Set Theory: Exploring the Unique Properties of the Cantor Set

1. Introduction

In mathematics, sets are a collection of mathematical objects where the set itself is also considered a mathematical object. Sets are one of the most fundamental concepts primitive to mathematics as they cannot be defined in more "simpler" terms. Hence to understand the importance of sets, we can use a common thought experiment that highlights how we came to the realisation of an important notion; that quantity is abstract. If we date back to prehistoric times we may intuitively think that we were able to count two sheep and two apples and see that both have something in common – a duality known today as "2". However, in order to be able to quantify a collection of sheep or apples, we must first be able to determine that there is a "collection" to quantify in the first place. Hence it is entirely possible to believe that the concept of a collection is as – if not more – fundamental as the concept of counting.

So, then comes the question, what are sets? What kind of properties do they have? We have postulated numerous axioms from our understandings of collections in the real world and *finite* collections, an example being the Zermelo-Fraenkel set theory (ZF; one of the most commonly used that consists of axioms such as the Axiom of Union which states that a union of sets exists. I will not delve deeply into the axioms of set theory, as that is worth another entire math exploration!). It is, however, the *infinite* collections that I find to be of most interest. My first encounter with infinite sets was in the form of a *TED-ED* video, where they described how **Hilbert's Paradox of the Grand Hotel** worked (something that I explore in the next page). When I had first watched it a while back, it took me some time to wrap my mind around the fact that different sizes of infinity could exist. Remembering that video, I was inspired to work with set theory, particularly in regards to infinity and **cardinality** and the countability of sets. While I will be exploring sets, I have chosen specifically to explore the **Cantor set** (named after Georg Cantor, a revolutionary mathematician known for his studies in the sizes of infinity) due to the unique properties that it possesses. In my exploration, I will be constructing the ternary Cantor Set before proving why it is 1) uncountable and 2) non-empty but with zero length.

2. Cardinality

To understand the properties of the cantor set, we must first understand some of the properties of set theory. Hence we must first establish an understanding in cardinality. In simple terms, the cardinality of a finite set X is a natural number (called the set's **cardinal number**) used to measure the number of elements present in that set, and it is usually denoted by a modulus[1]. An example where X is a set of even natural numbers in the interval]1, 10]:

$$X = \{2, 4, 6, 8, 10\}$$

[1] There are other forms of denoting cardinality, such as "#" however for consistency I will only be using the modulus.

$$|X| = 5$$

The set X is shown to have a cardinal number of 5, as it has five elements. The cardinality of a set goes hand in hand with the **countability** of sets, as in order to quantify the number of elements in a set, we must be able to count them. This is where the interesting part of sets arises, as we must then ask ourselves, are all sets countable? We can easily say that finite sets are countable as being finite implies that it has a finite number of elements[2], which can be numbered through from 1 to n where n is the cardinal number of that set. Set X above is a finite set. However, it is the countability of the infinite sets that are the question. This is where Hilbert's Paradox of the Grand Hotel (a.k.a Hilbert's Hotel) comes in play.

Hilbert's Hotel is a thought experiment that shows the counterintuitive properties of sets, as it demonstrates how a fully occupied hotel consisting of infinitely many rooms can accommodate more guests – infinitely more guests – and this process can be repeated an infinite amount of times. For example, if an additional guest wanted a room in the fully occupied hotel, we can simultaneously move the guest in Room 1, to Room 2, the guest in Room 2 to Room 3, and the guest in Room n to $n + 1$, hence making Room 1 free. Similarly, if an infinitely long bus with an infinite amount of passengers wanted accommodation in the hotel with the infinite amount of rooms, the guest in Room 1 can be moved to Room 2, the guest in Room 2 to Room 4 and the guest in Room n to Room $2n$. By doing so, the infinite guests that were already in the hotel will occupy the infinite even numbered rooms, leaving the infinite amount of odd numbered rooms to the infinite bus passengers that had just arrived.

