Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2015

Mathematics

Higher level

Paper 2

21 pages

International Baccalaureate®
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Section A

1. $\int_{-1}^{1} \pi\left(e^{-x^2}\right)^2 dx \quad \left(\int_{-1}^{1} \pi e^{-2x^2} dx \text{ or } \int_{0}^{1} 2\pi e^{-2x^2} dx\right)$ ***(M1)(A1)(A1)***

> **Note:** Award ***M1*** for integral involving the function given; ***A1*** for correct limits; ***A1*** for π and $\left(e^{-x^2}\right)^2$

$= 3.758249... = 3.76$ ***A1***

[4 marks]

2. (a) $X \sim N\left(210, 22^2\right)$

$P\left(X < 180\right) = 0.0863$ ***(M1)A1***

[2 marks]

(b) $P\left(X < T\right) = 0.9 \Rightarrow T = 238$ (mins) ***(M1)A1***

[2 marks]

Total [4 marks]

3. (a) $W \sim B(1000, 0.1)$ (accept $C_k^{1000}\left(0.1\right)^k\left(0.9\right)^{1000-k}$) ***A1A1***

> **Note:** First ***A1*** is for recognizing the binomial, second ***A1*** for both parameters if stated explicitly in this part of the question.

[2 marks]

(b) $\mu(= 1000 \times 0.1) = 100$ ***A1***

[1 mark]

(c) $P(W > 89) = P(W \geq 90) \quad \left(= 1 - P(W \leq 89)\right)$ ***(M1)***

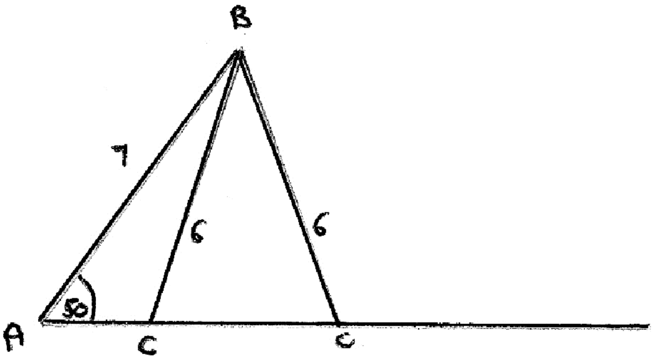
$= 0.867$ ***A1***

> **Notes:** Award ***M0A0*** for 0.889

[2 marks]

Total [5 marks]

4.



METHOD 1

$\frac{6}{\sin 50} = \frac{7}{\sin \text{C}} \Rightarrow \sin \text{C} = \frac{7\sin 50}{6}$ ***(M1)***

$C = 63.344...$ ***(A1)***

or $C = 116.655...$ ***(A1)***

$B = 13.344...$ (or $B = 66.656...$) ***(A1)***

area $= \frac{1}{2} \times 6 \times 7 \times \sin 13.344...$ (or $\frac{1}{2} \times 6 \times 7 \times \sin 66.656...$) ***(M1)***

$4.846...$ (or $= 19.281...$)

so answer is 4.85 (cm^2) ***A1***

METHOD 2

$6^2 = 7^2 + b^2 - 2 \times 7b\cos 50$ ***(M1)(A1)***

$b^2 - 14\,b\cos 50 + 13 = 0$ or equivalent method to solve the above equation ***(M1)***

$b = 7.1912821...$ or $b = 1.807744...$ (***A1***)

area $= \frac{1}{2} \times 7 \times 1.8077...\sin 50 = 4.846...$ ***(M1)***

(or $\frac{1}{2} \times 7 \times 7.1912821...\sin 50 = 19.281...$)

