Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2015

Mathematics

Higher level

Paper 1

Section A

1. arc length $= \frac{2}{x} = rx \left(\Rightarrow r = \frac{2}{x^2} \right)$ ***M1***

$16 = \frac{1}{2}\left(\frac{2}{x^2}\right)^2 x \left(\Rightarrow \frac{2}{x^3} = 16 \right)$ ***M1***

Note: Award M1s for attempts at the use of arc-length and sector-area formulae.

$x = \frac{1}{2}$ ***A1***

arc length $= 4(\text{cm})$ ***A1***

[4 marks]

2. attempt to integrate one factor and differentiate the other, leading to a sum of two terms ***M1***

$\int x \sin x \, dx = x(-\cos x) + \int \cos x \, dx$ ***(A1)(A1)***

$= -x \cos x + \sin x + c$ ***A1***

Note: Only award final ***A1*** if $+c$ is seen.

[4 marks]

3. (a) $(2+x)^4 = 2^4 + 4 \cdot 2^3 x + 6 \cdot 2^2 x^2 + 4 \cdot 2x^3 + x^4$ ***M1(A1)***

Note: Award ***M1*** for an expansion, by whatever method, giving five terms in any order.

$= 16 + 32x + 24x^2 + 8x^3 + x^4$ ***A1***

Note: Award ***M1A1A0*** for correct expansion not given in ascending powers of x.

[3 marks]

(b) let $x = 0.1$ (in the binomial expansion) ***(M1)***

$2.1^4 = 16 + 3.2 + 0.24 + 0.008 + 0.0001$ ***(A1)***

$= 19.4481$ ***A1***

Note: At most one of the marks can be implied.

[3 marks]

Total [6 marks]

4. (a) $\frac{dy}{dx} = (1-x)^{-2} \left(= \frac{1}{(1-x)^2} \right)$ ***(M1)A1***

[2 marks]

continued…

Question 4 continued

(b) gradient of Tangent $= \frac{1}{4}$ ***(A1)***

gradient of Normal $= -4$ ***(M1)***

$y + \frac{1}{2} = -4(x-3)$ or attempt to find c in $y = mx + c$ ***M1***

$8x + 2y - 23 = 0$ ***A1***

[4 marks]

Total [6 marks]

5. METHOD 1

$\int_{e}^{e^2} \frac{dx}{x \ln x} = [\ln(\ln x)]_{e}^{e^2}$ ***(M1)A1***

$= \ln(\ln e^2) - \ln(\ln e)$ $(= \ln 2 - \ln 1)$ ***(A1)***

$= \ln 2$ ***A1***

[4 marks]

METHOD 2

$u = \ln x, \frac{du}{dx} = \frac{1}{x}$ ***M1***

$= \int_1^2 \frac{du}{u}$ ***A1***

$= [\ln u]_1^2$ or equivalent in x $(= \ln 2 - \ln 1)$ ***(A1)***

$= \ln 2$ ***A1***

[4 marks]

6. (a) probability that Darren wins $P(W) + P(RRW) + P(RRRRW)$ ***(M1)***

Note: Only award ***M1*** if three terms are seen or are implied by the following numerical equivalent.

Note: Accept equivalent tree diagram for method mark.

$= \frac{2}{6} + \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} + \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \cdot \frac{2}{2}$ $\left(= \frac{1}{3} + \frac{1}{5} + \frac{1}{15}\right)$ ***A2***

Note: ***A1*** for two correct.

$= \frac{3}{5}$ ***A1***

[4 marks]

continued…

Question 6 continued

(b) **METHOD 1**

the probability that Darren wins is given by
$\mathrm{P}(W) + \mathrm{P}(RRW) + \mathrm{P}(RRRRW) + \ldots$ ***(M1)***

Note: Accept equivalent tree diagram with correctly indicated path for method mark.

$\mathrm{P}(\text{Darren Win}) = \frac{1}{3} + \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3} + \ldots$

or $= \frac{1}{3}\left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \ldots\right)$ ***A1***

$= \frac{1}{3}\left(\frac{1}{1-\frac{4}{9}}\right)$ ***A1***

$= \frac{3}{5}$ ***AG***

[3 marks]

METHOD 2

$\mathrm{P}(\text{Darren wins}) = \mathrm{P}$

$\mathrm{P} = \frac{1}{3} + \frac{4}{9}\mathrm{P}$ ***M1A2***

$\frac{5}{9}\mathrm{P} = \frac{1}{3}$

$\mathrm{P} = \frac{3}{5}$ ***AG***

[3 marks]

Total [7 marks]

7. (a) $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 2y\frac{\mathrm{d}y}{\mathrm{d}x}$ ***M1A1***

a horizontal tangent occurs if $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ so $y = 0$ ***M1***

we can see from the equation of the curve that this solution is not possible $(0 = 4)$ and so there is not a horizontal tangent ***R1***