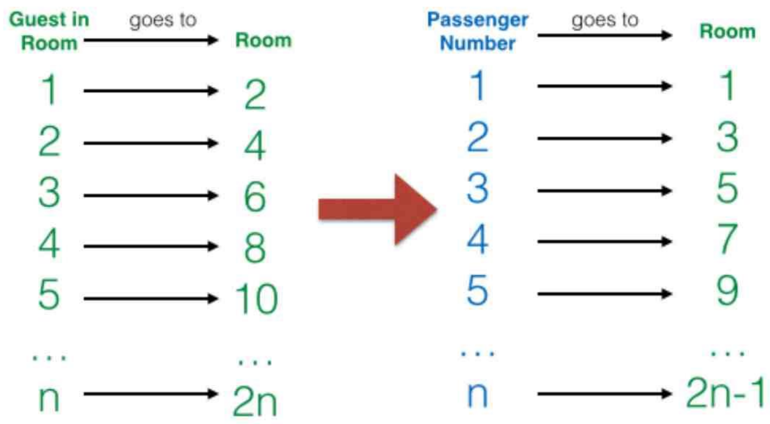

Image source: http://mathandmultimedia.com/2014/05/26/grand-hotel-paradox/

If we take this one step further and say that there are now an infinite number of buses with an infinite number of passengers each, then there is a solution to that as well. This time, we deal with prime numbers, as we assign the guest in Room 1 to

[2] An empty set (denoted as {} or ∅) is also considered a finite set as it has the cardinal number of 0, which in this context (as well as in this course) is considered a natural number.

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Room 2, and then every guest thereafter to Room 2^n. Then, every bus with their infinite number of passengers can be assigned to the next prime number, and their exponents, as seen in the next table.

Room Assignments	Passenger Number or Room Number						
	1	2	3	4	5	...	n
Hotel Guest	2^1	2^2	2^3	2^4	2^5	...	2^n
Bus 1	3^1	3^2	3^3	3^4	3^5	...	3^n
Bus 2	5^1	5^2	5^3	5^4	5^5	...	5^n
Bus 3	7^1	7^2	7^3	7^4	7^5	...	7^n
Bus 4	11^1	11^2	11^3	11^4	11^5	...	11^n
...							
Bus n	$P_{n+1}{}^1$	$P_{n+1}{}^2$	$P_{n+1}{}^3$	$P_{n+1}{}^4$	$P_{n+1}{}^5$	...	$P_{n+1}{}^n$

Image source: http://mathandmultimedia.com/2014/05/26/grand-hotel-paradox/

A very interesting notion that I had missed while I did my initial research was that Hilbert based his above solution on the assumption that there are an infinite number of prime numbers (otherwise his solution would not work). However, I found in the IB Discrete Maths[i] option book that this assumption can be easily proved using a proof by contradiction, attributed to Greek mathematician Euclid:

If we suppose there are finitely many prime numbers $p_1, p_2, p_3, \dots, p_k$, it means that every integer greater than 1 will either be a prime number or be divisible by a prime number. Now let P be a common multiple of these primes plus 1; $P = p_1 p_2 p_3 \dots p_k + 1$. We can see that P is greater than the prime numbers p_k hence it cannot be any of those prime numbers. Furthermore, since P is not a prime number P cannot be divided by each p_k as it would provide a remainder of 1 which does not work. Therefore the assumption that there are finitely many prime numbers is incorrect, which means that there are infinitely many prime numbers.