so answer is 4.85 (cm^2) ***A1***

METHOD 3

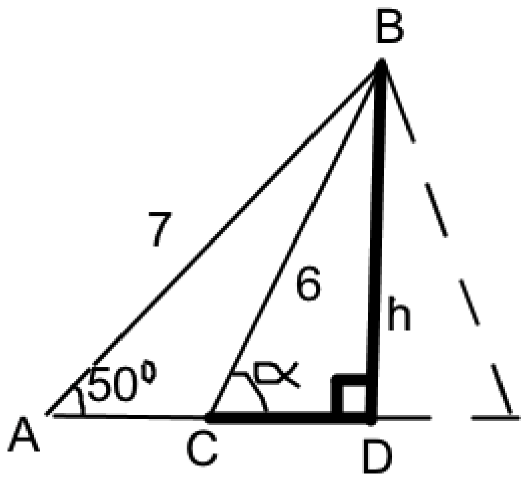


Diagram showing triangles ACB and ADB *(M1)*

$h = 7\sin(50) = 5.3623...$ (cm) *(M1)*

$\alpha = \arcsin\frac{h}{6} = 63.3442...$ *(M1)*

$\text{AC} = \text{AD} - \text{CD} = 7\cos 50 - 6\cos\alpha = 1.8077...$ (cm) *(M1)*

area $= \frac{1}{2} \times 1.8077... \times 5.3623...$ *(M1)*

$= 4.85$ (cm^2) *A1*

Total [6 marks]

5. $V = 200\pi r^2$ ***(A1)***

> **Note:** Allow $V = \pi h r^2$ if value of h is substituted later in the question.

EITHER

$\dfrac{dV}{dt} = 200\pi 2r \dfrac{dr}{dt}$ ***M1A1***

> **Note:** Award ***M1*** for an attempt at implicit differentiation.

at $r = 2$ we have $30 = 200\pi 4 \dfrac{dr}{dt}$ ***M1***

OR

$\dfrac{dr}{dt} = \dfrac{\frac{dV}{dt}}{\frac{dV}{dr}}$ ***M1***

$\dfrac{dV}{dr} = 400\pi r$ ***M1***

$r = 2$ we have $\dfrac{dV}{dr} = 800\pi$ ***A1***

THEN

$\dfrac{dr}{dt} = \dfrac{30}{800\pi}\left(= \dfrac{3}{80\pi} = 0.0119\right)$ (cm s^{-1}) ***A1***

Total [5 marks]

6. $f'(x) = 3x^2 + e^x$ ***A1***

> **Note:** Accept labelled diagram showing the graph $y = f'(x)$ above the x-axis; do not accept unlabelled graphs nor graph of $y = f(x)$.

EITHER

this is always > 0 ***R1***
so the function is (strictly) increasing ***R1***
and thus 1–1 ***A1***

OR

this is always > 0 (accept $\neq 0$) ***R1***
so there are no turning points ***R1***
and thus 1–1 ***A1***

> **Note:** ***A1*** is dependent on the first ***R1***.

Total [4 marks]

7. (a) $2\frac{e^{-m}m^4}{4!} = \frac{e^{-m}m^5}{5!}$ ***M1A1***

$\frac{2}{4!} = \frac{m}{5!}$ or other simplification ***M1***

> **Note:** accept a labelled graph showing clearly the solution to the equation**.** Do not accept simple verification that $m = 10$ is a solution.

$\Rightarrow m = 10$ ***AG***

[3 marks]

(b) $\mathrm{P}(X = 6 \mid X \le 11) = \frac{\mathrm{P}(X = 6)}{\mathrm{P}(X \le 11)}$ (***M1) (A1)***

$= \frac{0.063055...}{0.696776...}$ ***(A1)***

$= 0.0905$ ***A1***

[4 marks]

Total [7 marks]

8. (a) require $\begin{pmatrix} 4 \\ \lambda \\ 10 \end{pmatrix} = s\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ ***(M1)***

$\Rightarrow 4 = 2s \Rightarrow s = 2 \Rightarrow \lambda = 6$ ***A1***

> **Note:** Accept cross product solution.

[2 marks]

(b) require $\boldsymbol{v} \bullet \boldsymbol{w} = 2 \times 4 + 3 \times \lambda + 5 \times 10 = 0 \Rightarrow 3\lambda = -58 \Rightarrow \lambda = \frac{-58}{3}(-19.3)$ ***M1A1***

[2 marks]

(c) $\boldsymbol{v} \bullet \boldsymbol{w} = 2 \times 4 + 3 \times \lambda + 5 \times 10 = \sqrt{2^2 + 3^2 + 5^2} \times \sqrt{4^2 + \lambda^2 + 10^2} \times \cos 10^\circ$ (***M1)(A1)***

$58 + 3\lambda = \sqrt{38} \times \sqrt{116 + \lambda^2} \times \cos 10^\circ$

$\lambda = 3.73$ or 8.76 ***A1A1***

[4 marks]