[4 marks]

continued…

Question 7 continued

(b) $\frac{dy}{dx} = \frac{y}{2y - x}$ or equivalent with $\frac{dx}{dy}$

the tangent is vertical when $2y = x$ ***M1***

substitute into the equation to give $2y^2 = y^2 + 4$ ***M1***

$y = \pm 2$ ***A1***

coordinates are $(4, 2), (-4, -2)$ ***A1***

[4 marks]

Total [8 marks]

8. (a) $\sin\left(\theta + \frac{\pi}{2}\right) = \sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2}$ ***M1***

$= \cos\theta$ ***AG***

Note: Accept a transformation/graphical based approach.

[1 mark]

(b) consider $n = 1$, $f'(x) = a\cos(ax)$ ***M1***

since $\sin\left(ax + \frac{\pi}{2}\right) = \cos ax$ then the proposition is true for $n = 1$ ***R1***

assume that the proposition is true for $n = k$ so $f^{(k)}(x) = a^k \sin\left(ax + \frac{k\pi}{2}\right)$ ***M1***

$f^{(k+1)}(x) = \frac{d\left(f^{(k)}(x)\right)}{dx}\left(= a\left(a^k\cos\left(ax + \frac{k\pi}{2}\right)\right)\right)$ ***M1***

$= a^{k+1}\sin\left(ax + \frac{k\pi}{2} + \frac{\pi}{2}\right)$ (using part (a)) ***A1***

$= a^{k+1}\sin\left(ax + \frac{(k+1)\pi}{2}\right)$ ***A1***

given that the proposition is true for $n = k$ then we have shown that the proposition is true for $n = k + 1$. Since we have shown that the proposition is true for $n = 1$ then the proposition is true for all $n \in \mathbb{Z}^+$ ***R1***

Note: Award final ***R1*** only if all prior M and R marks have been awarded.

[7 marks]

Total [8 marks]

9. $(\sin 2x - \sin x) - (\cos 2x - \cos x) = 1$

attempt to use both double-angle formulae, in whatever form ***M1***

$(2\sin x \cos x - \sin x) - (2\cos^2 x - 1 - \cos x) = 1$

or $(2\sin x \cos x - \sin x) - \left(2\cos^2 x - \cos x\right) = 0$ for example ***A1***

Note: Allow any rearrangement of the above equations.

$\sin x(2\cos x - 1) - \cos x(2\cos x - 1) = 0$

$(\sin x - \cos x)(2\cos x - 1) = 0$ ***(M1)***

$\tan x = 1$ and $\cos x = \frac{1}{2}$ ***A1A1***

Note: These ***A*** marks are dependent on the ***M*** mark awarded for factorisation.

$x = -\frac{3\pi}{4}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{4}$ ***A2***

Note: Award ***A1*** for two correct answers, which could be for both tan or both cos solutions, for example.

[7 marks]

10. (a) the sum of the roots of the polynomial $= \frac{63}{16}$ ***(A1)***

$$2\left(\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right) = \frac{63}{16}$$ ***M1A1***

Note: The formula for the sum of a geometric sequence must be equated to a value for the ***M1*** to be awarded.

$$1-\left(\frac{1}{2}\right)^n = \frac{63}{64} \Rightarrow \left(\frac{1}{2}\right)^n = \frac{1}{64}$$

$n = 6$ ***A1***

[4 marks]

(b) $\frac{a_0}{a_n} = 2 \times 1 \times \frac{1}{2} \times \frac{1}{4} \times \frac{1}{8} \times \frac{1}{16}, \left(a_n = 16\right)$ ***M1***

$a_0 = 16 \times 2 \times 1 \times \frac{1}{2} \times \frac{1}{4} \times \frac{1}{8} \times \frac{1}{16}$

$a_0 = 2^{-5}\left(= \frac{1}{32}\right)$ ***A1***

[2 marks]

Total [6 marks]

Section B

11. (a) $z^3 = 8\left(\cos\left(\frac{\pi}{2} + 2\pi k\right) + \mathrm{i}\sin\left(\frac{\pi}{2} + 2\pi k\right)\right)$ ***(A1)***

attempt the use of De Moivre's Theorem in reverse ***M1***

$z = 2\left(\cos\left(\frac{\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{6}\right)\right);\ 2\left(\cos\left(\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(\frac{5\pi}{6}\right)\right);$

$2\left(\cos\left(\frac{9\pi}{6}\right) + \mathrm{i}\sin\left(\frac{9\pi}{6}\right)\right)$ ***A2***

Note: Accept cis form.