In short, this paradox illustrates the unique properties of infinity as it shows that there can be different sizes of infinity. Understanding this notion leads to the question of whether all infinite sets are countable. Within mathematics there is a general consensus that if the cardinality of some infinite set A is less than or equal to the cardinality of natural numbers it is then considered **countably infinite.** The cardinality of natural numbers is assigned (by Georg Cantor) and is (today) accepted to be the **aleph-null** (a.k.a aleph-naught) $\aleph_0$. This means that there must either be a

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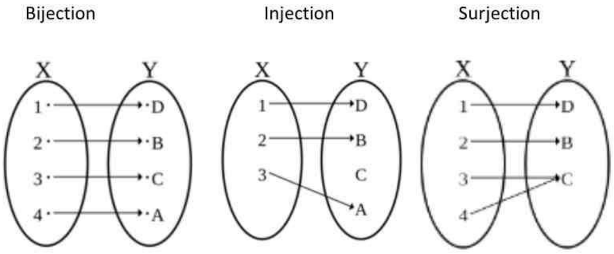
bijection – a one-to-one correspondence[3] between the two sets – or an **injection** (a one-to-one function) only from function f from A to the natural numbers. A simple example is the set of all even natural numbers, E:

$E = \{2, 4, 6, 8, 10, \dots\}$
$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$



If n is an element in the set of natural numbers, we can see that $f: \mathbb{N} \rightarrow E$ as $f(n) = 2n$ and $f^{-1}: E \rightarrow \mathbb{N}$ as $f^{-1}(n) = \frac{n}{2}$. Through the functions we can see that there is a one-to-one correspondence. This means that the functions are bijective, which implies that the set of even natural numbers are countably infinite, and more importantly, its cardinality is equal to aleph-null; $|E| = |\mathbb{N}| = \aleph_0$. This feels very counterintuitive, as clearly there must be more even numbers than natural numbers, and thus the cardinality of E should be more? However, if we think about it, through the functions we can see that distinct natural numbers get mapped to distinct even numbers, hence their cardinalities are the same.

Now the cardinality of the infinite set of even natural numbers is quite simple to understand, which is why I used it as an example so that understanding the cardinality – and thus the uncountability – of the Cantor set will be made easier. In the following section of this exploration, I will first construct the Cantor set, before using proofs to show that it is uncountable and has zero length.

3. Cantor Set Construction

The Cantor set has a variety of different constructions and representations, however the most well-known example (which ironically Cantor only mentioned fleetingly as an example in his work) is the **ternary** set construction. The Cantor set is **iterative** (repeating a process) with the set in actuality being the intersection of all the iterations. How? We start by letting $C_0 = [0,1]$. We then remove the open middle third from the interval C_0 to get the set C_1. This gives us $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. Then to get C_2 we remove the open middle thirds from each of the two segments within C_1 to get $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$. In general, C_n is constructed from C_{n-1} through removing the open middle thirds of any interval in C_{n-1}. Thus, we construct a sequence of sets:

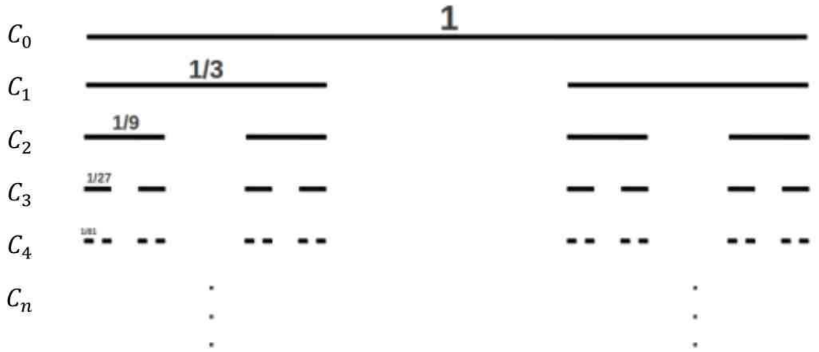
$$C_0 \supset C_1 \supset C_2 \supset \cdots \supset C_n$$

[3] To avoid confusion, keep in mind that one-to-one **correspondence** implies a bijection, whereas simply being called one-to-one implies an injection.