Total [8 marks]

9. $x = 0 \Rightarrow y = 1$ ***(A1)***

$y'(0) = 1.367879\ldots$ ***(M1)(A1)***

Note: The exact answer is $y'(0) = \frac{e+1}{e} = 1 + \frac{1}{e}$.

so gradient of normal is $\frac{-1}{1.367879\ldots}\left(= -0.731058\ldots\right)$ ***(M1)(A1)***

equation of normal is $y = -0.731058\ldots x + c$ ***(M1)***

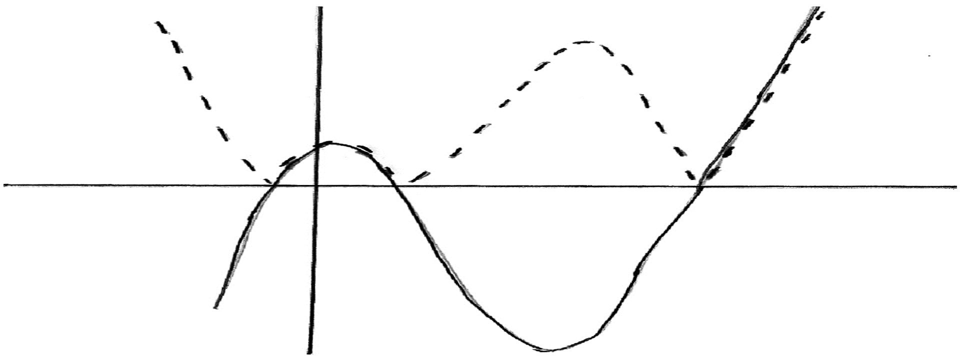
gives $y = -0.731x + 1$ ***A1***

Note: The exact answer is $y = -\frac{e}{e+1}x + 1$.

Accept $y - 1 = -0.731058\ldots(x - 0)$

Total [7 marks]

10. (a)



as roots of $f(x) = 0$ are $-1, 1, 5$ ***(M1)***

solution is $]-\infty, -1[\cup]1, 5[$ ($x < -1$ or $1 < x < 5$) ***A1A1***

Note: Award ***A1A0*** for closed intervals.

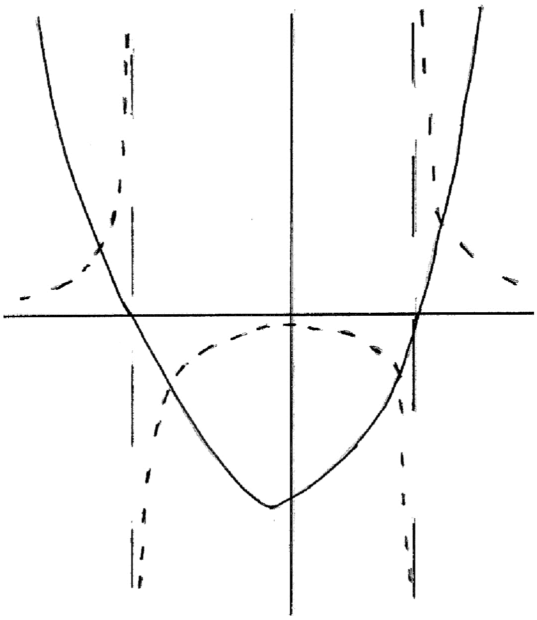
[3 marks]

continued…

Question 10 continued

(b) METHOD 1

(graphs of $g(x)$ and $\frac{1}{g(x)}$)



roots of $g(x) = 0$ are -3 and 2 ***(M1)(A1)***

Notes: Award ***M1*** if quadratic graph is drawn or two roots obtained. Roots may be indicated anywhere *eg* asymptotes on graph or in inequalities below.

the intersections of the graphs $g(x)$ and of $1/g(x)$
are $-3.19, -2.79, 1.79, 2.19$ ***(M1)(A1)***

Note: Award ***A1*** for at least one of the values above seen anywhere.

solution is $]-3.19,-3[\cup]-2.79,1.79[\cup]2,2.19[$
($-3.19 < x < -3$ or $-2.79 < x < 1.79$ or $2 < x < 2.19$) ***A1A1A1***