$z = \pm\sqrt{3} + \mathrm{i},\ -2\mathrm{i}$ ***A2***

Note: Award ***A1*** for two correct solutions in each of the two lines above.

[6 marks]

(b) (i) $z_1 = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right)\right)$ ***A1A1***

(ii) $\left(z_2 = \left(\sqrt{3} + \mathrm{i}\right)\right)$

$z_1 z_2 = (1 + \mathrm{i})\left(\sqrt{3} + \mathrm{i}\right)$ ***M1***

$= \left(\sqrt{3} - 1\right) + \mathrm{i}\left(1 + \sqrt{3}\right)$ ***A1***

(iii) $z_1 z_2 = 2\sqrt{2}\left(\cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right)\right)$ ***M1A1***

$\tan\frac{5\pi}{12} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$ ***A1***

$= 2 + \sqrt{3}$ ***M1A1***

Note: Award final ***M1*** for an attempt to rationalise the fraction.

(iv) $z_2{}^p = 2^p\left(\operatorname{cis}\left(\frac{p\pi}{6}\right)\right)$ ***(M1)***

$z_2{}^p$ is a positive real number when $p = 12$ ***A1***

[11 marks]

Total [17 marks]

12. (a) $f(-x) = (-x)\sqrt{1-(-x)^2}$ ***M1***

$= -x\sqrt{1-x^2}$

$= -f(x)$ ***R1***

hence f is odd ***AG***

[2 marks]

(b) $f'(x) = x \cdot \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}} \cdot -2x + \left(1-x^2\right)^{\frac{1}{2}}$ ***M1A1A1***

[3 marks]

(c) $f'(x) = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} \left(= \frac{1-2x^2}{\sqrt{1-x^2}}\right)$ ***A1***

Note: This may be seen in part (b).

$f'(x) = 0 \Rightarrow 1 - 2x^2 = 0$ ***M1***

$x = \pm\frac{1}{\sqrt{2}}$ ***A1***

[3 marks]

(d) y-coordinates of the Max Min Points are $y = \pm\frac{1}{2}$ ***M1A1***

so range of $f(x)$ is $\left[-\frac{1}{2}, \frac{1}{2}\right]$ ***A1***

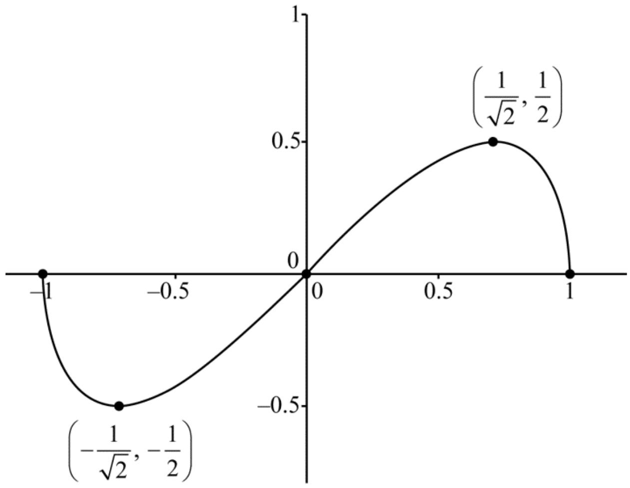
Note: Allow FT from (c) if values of x, within the domain, are used.

[3 marks]

continued…

Question 12 continued

(e)



Shape: The graph of an odd function, on the given domain, s-shaped, where the max(min) is the right(left) of $0.5(-0.5)$ ***A1***
x-intercepts ***A1***
turning points ***A1***

[3 marks]

(f) area $= \int_0^1 x\sqrt{1-x^2}\,dx$ ***(M1)***

attempt at "backwards chain rule" or substitution ***M1***

$= -\frac{1}{2}\int_0^1 (-2x)\,\sqrt{1-x^2}\,dx$

$= \left[\frac{2}{3}\left(1-x^2\right)^{\frac{3}{2}} \cdot -\frac{1}{2}\right]_0^1$ ***A1***

$= \left[-\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}\right]_0^1$

$= 0 - \left(-\frac{1}{3}\right) = \frac{1}{3}$ ***A1***

[4 marks]

(g) $\int_{-1}^{1}\left|x\sqrt{1-x^2}\right|dx > 0$ ***R1***

$\left|\int_{-1}^{1} x\sqrt{1-x^2}\,dx\right| = 0$ ***R1***

so $\int_{-1}^{1}\left|x\sqrt{1-x^2}\right|dx > \left|\int_{-1}^{1} x\sqrt{1-x^2}\,dx\right| = 0$ ***AG***

[2 marks]

Total [20 marks]

13. (a) $\vec{BR} = \vec{BA} + \vec{AR}$ $\left(= \vec{BA} + \frac{1}{2}\vec{AC}\right)$ ***(M1)***

$= (\boldsymbol{a} - \boldsymbol{b}) + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$

$= \frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$ ***A1***

[2 marks]

(b) (i) $\mathrm{r}_{BR} = \boldsymbol{b} + \lambda\left(\frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}\right)\left(= \frac{\lambda}{2}\boldsymbol{a} + (1-\lambda)\,\boldsymbol{b} + \frac{\lambda}{2}\boldsymbol{c}\right)$ ***A1A1***

Note: Award ***A1A0*** if the r = is omitted in an otherwise correct expression/equation.