Such that C_n is a union of 2^n disjoint closed intervals, i.e. we start with $2^0 = 1$ as the first line segment is continuous within the interval of [0,1] without any middle third removed, then in each iteration the number of intervals is doubled as each one is split into two parts, a third to each side. (Although some mathematicians and textbooks[ii] use the subset symbol $\supseteq$ I have chosen to use the proper subset symbol, as we can clearly see that C_1 is a proper subset of C_0 and C_2 of C_1 and so forth, since each subset is less than the preceding one as seen in the diagram below). Hence from the above, we can define the Cantor set C as the following:

$$C = \bigcap_{n=0}^{\infty} C_n$$

We can also conclude that each interval will have a length of $\left(\frac{1}{3}\right)^n$. This will be relevant later on when I show that the Cantor set has zero length. Here is an image representation of the Cantor set for clarity:



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3.1 Proof: The Cantor set is uncountable

We have seen with Hilbert's Hotel paradox how there are different sizes of infinity, and that the paradox has exemplified properties such as: $\infty + 1 = \infty$ and $\infty + \infty = \infty$ and so on. We also saw that the set of infinite even natural numbers was countable, as its cardinality is equal to aleph-null, the cardinality of the natural numbers. Most importantly, we understood that a cardinality larger than aleph-null implies an uncountable infinite set, which brings us to our next step. A well-known example of an uncountable infinite set is the set of real numbers $\mathbb{R}$, whose cardinality is higher than aleph-null. Thus, if we demonstrate a **surjective** (page 4 figure) function from the Cantor set to the set of real numbers between 0 and 1, $f: C \rightarrow [0,1]$ we can show that $|C| \geq |[0,1]|$ i.e. the cardinality of the Cantor set will be larger than or equal to the set of real numbers from 0 to 1. However, since $C \subseteq [0,1]$, its cardinality is also less than or equal to the cardinality of [0,1] which means

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that essentially the two sets have equal cardinality. Through this we can show that the Cantor set is uncountable.

In order to do so, we will need to work in a **ternary system.** Similar to the binary system where numbers are written using just the digits 0 and 1 or the decimal system with digits 0 to 9, the ternary system only uses three digits; 0, 1 and 2 (also known as base 3 notation). This means that every number will have a numerical representation using only 0, 1 and/or 2. For example, the number 45_{10} (base 10, standard number system) will be 1200_3. How do we do this? Well if we imagine columns, where the column headings (from right to left) would be powers of 3, i.e. 1, 3, 9, 27,… then we look at the largest power of three that is less than 45. In this case it is 27, and 45 goes into 27 once, so we write 1 in the 27 column. $45 - 27 = 18$ now the largest power less than 18 is 9, and 9 can go into 18 twice, so we write 2 under the 9 column. $18 - 18 = 0$, so in the remaining columns we simply write 0. So, our result is that $45_{10} = 1200_3$ (remember it's from right to left).

This same concept can be applied to the numbers between 0 and 1, except this time the column headings will be $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}$ etc. So, $\frac{1}{3}$ can be written as 0.1, and $\frac{2}{3}$ as 0.2. In our Cantor set, let $x \in [0,1]$ in ternary notation, so that it can be expressed as follows:

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \ where\ a_n = \{0,1,2\}$$

Hence: $x = (0.a_1a_2a_3 \ldots)_3$ and for example, $1 \ = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \ = (0.2222 \ldots)_3$

In C_1, our first middle third that is removed contains only base 3 numbers of the form $0.1xxx$… and $0.21xxx$… where xxx… can only be numbers 000… and/or 222…. So, if this first middle third has been removed, the ternary numbers that remain are in the form of $0.0xxx$… or $0.2xxx$… where the numbers thereafter, x, can be any of the ternary digits. The second iteration is similar as it removes numbers in the form of $0.01xxx$… and $0.21xxx$… leaving base 3 numbers where the first two digits a_1 and a_2 cannot be 1. As we can see, this pattern will continue with every iteration such that if n goes to infinity, a_n will never be equal to 1, and will only consist of the digits 0 and 2:

$$f(x) = f(0.a_1a_2a_3 \ldots) = \{n \in \mathbb{N} \mid a_n = 2\}$$

Thus we can make a surjection from the Cantor set to the real numbers, as all we have to do is map every 2 in any digit in the Cantor set to a 1. So, e.g. 0.202 will be 0.101, which means that we can get the full set of numbers in binary form in the interval [0,1]. This means that there is a surjection from the Cantor set to every real number in the interval [0,1]. And as explained earlier, the real numbers are uncountable which means that the Cantor set is therefore uncountable as well.