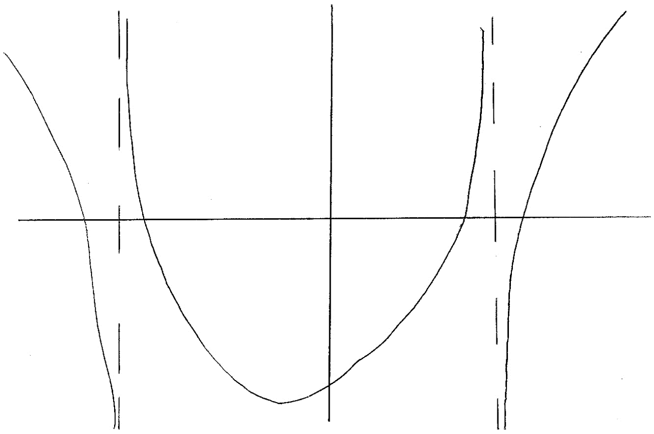
Note: Award ***A1A1A0*** for closed intervals.

continued…

Question 10 continued

METHOD 2

(graph of $g(x) - \frac{1}{g(x)}$)



asymptotes at $x = -3$ and $x = 2$ ***(M1)(A1)***

Note: May be indicated on the graph.

roots of graph are $-3.19, -2.79, 1.79, 2.19$ ***(M1)(A1)***

Note: Award ***A1*** for at least one of the values above seen anywhere.

solution is (when graph is negative)
$]-3.19, -3[\cup]-2.79, 1.79[\cup]2, 2.19[$
($-3.19 < x < -3$ or $-2.79 < x < 1.79$ or $2 < x < 2.19$) ***A1A1A1***

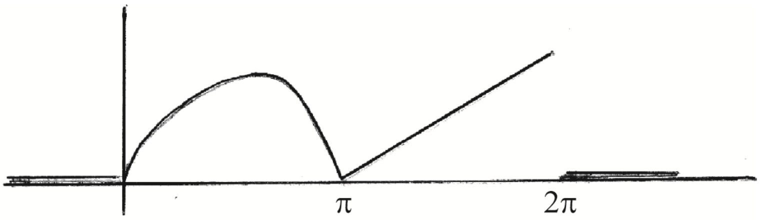
Note: Award ***A1A1A0*** for closed intervals.

[7 marks]

Total [10 marks]

Section B

11. (a)



Award ***A1*** for sine curve from 0 to π, award ***A1*** for straight line from π to 2π ***A1A1***

[2 marks]

(b) $\int_0^{\pi} \frac{\sin x}{4} \mathrm{d}x = \frac{1}{2}$ ***(M1)A1***

[2 marks]

(c) **METHOD 1**

require $\frac{1}{2} + \int_{\pi}^{2\pi} a(x-\pi)\,\mathrm{d}x = 1$ ***(M1)***

$\Rightarrow \frac{1}{2} + a\left[\frac{(x-\pi)^2}{2}\right]_{\pi}^{2\pi} = 1$ (or $\frac{1}{2} + a\left[\frac{x^2}{2} - \pi x\right]_{\pi}^{2\pi} = 1$) ***A1***

$\Rightarrow a\frac{\pi^2}{2} = \frac{1}{2}$ ***A1***

$\Rightarrow a = \frac{1}{\pi^2}$ ***AG***

Note: Must obtain the exact value. Do not accept answers obtained with calculator.

METHOD 2

0.5 + area of triangle = 1 ***R1***

area of triangle $= \frac{1}{2}\pi \times a\pi = 0.5$ ***M1A1***

Note: Award ***M1*** for correct use of area formula $= 0.5$, ***A1*** for $a\pi$.

$a = \frac{1}{\pi^2}$ ***AG***

[3 marks]

(d) median is π ***A1***

[1 mark]

continued…

Question 11 continued

(e) $\mu = \int_0^{\pi} x \cdot \frac{\sin x}{4} \mathrm{d}x + \int_{\pi}^{2\pi} x \cdot \frac{x - \pi}{\pi^2} \mathrm{d}x$ ***(M1)(A1)***

$= 3.40339... = 3.40$ (or $\frac{\pi}{4} + \frac{5\pi}{6} = \frac{13}{12}\pi$) ***A1***

[3 marks]