(ii) $\vec{AQ} = -\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$ ***(A1)***

$\mathrm{r}_{AQ} = \boldsymbol{a} + \mu\left(-\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} + \frac{1}{2}\boldsymbol{c}\right)\left(= (1-\mu)\,\boldsymbol{a} + \frac{\mu}{2}\boldsymbol{b} + \frac{\mu}{2}\boldsymbol{c}\right)$ ***A1***

(iii) when $\vec{AQ}$ and $\vec{BP}$ intersect we will have $\mathrm{r}_{BR} = \mathrm{r}_{AQ}$ ***(M1)***

$\frac{\lambda}{2}\boldsymbol{a} + (1-\lambda)\,\boldsymbol{b} + \frac{\lambda}{2}\boldsymbol{c} = (1-\mu)\,\boldsymbol{a} + \frac{\mu}{2}\boldsymbol{b} + \frac{\mu}{2}\boldsymbol{c}$

attempt to equate the coefficients of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ ***M1***

$$\left.\begin{aligned} \frac{\lambda}{2} &= 1-\mu \\ 1-\lambda &= \frac{\mu}{2} \\ \frac{\lambda}{2} &= \frac{\mu}{2} \end{aligned}\right\}$$ ***(A1)***

$\lambda = \frac{2}{3}$ or $\mu = \frac{2}{3}$ ***A1***

substituting parameters back into one of the equations ***M1***

$\vec{OG} = \frac{1}{2}\cdot\frac{2}{3}\boldsymbol{a} + \left(1 - \frac{2}{3}\right)\boldsymbol{b} + \frac{1}{2}\cdot\frac{2}{3}\boldsymbol{c} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$ ***AG***

[9 marks]

continued…

Question 13 continued

(c) $\vec{\mathrm{CP}} = \frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} - \boldsymbol{c}$ ***(M1)A1***

so we have that $\mathrm{r}_{\mathrm{CP}} = \boldsymbol{c} + \beta\left(\frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b} - \boldsymbol{c}\right)$ and when $\beta = \frac{2}{3}$ the line passes through

the point G (*ie*, with position vector $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$) ***R1***

hence $[\mathrm{AQ}]$, $[\mathrm{BR}]$ and $[\mathrm{CP}]$ all intersect in G ***AG***

[3 marks]

continued…

Question 13 continued

(d) $\overrightarrow{OG} = \frac{1}{3}\left(\begin{pmatrix}1\\3\\1\end{pmatrix} + \begin{pmatrix}3\\7\\-5\end{pmatrix} + \begin{pmatrix}2\\2\\1\end{pmatrix}\right) = \begin{pmatrix}2\\4\\-1\end{pmatrix}$ ***A1***

> **Note:** This independent mark for the vector may be awarded wherever the vector is calculated.

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix}2\\4\\-6\end{pmatrix} \times \begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}-6\\-6\\-6\end{pmatrix}$ ***M1A1***

$\overrightarrow{GX} = \alpha\begin{pmatrix}1\\1\\1\end{pmatrix}$ ***(M1)***

volume of Tetrahedron given by $\frac{1}{3} \times$ Area $\text{ABC} \times \text{GX}$

$= \frac{1}{3}\left(\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|\right) \times \text{GX} = 12$ ***(M1)(A1)***

> **Note:** Accept alternative methods, for example the use of a scalar triple product.

$= \frac{1}{6}\sqrt{(-6)^2 + (-6)^2 + (-6)^2} \times \sqrt{\alpha^2 + \alpha^2 + \alpha^2} = 12$ ***(A1)***

$= \frac{1}{6} 6\sqrt{3}\,|\alpha|\sqrt{3} = 12$

$\Rightarrow |\alpha| = 4$ ***A1***

> **Note:** Condone absence of absolute value.

this gives us the position of X as $\begin{pmatrix}2\\4\\-1\end{pmatrix} \pm \begin{pmatrix}4\\4\\4\end{pmatrix}$

$\text{X}(6, 8, 3)$ or $(-2, 0, -5)$ ***A1***

> **Note:** Award ***A1*** for either result.

[9 marks]

Total [23 marks]