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3.2 Proof: The Cantor is a nonempty set with length 0

This is one of the very fascinating properties of the Cantor set, as despite the Cantor set being uncountable, it has a length of {} or 0. To understand this counterintuitive property, if we think back to the image representation of the Cantor set on page 5, we can see that although line *segments* are being removed, the *endpoints* remain. So, each trisection of C_n to form C_{n+1} leaves two endpoints. We've established earlier (page 4) that the Cantor set is the infinite intersection of each C_n, so C contains the endpoints of each subinterval and therefore is nonempty. The fascinating part arises, when it can be proved that although the set is nonempty, it has 0 length.

As before, we let n equal the iterative step, where $n = 0$ is [0,1] without the middle third removed, we know from earlier that the length of each removal is $\left(\frac{1}{3}\right)^n$ and we can understand that the number of intervals as n goes to infinity is 2^{n-1} we can make a geometric series to find out the total length removed:

$$\sum_{n=1}^{\infty}(2^{n-1}) * (3^{-n}) = \left(1 * \frac{1}{3}\right) + \left(2 * \frac{1}{9}\right) + \cdots$$

$$\Rightarrow \sum_{n=1}^{\infty}(2^{n}) * (2^{-1}) * \left(\frac{1}{3^n}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right) * \left(\frac{2}{3}\right)^n$$

$$= \frac{\left(\frac{1}{3}\right)}{\left(1 - \frac{2}{3}\right)} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{1}{3}\right)} = 1$$

Due to this unique property, some mathematicians and textbooks call the Cantor set a **nonempty null set**. This essentially means that the Cantor set is "null" in the sense that it has no length, however since the endpoints still remain after the line segments have been removed, it is therefore non-empty as there are elements that are still within the set.

4. Conclusion

Despite this exploration being successful in its proofs, there are limitations that need to be considered. The most significant limitation is that I did not consider set theory – and thus by extension the Cantor Set– in relation to different systems of axiomatic set theory. Thus I cannot be sure that these proofs – although general – work within every axiomatic set theory like the briefly mentioned Zermelo-Fraenkel set theory. This exploration would then have been far more accurate and reliable, if it had been tested in the different systems of axiomatic set theory.

Essentially, set theory is a huge area of mathematics and encompasses far more than what I have explored here, however through my research with the properties of the Cantor set I believe I have come to a better appreciation of its implications in mathematics. In the introduction, I mentioned how sets and the concept of collections is perhaps more fundamental than counting itself, and through this

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exploration I believe I truly came to understand just how fundamental they are. Some of the most basic concepts in mathematics, like functions which we learn in school, relies on sets. Functions such as $f(x) = 4x$ are rules that essentially map numbers to other numbers. However simply saying “rules” is far too vague, and instead more formally we would say that a function is a relation between two or more *sets* where there is an input and an output. To take it even further, a function maps numbers to numbers however a simple algebraic structure consists of a *set* together with one or more binary operations. This is perhaps most important as the power of modern algebra lies in its ability to create theories that apply just to a collection of things which then follow certain rules, and this would simply not be possible without the mathematical notion of a *set.*

[i] Fannon, Paul, Vesna Kadelburg, Ben Woolley, and Stephen Ward. "Prime Numbers."*Option 10 Discrete Maths Cambridge Mathematics Higher Level HL for IB Diploma*. N.p.: Cambridge UP, 2012. 17. Print.

[ii] Lian, Tony. *FUNDAMENTALS OF ZERMELO-FRAENKEL SET THEORY* (n.d.): n. pag. Web. <http://www.math.uchicago.edu/~may/VIGRE/VIGRE2011/REUPapers/Lian.pdf>.

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