(f) For $\mu = 3.40339...$

EITHER

$\sigma^2 = \int_0^{\pi} x^2 \cdot \frac{\sin x}{4} \mathrm{d}x + \int_{\pi}^{2\pi} x^2 \cdot \frac{x - \pi}{\pi^2} \mathrm{d}x - \mu^2$ ***(M1)(A1)***

OR

$\sigma^2 = \int_0^{\pi} (x - \mu)^2 \cdot \frac{\sin x}{4} \mathrm{d}x + \int_{\pi}^{2\pi} (x - \mu)^2 \cdot \frac{x - \pi}{\pi^2} \mathrm{d}x$ ***(M1)(A1)***

THEN

$= 3.866277... = 3.87$ ***A1***

[3 marks]

(g) $\int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{4} \mathrm{d}x + \int_{\pi}^{\frac{3\pi}{2}} \frac{x - \pi}{\pi^2} \mathrm{d}x = 0.375$ (or $\frac{1}{4} + \frac{1}{8} = \frac{3}{8}$) ***(M1)A1***

[2 marks]

(h) $\mathrm{P}\left(\pi \le X \le 2\pi \,\middle|\, \frac{\pi}{2} \le X \le \frac{3\pi}{2} \right) = \frac{\mathrm{P}\left(\pi \le X \le \frac{3\pi}{2} \right)}{\mathrm{P}\left(\frac{\pi}{2} \le X \le \frac{3\pi}{2} \right)}$ ***(M1)(A1)***

$= \frac{\int_{\pi}^{\frac{3\pi}{2}} \frac{(x - \pi)}{\pi^2} \mathrm{d}x}{0.375} = \frac{0.125}{0.375}$ (or $= \frac{\frac{1}{8}}{\frac{3}{8}}$ from diagram areas) ***(M1)***

$= \frac{1}{3}$ (0.333) ***A1***

[4 marks]

Total [20 marks]

12. (a) (i) $(\cos\theta + \mathrm{i}\sin\theta)^5$

$= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta + 10\mathrm{i}^2\cos^3\theta\sin^2\theta +$

$10\mathrm{i}^3\cos^2\theta\sin^3\theta + 5\mathrm{i}^4\cos\theta\sin^4\theta + \mathrm{i}^5\sin^5\theta$ ***A1A1***

$(= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta -$

$10\mathrm{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta)$

> **Note:** Award first ***A1*** for correct binomial coefficients.

(ii) $(\mathrm{cis}\theta)^5 = \mathrm{cis}5\theta = \cos 5\theta + \mathrm{i}\sin 5\theta$ ***M1***

$= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10\mathrm{i}\cos^2\theta\sin^3\theta +$

$5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta$ ***A1***

> **Note:** Previous line may be seen in (i)

equating imaginary terms ***M1***

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$ ***AG***

(iii) equating real terms

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ ***A1***

[6 marks]

(b) $(r\,\mathrm{cis}\,\alpha)^5 = 1 \Rightarrow r^5\,\mathrm{cis}\,5\alpha = 1\,\mathrm{cis}\,0$ ***M1***

$r^5 = 1 \Rightarrow r = 1$ ***A1***

$5\alpha = 0 \pm 360k,\ k \in \mathbb{Z} \Rightarrow \alpha = 72k$ ***(M1)***

$\alpha = 72^\circ$ ***A1***

> **Note:** Award ***M1A0*** if final answer is given in radians.

[4 marks]

(c) use of $\sin(5\times 72) = 0$ **OR** the imaginary part of 1 is 0 ***(M1)***

$0 = 5\cos^4\alpha\sin\alpha - 10\cos^2\alpha\sin^3\alpha + \sin^5\alpha$ ***A1***

$\sin\alpha \neq 0 \Rightarrow\ 0 = 5\left(1-\sin^2\alpha\right)^2 - 10\left(1-\sin^2\alpha\right)\sin^2\alpha + \sin^4\alpha$ ***M1***

> **Note:** Award ***M1*** for replacing $\cos^2\alpha$.

