ***(M1)***

$\overrightarrow{PQ} = \boldsymbol{i} - \boldsymbol{j} - 4\boldsymbol{k} - \lambda(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) + \mu(\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

$= (1 - \lambda + \mu)\boldsymbol{i} + (-1 - \lambda - \mu)\boldsymbol{j} + (-4 - \lambda + 2\mu)\boldsymbol{k}$ ***A1***

[2 marks]

(b) **METHOD 1**

use of scalar product ***M1***

perpendicular to $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ gives

$(1 - \lambda + \mu) + (-1 - \lambda - \mu) + (-4 - \lambda + 2\mu) = 0$

$\Rightarrow -3\lambda + 2\mu = 4$ ***A1***

perpendicular to $\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ gives

$(1 - \lambda + \mu) - (-1 - \lambda - \mu) + 2(-4 - \lambda + 2\mu) = 0$

$\Rightarrow -2\lambda + 6\mu = 6$ ***A1***

solving simultaneous equations gives $\lambda = -\frac{6}{7},\ \mu = \frac{5}{7}$ ***A1A1***

METHOD 2

$\boldsymbol{v} \times \boldsymbol{w} = 3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ ***M1A1***

$\overrightarrow{PQ} = a(3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$

$1 - \lambda + \mu = 3a$

$-1 - \lambda - \mu = -a$ ***A1***

$-4 - \lambda + 2\mu = -2a$

solving simultaneous equations gives $\lambda = -\frac{6}{7},\ \mu = \frac{5}{7}$ ***A1A1***

[5 marks]

(c) $\overrightarrow{PQ} = \frac{18}{7}\boldsymbol{i} - \frac{6}{7}\boldsymbol{j} - \frac{12}{7}\boldsymbol{k}$ ***A1***

shortest distance $= \left|\overrightarrow{PQ}\right| = \frac{6}{7}\sqrt{3^2 + (-1)^2 + (-2)^2} = \frac{6}{7}\sqrt{14}$ ***M1A1***

[3 marks]

(d) **METHOD 1**

vector perpendicular to Π is given by vector product of $\boldsymbol{v}$ and $\boldsymbol{w}$ ***(R1)***

$\boldsymbol{v} \times \boldsymbol{w} = 3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$ ***(M1)A1***

so equation of Π is $3x - y - 2z + d = 0$

through $(1, 2, 3) \Rightarrow d = 5$ ***M1***

so equation is $3x - y - 2z + 5 = 0$ ***A1***

continued…

Question 13 continued

METHOD 2

from part (b) $\overrightarrow{PQ} = \frac{18}{7}\boldsymbol{i} - \frac{6}{7}\boldsymbol{j} - \frac{12}{7}\boldsymbol{k}$ is a vector perpendicular to Π ***R1A2***

so equation of Π is $\frac{18}{7}x - \frac{6}{7}y - \frac{12}{7}z + c = 0$

through $(1, 2, 3) \Rightarrow c = \frac{30}{7}$ ***M1***

so equation is $\frac{18}{7}x - \frac{6}{7}y - \frac{12}{7}z + \frac{30}{7} = 0 \quad (3x - y - 2z + 5 = 0)$ ***A1***

Note: Allow other methods *ie* via vector parametric equation.

[5 marks]

(e) $\overrightarrow{OT} = 2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + \eta(3\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$

$T = (2 + 3\eta, 1 - \eta, -1 - 2\eta)$ lies on Π implies

$3(2 + 3\eta) - (1 - \eta) - 2(-1 - 2\eta) + 5 = 0$ ***M1***

$\Rightarrow 12 + 14\eta = 0 \Rightarrow \eta = -\frac{6}{7}$ ***A1***

Note: If no marks awarded in (d) but correct vector product calculated in (e) award ***M1A1*** in (d).

[2 marks]

(f) $\left|\overrightarrow{BT}\right| = \frac{6}{7}\sqrt{3^2 + (-1)^2 + (-2)^2} = \frac{6}{7}\sqrt{14}$ ***M1A1***

[2 marks]

(g) they agree ***A1***

Note: FT is inappropriate here.

$\overrightarrow{BT}$ is perpendicular to both Π and l_2

so its length is the shortest distance between Π and l_2 which is the shortest distance between l_1 and l_2 ***R1***

[2 marks]

Total [21 marks]