$0 = 5\left(1 - 2\sin^2\alpha + \sin^4\alpha\right) - 10\sin^2\alpha + 10\sin^4\alpha + \sin^4\alpha$ ***A1***

> **Note:** Award ***A1*** for any correct simplification.

so $16\sin^4\alpha - 20\sin^2\alpha + 5 = 0$ ***AG***

[4 marks]

continued…

Question 12 continued

(d) $\sin^2\alpha = \dfrac{20 \pm \sqrt{400-320}}{32}$ ***M1A1***

$\sin\alpha = \pm\sqrt{\dfrac{20 \pm \sqrt{80}}{32}}$

$\sin\alpha = \dfrac{\pm\sqrt{10 \pm 2\sqrt{5}}}{4}$ ***A1***

Note: Award ***A1*** regardless of signs. Accept equivalent forms with integral denominator, simplification may be seen later.

as $72 > 60$, $\sin 72 > \dfrac{\sqrt{3}}{2} = 0.866\ldots$ so we have to take both positive signs (or equivalent argument) ***R1***

Note: Allow verification of correct signs with calculator if clearly stated

$\sin 72 = \dfrac{\sqrt{10+2\sqrt{5}}}{4}$ ***A1***

[5 marks]

Total [19 marks]

13. (a) (i) $a(t)=\frac{\mathrm{d}v}{\mathrm{d}t}=-10\ (\mathrm{m\,s^{-2}})$ ***A1***

(ii) $t=10\Rightarrow v=-100\ (\mathrm{m\,s^{-1}})$ ***A1***

(iii) $s=\int -10t\,\mathrm{d}t=-5t^2\ (+c)$ ***M1A1***

$s=1000$ for $t=0\Rightarrow c=1000$ **(*M1*)**

$s=-5t^2+1000$ ***A1***

at $t=10,\ s=500\ (\mathrm{m})$ ***AG***

Note: Accept use of definite integrals.

[6 marks]

(b) $\frac{\mathrm{d}t}{\mathrm{d}v}=\frac{1}{(-10-5v)}$ ***A1***

[1 mark]

(c) **METHOD 1**

$t=\int\frac{1}{-10-5v}\mathrm{d}v=-\frac{1}{5}\ln(-10-5v)(+c)$ ***M1A1***

Note: Accept equivalent forms using modulus signs.

$t=10,\ v=-100$

$10=-\frac{1}{5}\ln(490)+c$ ***M1***

$c=10+\frac{1}{5}\ln(490)$ ***A1***

$t=10+\frac{1}{5}\ln 490-\frac{1}{5}\ln(-10-5v)$ ***A1***

Note: Accept equivalent forms using modulus signs.

$t=10+\frac{1}{5}\ln\left(\frac{98}{-2-v}\right)$ ***AG***

Note: Accept use of definite integrals.

continued…

Question 13 continued

METHOD 2

$t=\int \frac{1}{-10-5v}\mathrm{d}v=-\frac{1}{5}\int \frac{1}{2+v}\mathrm{d}v=-\frac{1}{5}\ln\left|2+v\right|(+c)$ ***M1A1***

Note: Accept equivalent forms.

$t=10,\ v=-100$

$10=-\frac{1}{5}\ln\left|-98\right|+c$ ***M1***

Note: If $\ln\left(-98\right)$ is seen do not award further A marks.

$c=10+\frac{1}{5}\ln 98$ ***A1***

$t=10+\frac{1}{5}\ln 98-\frac{1}{5}\ln\left|2+v\right|$ ***A1***

Note: Accept equivalent forms.

$t=10+\frac{1}{5}\ln\left(\frac{98}{-2-v}\right)$ ***AG***

Note: Accept use of definite integrals.

[5 marks]

(d) $5(t-10)=\ln\frac{98}{(-2-v)}$

$\frac{2+v}{98}=-\mathrm{e}^{-5(t-10)}$ ***(M1)***

$v=-2-98\mathrm{e}^{-5(t-10)}$ ***A1***

[2 marks]

(e) $\frac{\mathrm{d}s}{\mathrm{d}t}=-2-98\mathrm{e}^{-5(t-10)}$

$s=-2t+\frac{98}{5}\mathrm{e}^{-5(t-10)}(+k)$ ***M1A1***

at $t=10,\ s=500\Rightarrow 500=-20+\frac{98}{5}+k\Rightarrow k=500.4$ ***M1A1***

$s=-2t+\frac{98}{5}\mathrm{e}^{-5(t-10)}+500.4$ ***A1***

Note: Accept use of definite integrals.

[5 marks]

continued…

Question 13 continued

(f) $t = 250$ for $s = 0$ ***(M1)A1***

[2 marks]

Total [21 marks]